Antidifferentiation and the Calculation of Feynman Amplitudes, DESY Zeuthen, Oct 4–9, 2020

Holonomic Relations for Hypergeometric Functions and Modular Forms

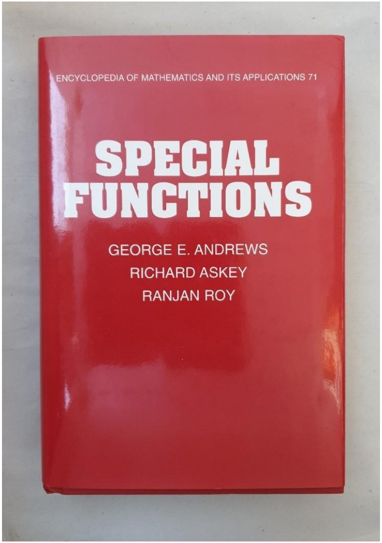
Peter Paule

Johannes Kepler University Linz
Research Institute for Symbolic Computation (RISC)

JOHANNES KEPLER UNIVERSITÄT LINZ

Introduction

Examples from



Pfaff-Saalschütz formula ($n \in \mathbb{Z}_{\geq 1}$):

$$S(n) := {}_3F_2\left(\begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix}; 1\right) = \sum_{k\geq 0} \frac{(-n)_k (a)_k (b)_k}{(c)_k (1+a+b-c-n)_k k!} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

Notation.

$$(a)_k := a(a+1)\dots(a+k-1),\ k \geq 1,\ (a)_0 := 1.$$

NOTE. Pfaff/Saalschütz (1797/1890): binomial version[1]

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \qquad \text{integers } m,n \geqslant 0.$$

[1] Taken from "Concrete Mathematics" by Graham, Knuth, Patashnik.

3

Ex. (Pfaff-Saalschütz)

$$S(n) = \underbrace{\sum_{k\geq 0} \frac{(-n)_k (a)_k (b)_k}{(c)_k (1+a+b-c-n)_k\, k!}}_{t_0(n,k)} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}$$

$$t_1(n,k) = t_0(n+1,k) = \frac{(n+1)(a+b-c-n+k)}{(n-k+1)(a+b-c-n)}\, t_0(n,k)$$

Note: $\mathbb{K} = \mathbb{C}(n)$

$$c_0\, t_0(n,k) + c_1\, t_1(n,k) = \Delta_k\, Cert(k)\, t_0(n,k)$$

$$c_0 = -(a-c-n)(b-c-n) \;(= c_0(n))$$

$$c_1 = c_1(n) = (c+n)(-a-b+c+n)$$

$$Cert(k) = k(c+k-1)(a+b-c+k-n)(n-k+1)^{-1}$$

$$\Rightarrow \quad c_0(n)\, S(n) + c_1(n)\, S(n+1) = 0$$

$\Delta_k f(k) := f(k+1) - f(k).$

4

Pfaff's Method (Ex.: Pfaff-Saalschütz)

$$S(n) = S(n,a,b,c) = \sum_{j=0}^{n} \frac{(-n)_j (a)_j (b)_j}{j!\,(c)_j\,(1+a+b-c-n)_j} = cf_n(a,b,c)$$

$$S(n,a,b,c) - S(n-1,a,b,c)$$

$$= \sum_{j=0}^{n} \left(\frac{(-n)_j (a)_j (b)_j}{j!\,(c)_j\,(1+a+b-c-n)_j} - \frac{(1-n)_j (a)_j (b)_j}{j!\,(c)_j\,(2+a+b-c-n)_j} \right)$$

$$= \sum_{j=0}^{n} \frac{(a)_j (b)_j (1-n)_{j-1}}{j!\,(c)_j\,(1+a+b-c-n)_{j+1}} \cdot \Big((-n)(1+a+b-c-n+j) - (-n+j)(1+a+b-c-n) \Big)$$

$$= -(1+a+b-c) \sum_{j=0}^{n} j \cdot \frac{(a)_j (b)_j (1-n)_{j-1}}{j!\,(c)_j\,(1+a+b-c-n)_{j+1}}$$

$$= -\frac{(1+a+b-c)\,a\,b}{c\,(1+a+b-c-n)(2+a+b-c-n)} \cdot \underbrace{\sum_{j=0}^{n-1} \frac{(a+1)_j (b+1)_j (1-n)_j}{j!\,(c+1)_j\,(3+a+b-c-n)_j}}_{S(n-1,\,a+1,\,b+1,\,c+1)}$$

□

cf = "closed form"

NOTE. This is Pfaff's proof.

5

"There is a somewhat different summation method due to Pfaff. This method is less algorithmic than the WZ method. However, it spreads out the algebraic complications to systems of recurrences. Consequently, it may provide new summations in addition to the one we wish to prove and it may allow the required algebra to be considerably simpler than that required by the WZ method."

6

Ex. (Bailey's ${}_4F_3$ summation)

$$S(n,a,b)=\sum_{j=0}^{n}\frac{(a)_{2j}(b+n)_j(-n)_j}{j!\,(b)_{2j}\,(a+1)_j}=\frac{(b-a)_n}{(b)_n}$$

$$S(n,a,b)-S(n-1,a,b)=\frac{a(1-b-2n)}{b(b+1)}\,T(n-1,a+2,b+2)$$

$$T(n,a,b)=\sum_{j=0}^{n}\frac{(a)_{2j}(b+n-1)_j(-n)_j}{j!\,(b)_{2j}\,(a)_j}$$

$$T(n,a,b)-S(n-1,a,b)=-\frac{(b+n-1)(a+n)}{b(b+1)}\,T(n-1,a+2,b+2)$$

$$T(n,a,b)=\frac{(b-a)_n}{(b+2n-1)(b)_{n-1}}$$ $(S(0,a,b)=T(0,a,b)=1)$

□

SUMMARY:

