A novel algorithm for calculating Feynman integrals via differential equations

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Antidifferentiation and the Calculation of Feynman Amplitudes
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Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

European Research Council
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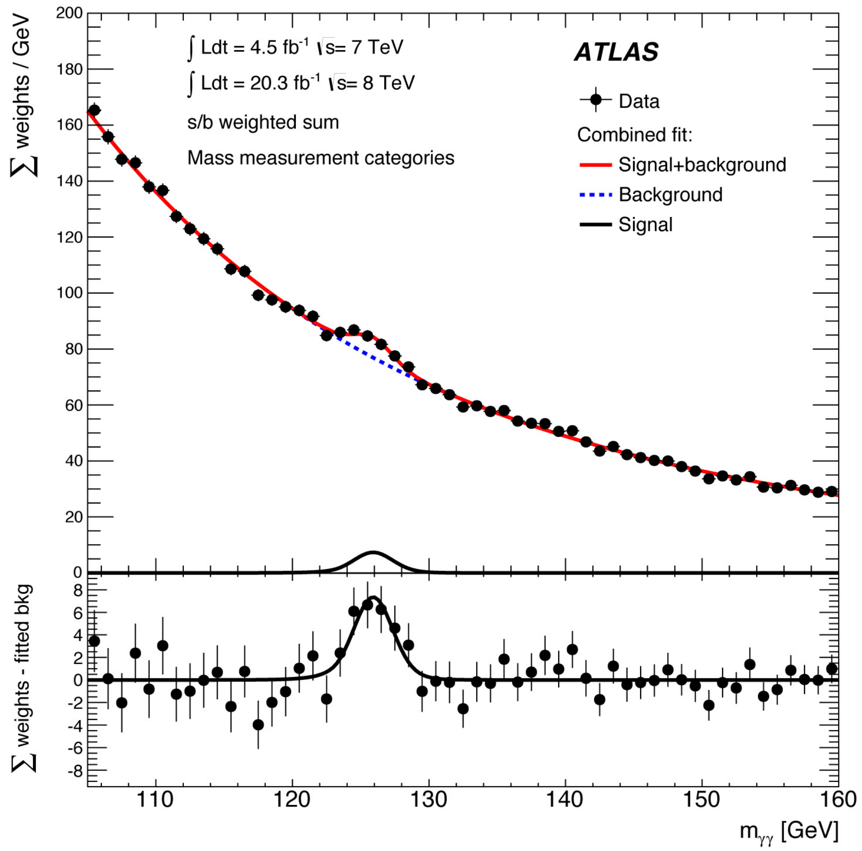
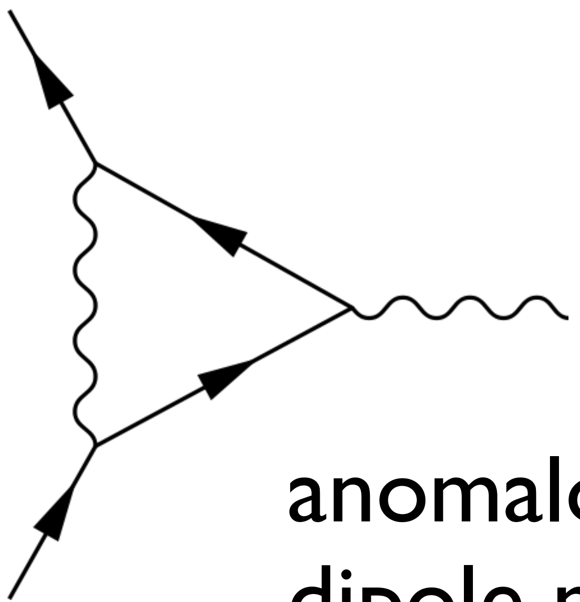
Feynman integrals are important in QFT