Pfaff-Saalschütz : $\boxed{S(n,a,b,c) = cf}$

(zb) $c_0(n)\, S(n,a,b,c) + c_1(n)\, S(n+1,a,b,c) = 0$

(Pfaff) $S(n,a,b,c) - S(n-1,a,b,c)$

$= r(n)\, S(n-1, a+1, b+1, c+1)$

Bailey's ${}_4F_3$-Σ : $\boxed{S(n,a,b) = cf}$

(zb) $c_0(n)\, S(n,a,b) + c_1(n)\, S(n+1,a,b) + c_2(n)\, S(n+2,a,b) = 0$

(Pfaff) $S(n,a,b) - S(n-1,a,b)$

$= r_1(n)\, T(n-1, a+2, b+2)$

& $T(n,a,b) - S(n-1,a,b)$

$= r_2(n)\, T(n-1, a+2, b+2)$

SUMMARY:

Pfaff-Saalschütz : $\boxed{S(n,a,b,c) = cf}$

(Zb) $c_0(n)\, S(n,a,b,c) + c_1(n)\, S(n+1,a,b,c) = 0$

(Pfaff) $S(n,a,b,c) - S(n-1,a,b,c)$

$$= r(n)\, S(n-1,a+1,b+1,c+1)$$

Bailey's ${}_4F_3$-Σ : $\boxed{S(n,a,b) = cf}$

(Zb) $c_0(n)\, S(n,a,b) + c_1(n)\, S(n+1,a,b) + c_2(n)\, S(n+2,a,b) = 0$

(Pfaff) $S(n,a,b) - S(n-1,a,b)$

$$= r_1(n)\, T(n-1,a+2,b+2)$$

& $T(n,a,b) - S(n-1,a,b)$

$$= r_2(n)\, T(n-1,a+2,b+2)$$

QUESTIONS:

- Zb $\overset{?}{\longleftrightarrow}$ Pfaff

SUMMARY:

Pfaff-Saalschütz : $S(n,a,b,c) = cf$

(Zb) $c_0(n)\, S(n,a,b,c) + c_1(n)\, S(n+1,a,b,c) = 0$

(Pfaff) $S(n,a,b,c) - S(n-1,a,b,c)$

$= r(n)\, S(n-1,a+1,b+1,c+1)$

Bailey's ${}_4F_3$-Σ : $S(n,a,b) = cf$

(Zb) $c_0(n)\, S(n,a,b) + c_1(n)\, S(n+1,a,b) + c_2(n)\, S(n+2,a,b) = 0$

(Pfaff) $S(n,a,b) - S(n-1,a,b)$

$= r_1(n)\, T(n-1,a+2,b+2)$

& $T(n,a,b) - S(n-1,a,b)$

$= r_2(n)\, T(n-1,a+2,b+2)$

QUESTIONS:

- Zb $\overset{?}{\longleftrightarrow}$ Pfaff
- Why
  Zb-order$=2$?

SUMMARY:

Pfaff-Saalschütz : $S(n,a,b,c) = \text{cf}$

(Zb) $c_0(n)\, S(n,a,b,c) + c_1(n)\, S(n+1,a,b,c) = 0$

(Pfaff) $S(n,a,b,c) - S(n-1,a,b,c)$

$= r(n)\, S(n-1,a+1,b+1,c+1)$

Bailey's ${}_4F_3$-Σ : $S(n,a,b) = \text{cf}$

(Zb) $c_0(n)\, S(n,a,b) + c_1(n)\, S(n+1,a,b) + c_2(n)\, S(n+2,a,b) = 0$

(Pfaff) $S(n,a,b) - S(n-1,a,b)$

$= r_1(n)\, T(n-1,a+2,b+2)$

& $T(n,a,b) - S(n-1,a,b)$

$= r_2(n)\, T(n-1,a+2,b+2)$

QUESTIONS:

- Zb $\overset{?}{\longleftrightarrow}$ Pfaff
- Why
  Zb-order$= 2$?
- Pfaff's method
  less algorithmic?

SUMMARY:

Pfaff-Saalschütz : $S(n,a,b,c) = cf$

(Zb) $c_0(n)\, S(n,a,b,c) + c_1(n)\, S(n+1,a,b,c) = 0$

(Pfaff) $S(n,a,b,c) - S(n-1,a,b,c)$

$$= r(n)\, S(n-1,a+1,b+1,c+1)$$

Bailey's ${}_4F_3$-Σ : $S(n,a,b) = cf$

(Zb) $c_0(n)\, S(n,a,b) + c_1(n)\, S(n+1,a,b) + c_2(n)\, S(n+2,a,b) = 0$

(Pfaff) $S(n,a,b) - S(n-1,a,b)$

$$= r_1(n)\, T(n-1,a+2,b+2)$$

& $T(n,a,b) - S(n-1,a,b)$

$$= r_2(n)\, T(n-1,a+2,b+2)$$

QUESTIONS:

- Zb $\overset{?}{\longleftrightarrow}$ Pfaff
- Why
  Zb-order$= 2$?
- Pfaff's method
  less algorithmic?

All this is explained by CONTIGUOUS RELATIONS!

Contiguous Relations: History

DISQUISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

PARS PRIOR

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE SCIENTIARUM TRADITA 1812. JAN. 30.

Commentationes societatis regiae scientiarum Gottingensis recentiores. Vol. II.

Gottingae MDCCCXIII.

Gauß Werke, Vol. 3, page "131:123" at https://gdz.sub.uni-goettingen.de/id/PPN235999628

SECTIO PRIMA.

Relationes inter functiones contiguas:

7.

Functionem ipsi $F(\alpha, \beta, \gamma, x)$ *contiguam* vocamus, quae ex illa oritur, dum elementum primum, secundum, vel tertium unitate vel augetur vel diminuitur, manentibus tribus reliquis elementis. Functio itaque primaria $F(\alpha. \beta, \gamma, x)$ sex contiguas suppeditat, inter quarum binas ipsamque primariam aequatio persimplex linearis datur. Has aequationes, numero quindecim, hic in conspectum producimus, brevitatis gratia elementum quartum quod semper subintelligitur $= x$ omittentes, functionemque primariam simpliciter per F denotantes.