Required to compute physical observables

Higgs discovery





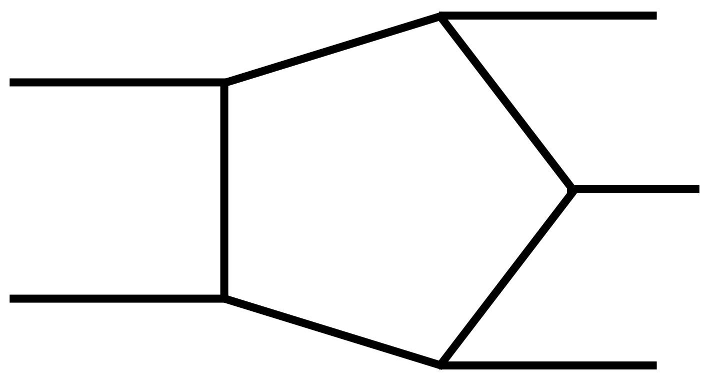
anomalous magnetic dipole moment

Fruitful interplay with mathematics

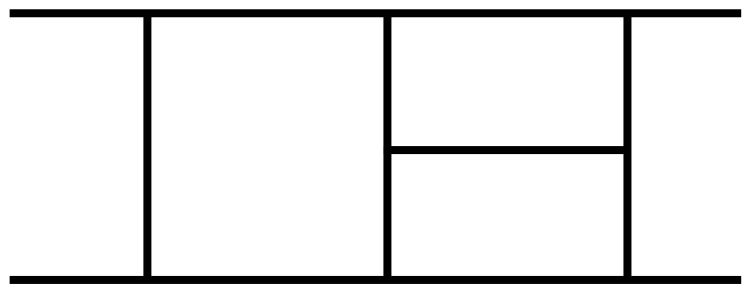
special functions; differential equations; algebraic geometry

Technical challenges

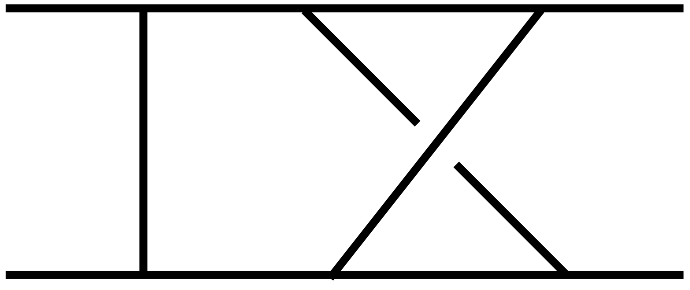
Scales (masses, energies) — multi-variable functions



Multi-loop integrals — multifold iterated integrals



Non-planar diagrams — complicated analytic structure



Challenge: multi-loop Feynman integrals

$$f(p_i \cdot p_j) = \int d^4k_1 d^4k_2 \ldots d^4k_L \, \mathcal{I}(p_i; k_j)$$

Transcendental function

rational loop integrand

- What special functions do Feynman integrals evaluate to?
- What singularities do they have?
- How can we determine the functions efficiently?

Usefulness of dimensional regularisation

Integrals in non-integer dimensions:

$$d^4k \longrightarrow d^{4-2\epsilon}k$$

Physical use: serves to regulate divergences:

$$f(p_i;\epsilon) = \frac{1}{\epsilon^{k_{\max}}} \sum_{k\geq 0} \epsilon^k f^{(k)}(p_i)$$

Mathematical use: organizing principle (grading) $f^{(k)}$ are k-fold iterated integrals (under some assumptions)
We say they have uniform transcendental (UT) weight.

Canonical differential equation (DE) method

n-th order partial differential equations (Picard-Fuchs)

↔

system of 1st order DE

Typically complicated

↓

'canonical' DE define special functions

Very simple

Canonical differential equations [Henn, 2013]

Generic form:

$$\frac{d}{dx} f(x,\epsilon) = A(x,\epsilon) \quad f(x,\epsilon)$$

Matrix with complicated rational entries

$$f \longrightarrow T^{-1} f \qquad A \longrightarrow T^{-1} A T - T^{-1} \partial_x T$$

Canonical form:

$$\frac{d}{dx} f(x,\epsilon) = \epsilon \left[\sum_k m_k \, \frac{1}{x - x_k} \right] f(x,\epsilon)$$

Basis of uniform weight Feynman integrals (UT)

Constant matrices

Singular points

How to find UT integrals?