[1] $0 = (\gamma - 2\alpha - (\beta - \alpha)x)F + \alpha(1-x)F(\alpha+1, \beta, \gamma) - (\gamma-\alpha)F(\alpha-1, \beta, \gamma)$

[2] $0 = (\beta - \alpha)F + \alpha F(\alpha+1, \beta, \gamma) - \beta F(\alpha, \beta+1, \gamma)$

[3] $0 = (\gamma - \alpha - \beta)F + \alpha(1-x)F(\alpha+1, \beta, \gamma) - (\gamma - \beta)F(\alpha, \beta-1, \gamma)$

[4] $0 = \gamma(\alpha - (\gamma - \beta)x)F - \alpha\gamma(1-x)F(\alpha+1, \beta, \gamma) + (\gamma-\alpha)(\gamma-\beta)xF(\alpha, \beta, \gamma+1)$

[5] $0 = (\gamma - \alpha - 1)F + \alpha F(\alpha+1, \beta, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma-1)$

[6] $0 = (\gamma - \alpha - \beta)F - (\gamma - \alpha)F(\alpha-1, \beta, \gamma) + \beta(1-x)F(\alpha, \beta+1, \gamma)$

[7] $0 = (\beta - \alpha)(1-x)F - (\gamma - \alpha)F(\alpha-1, \beta, \gamma) + (\gamma - \beta)F(\alpha, \beta-1, \gamma)$

[8] $0 = \gamma(1-x)F - \gamma F(\alpha-1, \beta, \gamma) + (\gamma - \beta)xF(\alpha, \beta, \gamma+1)$

[9] $0 = (\alpha - 1 - (\gamma - \beta - 1)x)F + (\gamma - \alpha)F(\alpha-1, \beta, \gamma) - (\gamma - 1)(1-x)F(\alpha, \beta, \gamma-1)$

[10] $0 = (\gamma - 2\beta + (\beta - \alpha)x)F + \beta(1-x)F(\alpha, \beta+1, \gamma) - (\gamma - \beta)F(\alpha, \beta-1, \gamma)$

[11] $0 = \gamma(\beta - (\gamma - \alpha)x)F - \beta\gamma(1-x)F(\alpha, \beta+1, \gamma) - (\gamma - \alpha)(\gamma - \beta)F(\alpha, \beta, \gamma+1)$

[12] $0 = (\gamma - \beta - 1)F + \beta F(\alpha, \beta+1, \gamma) - (\gamma - 1)F(\alpha, \beta, \gamma-1)$

[13] $0 = \gamma(1-x)F - \gamma F(\alpha, \beta-1, \gamma) + (\gamma - \alpha)xF(\alpha, \beta, \gamma+1)$

[14] $0 = (\beta - 1 - (\gamma - \alpha - 1)x)F + (\gamma - \beta)F(\alpha, \beta-1, \gamma) - (\gamma - 1)(1-x)F(\alpha, \beta, \gamma-1)$

[15] $0 = \gamma(\gamma - 1 - (2\gamma - \alpha - \beta - 1)x)F + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma+1)$
$\qquad - \gamma(\gamma - 1)(1-x)F(\alpha, \beta, \gamma-1)$

Gauß Werke, Vol. 3, page "138:130" at https://gdz.sub.uni-goettingen.de/id/PPN235999628

10.

Propositae sint e. g. functiones

$$F(\alpha, \beta, \gamma),\ F(\alpha+1, \beta+1, \gamma+1),\ F(\alpha+2, \beta+2, \gamma+2)$$

inter quas aequationem linearem invenire oporteat. Iungamus ipsas per functiones contiguas sequenti modo:

$$F(\alpha, \beta, \gamma) = F$$
$$F(\alpha+1, \beta, \gamma) = F'$$
$$F(\alpha+1, \beta+1, \gamma) = F''$$
$$F(\alpha+1, \beta+1, \gamma+1) = F'''$$
$$F(\alpha+2, \beta+1, \gamma+1) = F''''$$
$$F(\alpha+2, \beta+2, \gamma+1) = F'''''$$
$$F(\alpha+2, \beta+2, \gamma+2) = F''''''$$

Habemus itaque quinque aequationes lineares (e formulis 6, 13, 5 art. 7):

I. $0 = (\gamma-\alpha-1)F - (\gamma-\alpha-1-\beta)F' - \beta(1-x)F''$

II. $0 = \gamma F' - \gamma(1-x)F'' - (\gamma-\alpha-1)xF'''$

III. $0 = \gamma F'' - (\gamma-\alpha-1)F''' - (\alpha+1)F''''$

IV. $0 = (\gamma-\alpha-1)F''' - (\gamma-\alpha-2-\beta)F'''' - (\beta+1)(1-x)F'''''$

V. $0 = (\gamma+1)F'''' - (\gamma+1)(1-x)F''''' - (\gamma-\alpha-1)xF''''''$

Gauß Werke, Vol. 3, page "140:132" at https://gdz.sub.uni-goettingen.de/id/PPN235999628

VI. $0 = \gamma F - \gamma(1-x)F'' - (\gamma-\alpha-\beta-1)xF'''$

Hinc atque ex III, eliminando F''

VII. $0 = \gamma F - (\gamma-\alpha-1-\beta x)F''' - (\alpha+1)(1-x)F''''$

Porro ex IV atque V, eliminando F''''

VIII. $0 = (\gamma+1)F''' - (\gamma+1)F'''' + (\beta+1)xF'''''$

Hinc atque ex VII, eliminando F'''',

IX. $0 = \gamma(\gamma+1)F - (\gamma+1)(\gamma-(\alpha+\beta+1)x)F''' - (\alpha+1)(\beta+1)x(1-x)F'''''$

11.

Si omnes relationes inter ternas functiones $F(\alpha,\beta,\gamma)$, $F(\alpha+\lambda,\beta+\mu,\gamma+\nu)$, $F(\alpha+\lambda',\beta+\mu',\gamma+\nu')$, in quibus $\lambda, \mu, \nu, \lambda', \mu', \nu'$ vel $=0$ vel $=+1$ vel $=-1$, exhaurire vellemus, formularum multitudo usque ad 325 ascenderet. Haud inutilis foret talis collectio, saltem simpliciorum ex his formulis: hoc vero loco sufficiat, paucas tantummodo apposuisse, quas vel ex formulis art. 7, vel si magis placet, simili modo ut duae priores ex illis in art. 8 erutae sunt, quivis nullo negotio sibi demonstrare poterit.