Solution of canonical DE are iterated integrals of uniform transcendental weight (UT)

- Analysis of singularity structure of DE matrix A

$$f \longrightarrow T^{-1} f \qquad\qquad A \longrightarrow T^{-1} A T - T^{-1}\partial_x T$$

[Lee '14; JMH, '14; Prausa '17; Meyer '17; Gituliar, Magerya '17]

- Analysis of residues of rational loop integrand

[Arkani-Hamed, Bourjaily, Cachzo, Goncharov, Postnikov, Trnka, 2012]

related to `d-log integrand`:

$$\frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$
$$= d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

Practical need for refined methods

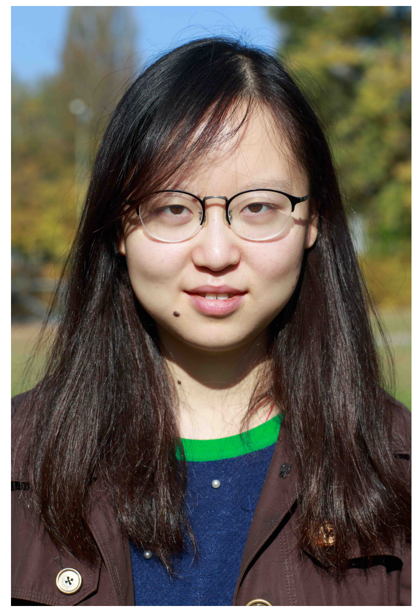
Often easy to find a few UT integrals,
but hard to find a complete UT basis.

Some methods restricted to small matrices,
or to few variables.

Our new method:

- needs only one UT integral as starting point
- can deal with larger matrices
- applies to multiple variables

The collaboration at the Max Planck Institute for Physics



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Based on JHEP 05 (2020) 025

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We develop further an idea by [Höschele, Hoff, Ueda ’14]

- First-order DE only canonical if all integrals are UT

- Picard-Fuchs eq. for a single integral is unique, and contains valuable information.

- They applied this knowledge to find the remaining UT integrals, for cases with 2-3 master integrals, and outlined a general procedure.

- We formulate the method in matrix form, and solve the equations systematically

Public algorithm: [Dlapa, Henn, Yan ’19]

https://github.com/UT-team/INITIAL

Step 1: Picard-Fuchs eq. of a UT integral

Assumption: know one UT integral f_1

- Complete to (any) basis $\vec{f}$, and compute DE

$$\frac{d}{dx}\vec{f} = A(x,\epsilon)\vec{f}\,.$$

- Differentiate $(f_1', f_1'', \cdots f_1^{(n)})^T = \Psi(x,\epsilon)\,\vec{f}$

given by derivatives of A

- Ψ^{-1} yields Picard-Fuchs eq. for f_1

Idea: use the infinite amount of information provided by f_1 being UT.

Step 2: Ansatz for canonical system

Assume existence of UT basis $\vec{g}$ with $g_1 = f_1$.

$$\frac{d}{dx}\vec{g} = \epsilon\,\tilde{A}(x)\,\vec{g}\,.$$

$$(g_1', g_1'', \cdots g_1^{(n)})^T = \Phi(x,\epsilon)\,\vec{g}.$$

Basis transformation: $\vec{f} = T\,\vec{g}\,. \quad \text{where } T \equiv \Psi^{-1}\Phi\,.$

$$\vec{v}_0\Psi^{-1}\Phi = \vec{v}_0$$

← Unit vector

Constraint on Φ from expected canonical form:

$$\tilde{A}(x) = \sum_i \frac{d\ln a_i(x)}{dx}\,m_i$$

$\frac{d\ln a_i(x)}{dx}$: Follow from singularities of A(x)

m_i: constant matrices: to be found

Step 3: Solve equations algorithmically

$$\vec{v}_0 \Psi^{-1} \Phi = \vec{v}_0$$

Unit vector

Known matrix
x, ϵ

Parametrized by set of unknown
constant matrices m_i
Known polynomial ϵ dependence

Equations valid for any x : can use finite field methods.

Solve at each order in ϵ .

Higher orders in ϵ provide consistency check.

Public algorithm available

All steps implemented in public algorithm

https://github.com/UT-team/INITIAL

Presentation was for single-variable case, but method works for multiple variables; key advantage over other methods!

Corollary of our method: test of UT property of a given integral

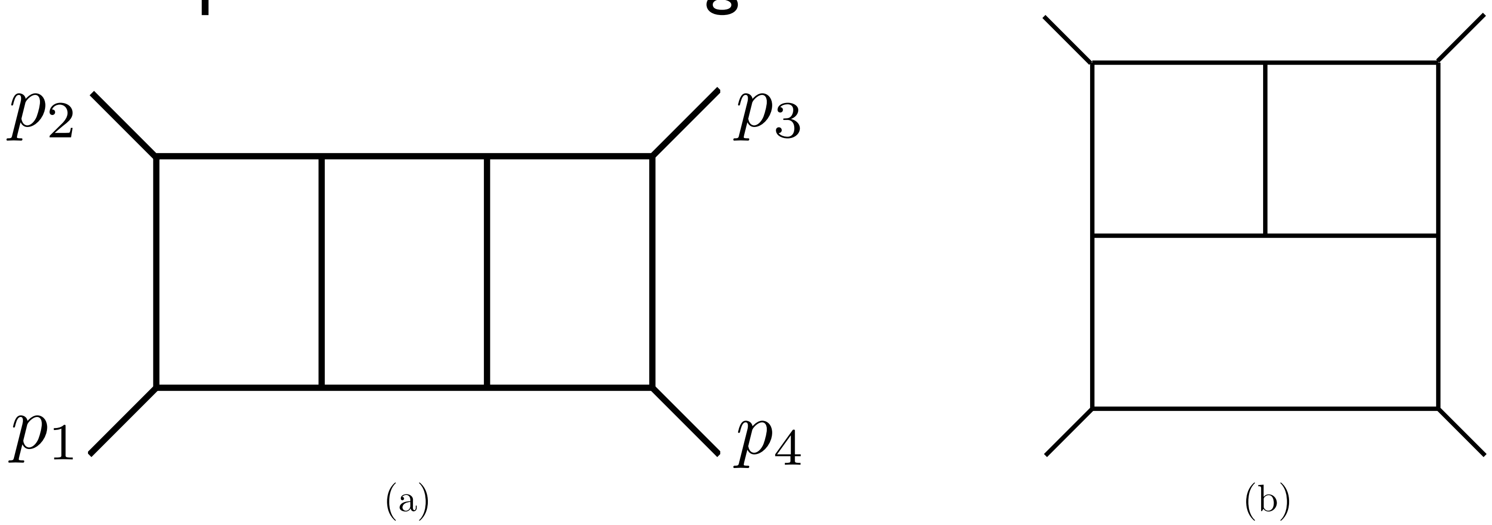
- Algorithm can be used to test whether candidate integrals can be UT
- Can suggest modifications if an integral is 'almost' UT
- Very useful to algorithmically search for UT integrals

State-of-the art applications

Application 1: Planar three-loop on-shell integrals

Four-particle scattering

[Henn, Smirnov^2, '14]