[16] $$F(\alpha,\beta,\gamma) - F(\alpha,\beta,\gamma-1) = -\frac{\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1,\gamma+1)$$

[17] $$F(\alpha,\beta+1,\gamma) - F(\alpha,\beta,\gamma) = \frac{\alpha x}{\gamma}F(\alpha+1,\beta+1,\gamma+1)$$

7

Contiguous Relations

Gauss: $\,_2F_1\left(\begin{matrix} a\pm 1,\ b\pm 1 \\ c\pm 1 \end{matrix};z\right)$

$\,_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};z\right)$ is contiguous to any 2 of these.

Ex.: $$z(1-z)\frac{(a+1)(b+1)}{c(c+1)}\,_2F_1\left(\begin{matrix} a+2,\ b+2 \\ c+2 \end{matrix};z\right)$$

$$+\frac{c-(a+b+1)z}{c}\,_2F_1\left(\begin{matrix} a+1,\ b+1 \\ c+1 \end{matrix};z\right)-\,_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};z\right)=0 \quad \Box$$

Ex.: $$(c-2a-(b-a)z)\,_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};z\right)$$

$$+a(1-z)\,_2F_1\left(\begin{matrix} a+1,\ b \\ c \end{matrix};z\right)-(c-a)\,_2F_1\left(\begin{matrix} a-1,\ b \\ c \end{matrix};z\right)$$

$=0$, etc. $\Box$

NACHLASS.

DETERMINATIO SERIEI NOSTRAE

PER AEQUATIONEM DIFFERENTIALEM SECUNDI ORDINIS.

38.

Statuendo brevitatis causa $F(\alpha, \beta, \gamma, x) = P$, habemus per art. 4

$$\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x)$$

atque hinc differentiando denuo

$$\frac{\mathrm{d}\mathrm{d}P}{\mathrm{d}x^2} = \frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma(\gamma+1)} F(\alpha+2, \beta+2, \gamma+2, x)$$

Hinc aequatio IX art. 10 suppeditat

[80] $$0 = \alpha\beta P - (\gamma - (\alpha+\beta+1)x)\frac{\mathrm{d}P}{\mathrm{d}x} - (x - xx)\frac{\mathrm{d}\mathrm{d}P}{\mathrm{d}x^2}$$

Gauß Werke, Vol. 3, page "215:207" at https://gdz.sub.uni-goettingen.de/id/PPN235999628

Holonomic Functions & Sequences

Def. $(a(n))_{n\geq 0}$ holonomic (P-recursive) $:\Leftrightarrow$

$$p_d(n)a(n+d) + \cdots + p_0(n)a(n) = 0,\ n \geq 0.$$

Def. $(a(n))_{n\geq 0}$ holonomic (P-recursive) :⇔

$$p_d(n)a(n+d)+\cdots+p_0(n)a(n)=0,\ n\geq 0.$$

Def. $y(x)=\sum_{n=0}^{\infty} a(n)x^n$ holonomic (D-finite) :⇔

$$P_m(x)y^{(m)}(x)+\cdots+P_0(x)y(x)=0.$$

Def. $(a(n))_{n\geq 0}$ holonomic (P-recursive) $:\Leftrightarrow$

$$p_d(n)a(n+d)+\cdots+p_0(n)a(n)=0,\ \ n\geq 0.$$

Def. $y(x)=\sum_{n=0}^{\infty} a(n)x^n$ holonomic (D-finite) $:\Leftrightarrow$

$$P_m(x)y^{(m)}(x)+\cdots+P_0(x)y(x)=0.$$

FACT. $\boxed{(a(n))_{n\geq 0} \text{ holonomic}}$ $\Leftrightarrow$ $\boxed{y(x)=\sum_{n=0}^{\infty} a(n)x^n \text{ holonomic.}}$

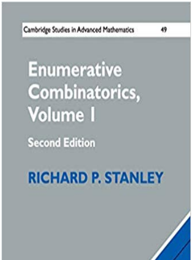
Def. $(a(n))_{n\geq 0}$ holonomic (P-recursive) $:\Leftrightarrow$

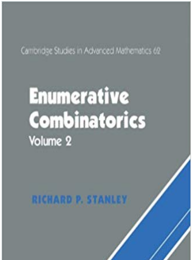
$$p_d(n)a(n+d) + \cdots + p_0(n)a(n) = 0, \;\; n \geq 0.$$

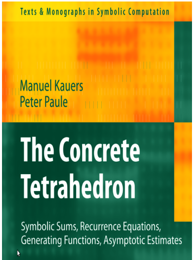
Def. $y(x) = \sum_{n=0}^{\infty} a(n)x^n$ holonomic (D-finite) $:\Leftrightarrow$

$$P_m(x)y^{(m)}(x) + \cdots + P_0(x)y(x) = 0.$$

FACT. $\boxed{(a(n))_{n\geq 0} \text{ holonomic}}$ $\Leftrightarrow$ $\boxed{y(x) = \sum_{n=0}^{\infty} a(n)x^n \text{ holonomic.}}$







Example.

$$y(x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$$

Example.

$$y(x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$$

```
In[3]:= (n_ k_)_* := Binomial[n, k]

In[4]:= (a 3)_*

Out[4]= 1/6 (-2 + a) (-1 + a) a
```

Example.

$$y(x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$$

In[3]:= $\binom{n\_}{k\_}_*$:= Binomial[n, k]

In[4]:= $\binom{a}{3}_*$

Out[4]= $\frac{1}{6}$ (-2 + a) (-1 + a) a

In[5]:= $(a\_)_{k\_}$:= Pochhammer[a, k]

In[6]:= {$(a)_0$, $(a)_1$, $(a)_2$, $(a)_5$}

Out[6]= {1, a, a (1 + a), a (1 + a) (2 + a) (3 + a) (4 + a)}

In[7]:= FullSimplify[$\binom{n}{k}_* - \frac{(n - k + 1)_k}{(1)_k}$]

For holonomic functions and sequences we will use the RISC package:

```
In[16]:= << RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

For holonomic functions and sequences we will use the RISC package:

```
In[16]:= << RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

For hypergeometric summation (Gosper & Zeilberger algo.) we will use the RISC package:

```
In[17]:= << RISC`fastZeil`
```