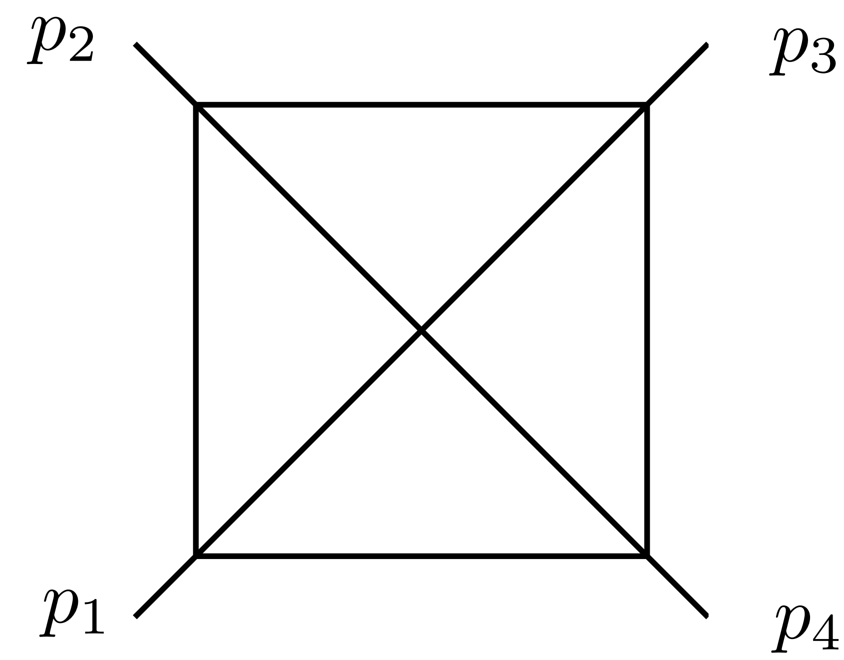
Kinematics: $\sum_{i=1}^{4} p_i = 0\,, \quad p_i^2 = 0\,. \qquad s = (p_1+p_2)^2\,, \quad t = (p_2+p_3)^2\,, \quad x = t/s\,.$

- UT integral in top sector easily found using d-log integrand analysis [Arkani-Hamed et al '11, Henn '13, Wasser, '16]
- Obtained full system of differential equations
  Matrix size 26x26 for case (a), 41x41 for case (b)

$$\tilde{A}(x) = m_0 \frac{d}{dx} \ln x + m_1 \frac{d}{dx} \ln(1+x)$$

Matrix block structure: at most 3 master integrals per sector.

Application 2: Four-loop four-particle scattering



1. Solved the system on the cut (8 MI)

2. Found further UT integrals (off the cut) by testing a large list of candidates

3. Full canonical system (19 MI) of DE

4. Fix boundary constants from analyticity + one trivial integral

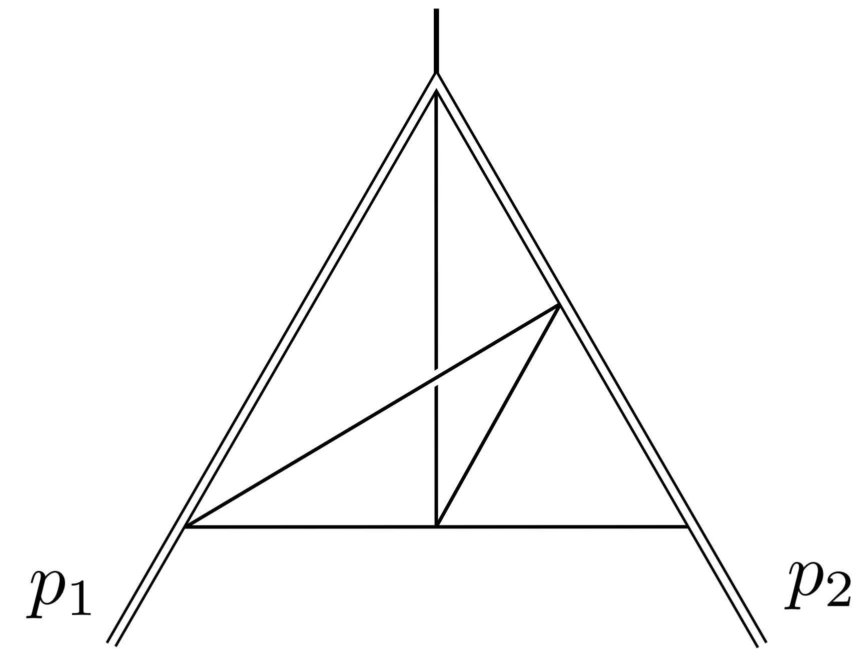
5. Analytic result for scalar integral:

$$\epsilon^6(1-5\epsilon)(1-6\epsilon)\,G_{1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0} = 5\epsilon^5\zeta_5 + \epsilon^6\Big\{\frac{\pi^6}{30} + 3\zeta_3^2 + \frac{1}{6}\pi^4 H_{-1,-1}(x) + \frac{1}{5}\pi^4\,H_{-1,0}(x)$$
$$-\frac{8}{45}\pi^4\,H_{0,-1}(x) + \frac{2}{3}\pi^2\,H_{-1,-1,0,0}(x) - 2\pi^2\,H_{-1,0,0,-1}(x) + \frac{2}{3}\pi^2\,H_{-1,0,0,0}(x) - \frac{2}{3}\pi^2\,H_{0,-1,-1,0}(x)$$
$$+2\pi^2\,H_{0,0,-1,-1}(x) - \frac{2}{3}\pi^2\,H_{0,0,-1,0}(x) + 4\,H_{-1,-1,0,0,0,0}(x) - 4\,H_{-1,0,0,-1,0,0}(x) - 4\,H_{0,-1,-1,0,0,0}(x)$$
$$+4\,H_{0,0,-1,-1,0,0}(x) - \frac{4}{3}\pi^2\zeta_3\,H_{-1}(x) - 4\zeta_3\,H_{0,-1,0}(x) + 4\zeta_3\,H_{0,0,-1}(x) - 20\zeta_5\,H_{-1}(x)\Big\}$$

Application 3: many coupled master integrals

- State of the art for heavy quark effective theory (HQET) integrals: 3 loops [Henn, Korchemsky Marquard '15]

- Here: four-loop non-planar integrals
- 17 coupled master integrals



Kinematics $$\cos\phi = \frac{p_1 \cdot p_2}{\sqrt{p_1^2 p_2^2}} = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

- Form of singularities:

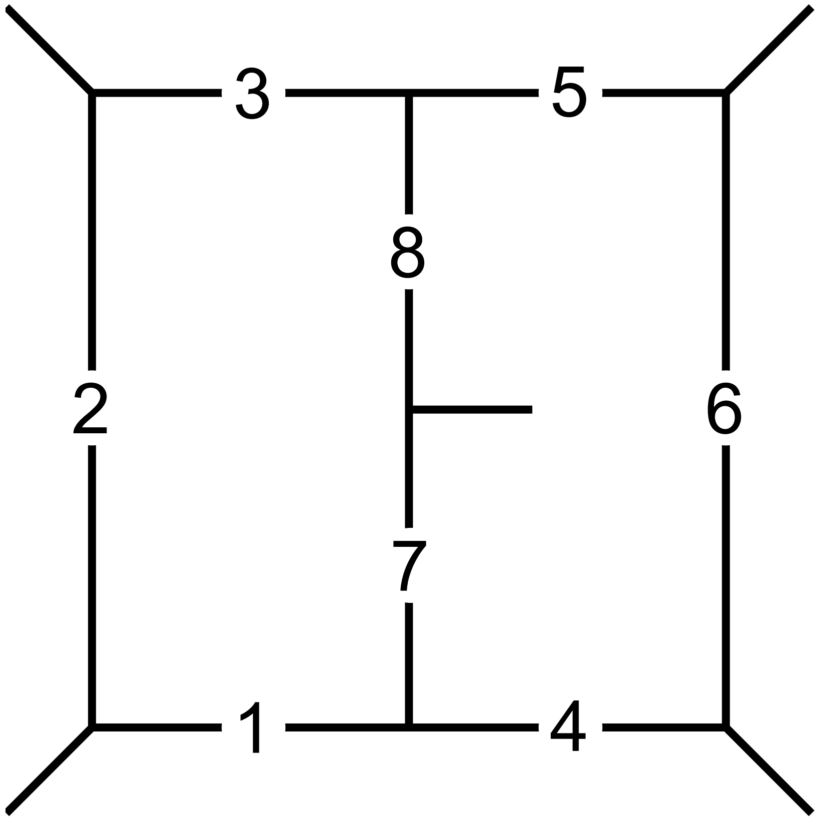
$$\tilde{A}(x) = m_0 \frac{d}{dx} \ln x + m_1 \frac{d}{dx} \ln(1+x) + m_{-1} \frac{d}{dx} \ln(1-x)$$