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Example.

$$y(x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

In[19]:=
```
sdList[N_] := Table[(a)_k (b)_k / ((c)_k k!), {k, 0, N}]
```

$$\texttt{sdList[N\_] := Table}\left[\frac{(a)_k\,(b)_k}{(c)_k\,k!},\ \{k, 0, N\}\right]$$

In[20]:=
```
sdList[3]
```

Out[20]=

$$\left\{1, \frac{a\,b}{c}, \frac{a\,(1+a)\,b\,(1+b)}{2\,c\,(1+c)}, \frac{a\,(1+a)\,(2+a)\,b\,(1+b)\,(2+b)}{6\,c\,(1+c)\,(2+c)}\right\}$$

In[23]:=
```
GuessRE[sdList[8], f[n]]
```

Out[23]=
```
{{-(a + n) (b + n) f[n] + (1 + n) (c + n) f[1 + n] == 0, f[0] == 1}, ogf}
```

In[24]:=
```
RE2DE[%[[1]], f[n], y[x]]
```

Out[24]=

$$\left\{-a\,b\,y[x] + (c - x - a\,x - b\,x)\,y'[x] - \left(-x + x^2\right) y''[x] == 0,\ y[0] == 1,\ y'[0] == \frac{a\,b}{c}\right\}$$

Creative/Parameterized Telescoping

Example. $${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k = \sum_{k=0}^{\infty} \mathtt{sd[k]}\, x^k.$$

Find A, B, C such that

$$A \cdot {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; x\right) + B \cdot {}_2F_1\left(\begin{matrix}a+1, b\\ c\end{matrix}; x\right) + C \cdot {}_2F_1\left(\begin{matrix}a-1, b\\ c\end{matrix}; x\right) = 0.$$

In[56]:= $\mathtt{Mults} = \left\{\frac{(a)_k (b)_k}{(c)_k\, k!} \frac{1}{\mathtt{sd[k]}}, \frac{(a+1)_k (b)_k}{(c)_k\, k!} \frac{1}{\mathtt{sd[k]}}, \frac{(a-1)_k (b)_k}{(c)_k\, k!} \frac{1}{\mathtt{sd[k]}}\right\}$ // FullSimplify

Out[56]=

$\left\{1, \frac{a+k}{a}, \frac{-1+a}{-1+a+k}\right\}$

In[59]:= Gosper[sd[k] x^k, {k, 0, N}, Parameterized → Mults]

If `N' is a natural number, then:

Out[59]=

$\Big\{\mathrm{Sum}[(2\,a - c - a\,x + b\,x)\, F_0[k] + a\,(-1+x)\, F_1[k] + (-a+c)\, F_2[k], \{k, 0, N\}] ==$

$\frac{(b+N)\, x^{1+N}\, \mathrm{Pochhammer}[a, N]\, \mathrm{Pochhammer}[b, N]}{N!\, \mathrm{Pochhammer}[c, N]}\Big\}$

Out[56]=

$$\left\{1, \frac{a+k}{a}, \frac{-1+a}{-1+a+k}\right\}$$

In[59]:=
```
Gosper[sd[k] x^k, {k, 0, N}, Parameterized → Mults]
```

```
If `N' is a natural number, then:
```

Out[59]=

$$\left\{\text{Sum}[(2a-c-ax+bx)\, F_0[k] + a\,(-1+x)\, F_1[k] + (-a+c)\, F_2[k], \{k, 0, N\}] == \frac{(b+N)\, x^{1+N}\, \text{Pochhammer}[a, N]\, \text{Pochhammer}[b, N]}{N!\, \text{Pochhammer}[c, N]}\right\}$$

This translates into:

$$(2a-c-ax+bx)\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(c)_k k!}x^k + a(-1+x)\sum_{k=0}^{N}\frac{(a+1)_k(b)_k}{(c)_k k!}x^k$$

$$+(-a+c)\sum_{k=0}^{N}\frac{(a-1)_k(b)_k}{(c)_k k!}x^k = (b+N)x^{1+N}\frac{(a)_N(b)_N}{(c)_N N!}.$$

[1] $$0 = (\gamma - 2\alpha - (\beta-\alpha)x)F + \alpha(1-x)F(\alpha+1,\beta,\gamma) - (\gamma-\alpha)F(\alpha-1,\beta,\gamma)$$

We computed:

$$(2a-c-ax+bx)\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(c)_k k!}x^k + a(-1+x)\sum_{k=0}^{N}\frac{(a+1)_k(b)_k}{(c)_k k!}x^k$$
$$+(-a+c)\sum_{k=0}^{N}\frac{(a-1)_k(b)_k}{(c)_k k!}x^k = (b+N)x^{1+N}\frac{(a)_k(b)_k}{(c)_k k!}.$$

NOTE. This relation can be also obtained by Zeilberger's algorithm:

In[60]:= Zb[sd[k] x^k, {k, 0, N}, a]

```
If `N' is a natural number, then:
```

Out[60]=

$$\Big\{(-1-a+c)\,\mathrm{SUM}[a]+(2+2a-c-x-ax+bx)\,\mathrm{SUM}[1+a]+(1+a)(-1+x)\,\mathrm{SUM}[2+a]==\frac{(a+N)(b+N)x^{1+N}\,\mathrm{Pochhammer}[a,N]\,\mathrm{Pochhammer}[b,N]}{a\,N!\,\mathrm{Pochhammer}[c,N]}\Big\}$$

Example. $${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k=\sum_{k=0}^{\infty}\mathtt{sd[k]}\,x^k.$$

Find A,B,C such that

$$A\cdot{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)+B\cdot{}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix};x\right)+C\cdot{}_2F_1\left(\begin{matrix}a,b+1\\c\end{matrix};x\right)=0.$$

In[61]:= $\mathtt{Mults}=\left\{\frac{(a)_k\,(b)_k}{(c)_k\,k!}\,\frac{1}{\mathtt{sd[k]}},\ \frac{(a+1)_k\,(b)_k}{(c)_k\,k!}\,\frac{1}{\mathtt{sd[k]}},\ \frac{(a)_k\,(b+1)_k}{(c)_k\,k!}\,\frac{1}{\mathtt{sd[k]}}\right\}$ // FullSi