Solved easily (~10 min) using our algorithm.

Recently applied to four-loop cusp anomalous dimension!

[Brüser, Dlapa, Henn, Yan, 2007.04851 [hep-th] (submitted to PRL)]

Application 4: multi-variable case



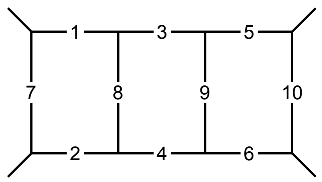
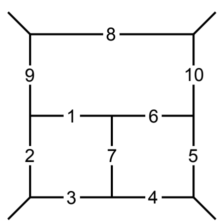
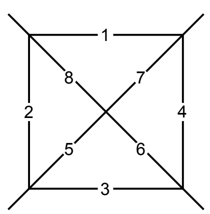
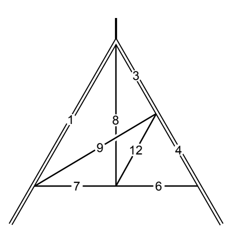
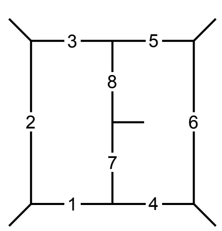
- Two-loop double pentagon integrals
- Computed only 1 year ago using state-of-the-art methods (e.g. D-dimensional leading singularities)

  [Abreu, Dixon, Herrmann, Zeng '18; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]

- 9 coupled integrals in top sector
- Four different kinematic variables
- 17 relevant alphabet letters

Our algorithm takes 5 minutes to find the UT basis on the cut.

Algorithm is efficient for many coupled integrals, and in multi-variable case

Type of problem		#MI	#vars	#letters	time [min.]	Memory [MB]
Full three-loop DE		26 | 3	1	2	2	330
		41 | 3	1	2	34	1710
Full four-loop DE		19 | 12	1	2	1	240
HQET DE on cut		17 | 17	1	3	2	390
Five-point integrals DE on cut		9 | 9	4	17	5	510

Discussion

Our work provides an automated public tool for the calculation of canonical differential equations. It removes an important bottleneck in the calculation of Feynman integrals.

- Our method is efficient for solving large systems of coupled integrals
- Applies to multi-variables case
- Corollary: test of the UT property

Outlook: more complicated integrals

The idea of canonical form of differential equations has also been explicitly applied for elliptic polylogarithms. We find it conceivable that our new ideas can be applied here as well.

'Pre-canonical form':

$$d\,g(\mathbf{x},\epsilon) = [dA_0(\mathbf{x}) + \epsilon\, dA_1(\mathbf{x})]\, g(\mathbf{x},\epsilon)$$

[Henn '14; Mizera, Pokraka '19]

Canonical form

Integrating out A_0 introduces elliptic functions:

$$d\,f(\mathbf{x},\epsilon) = \epsilon\, dA(\mathbf{x}) f(\mathbf{x},\epsilon)$$

Beyond logarithmic kernels.
[Broedel, Duhr, Dulat, Penante, Tancredi '18; Adams and Weinzierl '18]

Outlook: finite integrals (e.g. D=4)

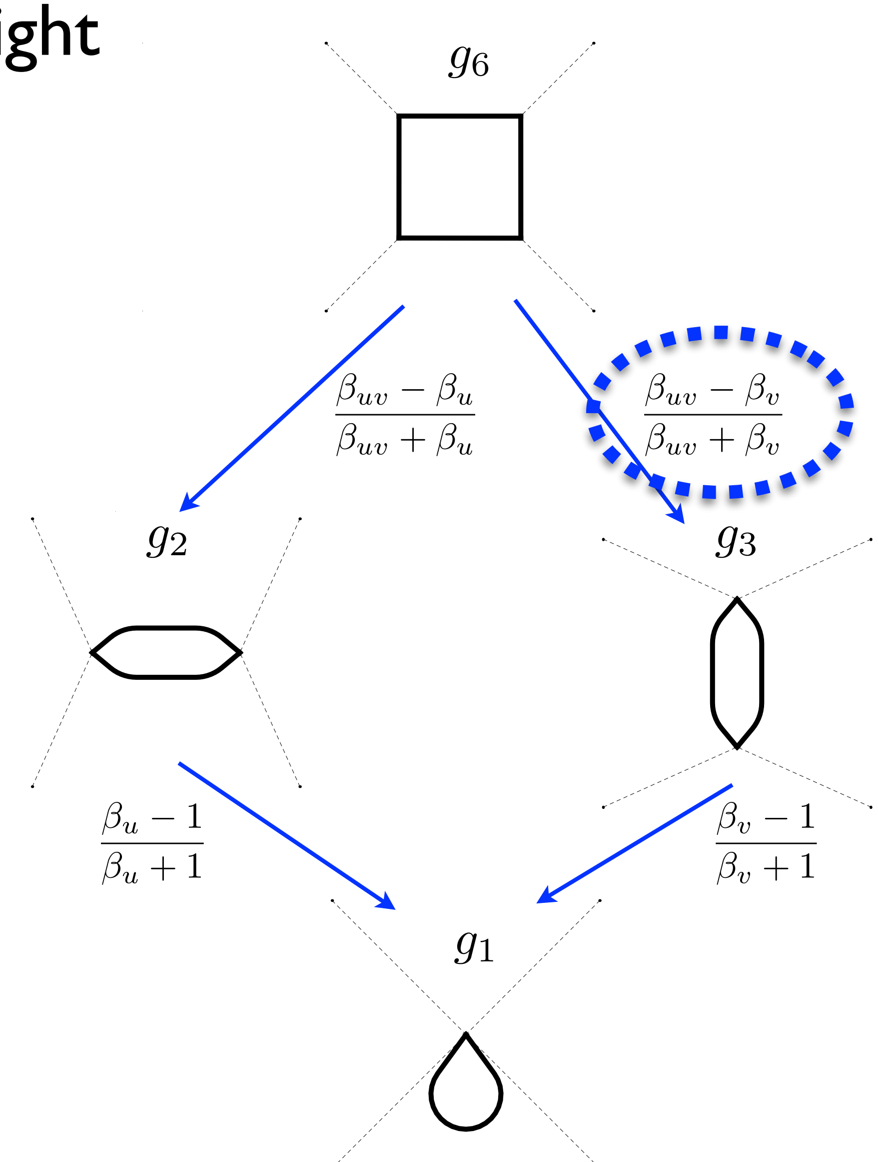
For finite integrals, simplifications occur, and the matrices become nilpotent [Caron-Huot, Henn `14].

Transcendental weight

2

1

0



Arrows represent non-zero matrix entries

$$d\begin{pmatrix} g_6 \\ g_3 \\ g_2 \\ g_1 \end{pmatrix} = d\begin{pmatrix} 0 & \log\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v} & \log\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u} & 0 \\ 0 & 0 & 0 & \log\frac{\beta_u-1}{\beta_u+1} \\ 0 & 0 & 0 & \log\frac{\beta_v-1}{\beta_v+1} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_6 \\ g_3 \\ g_2 \\ g_1 \end{pmatrix}$$

Exploit simpler structure to solve larger systems?

Thank you for your attention!