Out[61]=

$\left\{1,\ \frac{a+k}{a},\ \frac{b+k}{b}\right\}$

In[62]:= Gosper[sd[k] x^k, {k, 0, N}, Parameterized → Mults]

```
If `N' is a natural number, then:
```

Out[62]=

{Sum[(a - b) F_0[k] - a F_1[k] + b F_2[k], {k, 0, N}] == 0}

Out[61]=

$$\left\{1, \frac{a+k}{a}, \frac{b+k}{b}\right\}$$

In[62]:=
```
Gosper[sd[k] x^k, {k, 0, N}, Parameterized → Mults]
```

```
If `N' is a natural number, then:
```

Out[62]=
```
{Sum[(a - b) F0[k] - a F1[k] + b F2[k], {k, 0, N}] == 0}
```

This translates into:

$$(a-b)\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(c)_k k!}x^k - a(-1+x)\sum_{k=0}^{N}\frac{(a+1)_k(b)_k}{(c)_k k!}x^k$$
$$+ b\sum_{k=0}^{N}\frac{(a)_k(b+1)_k}{(c)_k k!}x^k = 0.$$

[2] $\quad 0 = (\beta-\alpha)F + \alpha F(\alpha+1,\beta,\gamma) - \beta F(\alpha,\beta+1,\gamma)$

We computed:

$$(a-b)\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(c)_k k!}x^k - a(-1+x)\sum_{k=0}^{N}\frac{(a+1)_k(b)_k}{(c)_k k!}x^k$$
$$+\,b\sum_{k=0}^{N}\frac{(a)_k(b+1)_k}{(c)_k k!}x^k = 0.$$

NOTE 1. This relation cannot be obtained by Zeilberger's algorithm!

We computed:

$$(a-b)\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(c)_k k!}x^k - a(-1+x)\sum_{k=0}^{N}\frac{(a+1)_k(b)_k}{(c)_k k!}x^k + b\sum_{k=0}^{N}\frac{(a)_k(b+1)_k}{(c)_k k!}x^k = 0.$$

NOTE 1. This relation cannot be obtained by Zeilberger's algorithm!

NOTE 2. But in contrast to other contiguous relations, in this case the identity is trivial:

```
Out[62]=
        {Sum[(a - b) F0[k] - a F1[k] + b F2[k], {k, 0, N}] == 0}

In[63]:= ((a - b) * 1 - a * (a + k)/a + b * (b + k)/b) sd[k]

Out[63]=
        0
```

CLAIM.

All classical (like those of Gauß) and most of of the specialized contiguous relations (i.e., the argument is set to a concrete value) can be computed by parameterized telescoping (i.e., by an extension of Zeilberger's version of creative telescoping).

CLAIM.

All classical (like those of Gauß) and most of of the specialized contiguous relations (i.e., the argument is set to a concrete value) can be computed by parameterized telescoping (i.e., by an extension of Zeilberger's version of creative telescoping).

NOTE.

This CLAIM encloses also relations between ${}_rF_s$-series and also their q-analogues.

- EXAMPLE (specialized contiguous relation):

Contiguous Relations imply $\zeta(\mathbf{2}) = \frac{\pi^{\mathbf{2}}}{\mathbf{6}}$

NT1

Nonterminating hg series

Ex. Explaining how Euler missed an opportunity, K. Knopp & I. Schur proved: *)

$$\zeta(2) = 3 \cdot \sum_{k=1}^{N} \frac{1}{k^2}\binom{2k}{k}^{-1} + \sum_{k=N+1}^{\infty} \frac{1}{k^2}\binom{N+k}{k}^{-1} \quad (N \geq 1)$$

With $c(k) := \frac{1}{k^2}\binom{2k}{k}^{-1}$ this is equiv. to

$$\zeta(2) = {}_3F_2\left(\begin{matrix}1, 1, 1\\ 2, 2\end{matrix}; 1\right) = -c(N) + 3 \cdot \sum_{k=1}^{N} c(k)$$

$$+ c(N)\, \underbrace{{}_3F_2\left(\begin{matrix}N, N, 1\\ N+1, 2N+1\end{matrix}; 1\right)}_{=: F(N)}$$

Case $N = 1$

$$ {}_3F_2\left(\begin{matrix}1, 1, 1\\ 2, 2\end{matrix}; 1\right) = 1 + \frac{1}{2}\, \underbrace{{}_3F_2\left(\begin{matrix}1, 1, 1\\ 2, 3\end{matrix}; 1\right)}_{F(1)}$$

NOTE.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}\binom{2k}{k}^{-1} = 2\arcsin(1/2)^2$$

Remark:

$$ {}_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};z\right) = \frac{d-1}{e}\,{}_3F_2\left(\begin{matrix}a,b,c\\d-1,e+1\end{matrix};z\right) + \frac{1+e-d}{e}\,{}_3F_2\left(\begin{matrix}a,b,c\\d,e+1\end{matrix};z\right) $$

(C46) in

C. Krattenthaler's HYP [List of 3-term contig. rel.]

In addition, by Z's alg.,

$$ F(N) = 1 + \frac{N^2}{(N+1)(2N+1)} + \frac{N^2}{2(N+1)(2N+1)} F(N+1) $$

Iterating this relation gives Knopp/Schur. □

$$ {}_3F_2\left(\begin{matrix}1,1,1\\2-1,2+1\end{matrix};1\right) $$

$$ = {}_2F_1\left(\begin{matrix}1,1\\3\end{matrix};1\right) $$

$$ [\mathrm{Re}(3-1-1) > 0] $$

$$ = \frac{\Gamma(3)\Gamma(3-1-1)}{\Gamma(3-1)\Gamma(2-1)} $$

$$ = 2. $$

SUMMARY.

$$\zeta(2) = {}_3F_2\left({1,1,1 \atop 2,2};1\right) = -c(N) + 3\sum_{k=1}^{N} c(k) + c(N)\,F(N)$$

$$\begin{cases} c(N) = \dfrac{1}{N^2}\dbinom{2N}{N}^{-1} \\ F(N) = {}_3F_2\left({1,1,1 \atop N+1,2N+1};1\right) \end{cases}$$

SUMMARY.

$$\zeta(2) = {}_3F_2\left(\begin{matrix}1,1,1\\2,2\end{matrix};1\right) = -c(N) + 3\sum_{k=1}^{N} c(k) + c(N)\,F(N)$$

$$\begin{cases} c(N) = \dfrac{1}{N^2}\dbinom{2N}{N}^{-1} \\ F(N) = {}_3F_2\left(\begin{matrix}1,1,1\\N+1,2N+1\end{matrix};1\right) \end{cases}$$

$$F(N) = 1 + 2\frac{c(N+1)}{c(N)} + \frac{c(N+1)}{c(N)}F(N+1)$$

SUMMARY.

$$\zeta(2) = {}_3F_2\left(\begin{matrix}1,1,1\\2,2\end{matrix};1\right) = -c(N) + 3\sum_{k=1}^{N} c(k) + c(N)\,F(N)$$

$$\begin{cases} c(N) = \dfrac{1}{N^2}\dbinom{2N}{N}^{-1} \\ F(N) = {}_3F_2\left(\begin{matrix}1,1,1\\N+1,2N+1\end{matrix};1\right) \end{cases}$$

$$F(N) = 1 + 2\frac{c(N+1)}{c(N)} + \frac{c(N+1)}{c(N)}F(N+1)$$

$$\zeta(2) \overset{N=1}{=} 2c(1) + c(1)F(1) = 2c(1) + c(1)\left(1 + 2\frac{c(2)}{c(1)} + \frac{c(2)}{c(1)}F(2)\right)$$

$$= 3c(1) + 2c(2) + c(2)F(2) = \ldots$$

CLAIM.

All classical (like those of Gauß) and most of of the specialized contiguous relations (i.e., the argument is set to a concrete value) can be computed by parameterized telescoping (i.e., by an extension of Zeilberger's version of creative telescoping).

NOTE.

This CLAIM encloses also relations between ${}_rF_s$-series and also their q-analogues.

CLAIM.

All classical (like those of Gauß) and most of of the specialized contiguous relations (i.e., the argument is set to a concrete value) can be computed by parameterized telescoping (i.e., by an extension of Zeilberger's version of creative telescoping).

NOTE.

This CLAIM encloses also relations between ${}_rF_s$-series and also their q-analogues.

- EXAMPLE (q-contiguous relation):

q-Contiguous Relations

From the Digital Library of Mathematical Functions (DLMF):

https://dlmf.nist.gov/17.6#iii

§17.6(iii) Contiguous Relations

Heine's Contiguous Relations

17.6.17
$$ {}_2\phi_1\left({a,b \atop c/q};q,z\right) - {}_2\phi_1\left({a,b \atop c};q,z\right) = cz\frac{(1-a)(1-b)}{(q-c)(1-c)}\,{}_2\phi_1\left({aq,bq \atop cq};q,z\right), $$

17.6.18
$$ {}_2\phi_1\left({aq,b \atop c};q,z\right) - {}_2\phi_1\left({a,b \atop c};q,z\right) = az\frac{1-b}{1-c}\,{}_2\phi_1\left({aq,bq \atop cq};q,z\right), $$

17.6.19
$$ {}_2\phi_1\left({aq,b \atop cq};q,z\right) - {}_2\phi_1\left({a,b \atop c};q,z\right) = az\frac{(1-b)(1-(c/a))}{(1-c)(1-cq)}\,{}_2\phi_1\left({aq,bq \atop cq^2};q,z\right), $$

17.6.20
$$ {}_2\phi_1\left({aq,b/q \atop c};q,z\right) - {}_2\phi_1\left({a,b \atop c};q,z\right) = az\frac{(1-b/(aq))}{1-c}\,{}_2\phi_1\left({aq,b \atop cq};q,z\right), $$

17.6.21
$$ b(1-a)\,{}_2\phi_1\left({aq,b \atop c};q,z\right) - a(1-b)\,{}_2\phi_1\left({a,bq \atop c};q,z\right) = (b-a)\,{}_2\phi_1\left({a,b \atop c};q,z\right), $$

17.6.22
$$ a\left(1-\frac{b}{c}\right){}_2\phi_1\left({a,b/q \atop c};q,z\right) - b\left(1-\frac{a}{c}\right){}_2\phi_1\left({a/q,b \atop c};q,z\right) = (a-b)\left(1-\frac{abz}{cq}\right){}_2\phi_1\left({a,b \atop c};q,z\right), $$

NOTE. Besides classical relations between ${}_r\phi_s$, there are important contiguous relations between related types of q-series. For example:

Q1

q-Series

$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$

$$F(a,b,t) := 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n$$

Manipulatorics gives: (N. Fine)

(1) $F(a,b,t) = 1 + \frac{1-aq}{1-bq} t\, F(aq,bq,t)$

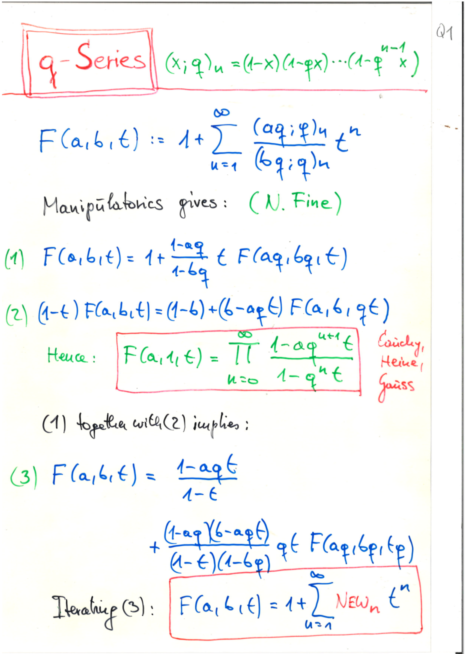
(2) $(1-t)F(a,b,t) = (1-b) + (b-aqt)\, F(a,b,qt)$

Hence: $F(a,1,t) = \prod_{n=0}^{\infty} \frac{1-aq^{n+1}t}{1-q^n t}$ Cauchy, Heine, Gauss

(1) together with (2) implies:

(3) $$F(a,b,t) = \frac{1-aqt}{1-t} + \frac{(1-aq)(b-aqt)}{(1-t)(1-bq)} qt\, F(aq,bq,tq)$$

Iterating (3): $F(a,b,t) = 1 + \sum_{n=1}^{\infty} \mathrm{NEW}_n\, t^n$



Q1

q-Series $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$

$$F(a,b,t) := 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n$$

Manipulatorics gives: (N. Fine)

(1) $F(a,b,t) = 1 + \frac{1-aq}{1-bq} t\, F(aq,bq,t)$

(2) $(1-t)F(a,b,t) = (1-b) + (b-aqt)\, F(a,b,qt)$

Hence: $F(a,1,t) = \prod_{n=0}^{\infty} \frac{1-aq^{n+1}t}{1-q^n t}$ Cauchy, Heine, Gauss

(1) together with (2) implies:

(3) $$F(a,b,t) = \frac{1-aqt}{1-t} + \frac{(1-aq)(b-aqt)}{(1-t)(1-bq)} qt\, F(aq,bq,tq)$$

Iterating (3): $F(a,b,t) = 1 + \sum_{n=1}^{\infty} \mathrm{NEW}_n\, t^n$

All these relations can be computed algorithmically!

To this end we use the RISC package:

In[1]:=
```
<< RISC`qZeil`
```

Package q-Zeilberger version 4.50 written by Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[2]:=
```
<< RISC`qSimplify`
```

In[9]:=
```
? qPochhammer*
```

qPochhammer[a, q] denotes the limit of qPochhammer[a, q, k] for k approaching (positive) infinity.

qPochhammer[a, q, k] represents the q–shifted factorial of a in base q with index k given by

$$(a\,;\,q)_k := \begin{cases} (1-a)\,(1-a\,q)\cdots\left(1-a\,q^k\right), & \text{if } k>0,\\ 1, & \text{if } k=0,\\ \dfrac{1}{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)\cdots\left(1-\frac{a}{q^k}\right)}, & \text{if } k<0. \end{cases}$$

In[10]:=
```
qP = qPochhammer;
```

Relation (1):

In[11]:=
```
qsd[n_] := qP[a q, q, n] / qP[b q, q, n]
```

In[12]:=
```
qP[a q^2, q, n] / qP[b q^2, q, n] * 1/qsd[n] // qSimplify
```

Out[12]=

$$\frac{(1-b\,q)\,(1-a\,q^{1+n})}{(1-a\,q)\,(1-b\,q^{1+n})}$$

In[13]:=
```
qTelescope[qsd[n] t^n, {n, 0, N},
  qParameterized -> {1, ((1 - b q) (1 - a q^(1+n)))/((1 - a q) (1 - b q^(1+n)))}]
```

Out[13]=

$$\text{Sum}\left[F_0[n] - \frac{(-1+a\,q)\,t\,F_1[n]}{-1+b\,q},\ \{n,0,N\}\right] == 1 - \frac{t^{1+N}\,\text{qPochhammer}[a\,q,q,1+N]}{\text{qPochhammer}[b\,q,q,1+N]}$$

$$F(a,b,t)$$

$$= 1 + \sum_{n\geq 1}\frac{(aq;q)_n}{(bq;q)_n}t^n$$

$$= 1 + \frac{1-aq}{1-bq}tF(aq,bq,t)$$

$\left\{\begin{array}{l}\text{apply}\\ N\to\infty\end{array}\right.$

In[14]:= $\frac{\texttt{qP[a q, q, n]}}{\texttt{qP[b q, q, n]}} q^n \frac{1}{\texttt{qsd[n]}}$ // qSimplify

Out[14]=

q^n

In[15]:= qTelescope[qsd[n] t^n, {n, 0, N}, qParameterized → {1, q^n}]

Out[15]=

$$\text{Sum}[(-1+t)\,F_0[n] + (b - a\,q\,t)\,F_1[n],\ \{n, 0, N\}] == -1 + b + \frac{t^{1+N}\,\text{qPochhammer}[a\,q, q, 1+N]}{\text{qPochhammer}[b\,q, q, N]}$$

```
In[14]:= qP[a q, q, n] / qP[b q, q, n] q^n 1/qsd[n] // qSimplify

Out[14]=
q^n

In[15]:= qTelescope[qsd[n] t^n, {n, 0, N}, qParameterized → {1, q^n}]

Out[15]=
Sum[(-1 + t) F0[n] + (b - a q t) F1[n], {n, 0, N}] ==
 -1 + b + t^(1+N) qPochhammer[a q, q, 1 + N] / qPochhammer[b q, q, N]
```

Relation (2). $(1-t)F(a,b,t) = 1-b+(b-aqt)F(a,b,qt)$

In[14]:=
```
qP[a q, q, n]      n     1
------------- q   ------- // qSimplify
qP[b q, q, n]     qsd[n]
```

Out[14]=

q^n

In[15]:=
```
qTelescope[qsd[n] t^n, {n, 0, N}, qParameterized → {1, q^n}]
```

Out[15]=

$$\text{Sum}[(-1+t)\,F_0[n] + (b - a\,q\,t)\,F_1[n],\ \{n, 0, N\}] == -1 + b + \frac{t^{1+N}\,\text{qPochhammer}[a\,q,\ q,\ 1+N]}{\text{qPochhammer}[b\,q,\ q,\ N]}$$

Relation (2). $(1-t)F(a,b,t) = 1-b+(b-aqt)F(a,b,qt)$

$$F(a,1,t) = \frac{1-aqt}{1-t}F(a,1,qt)$$
$$= \frac{1-aqt}{1-t}\frac{1-aq^2t}{1-qt}F(a,1,q^2t) = \dots$$

Telescoping Contiguous Relations

Existence of Telescoping Contiguous Relations

CONVERGENCE ${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right)$ non-terminating

$p \leq q$	$p \geq q+2$	$p = q+1$
abs. conv. $\forall z$	div. $z \neq 0$	div. $\lvert z\rvert > 1$
	abs. conv. $z = 0$	abs. conv. $\lvert z\rvert < 1$
		$\lvert z\rvert = 1$ Case

$\lvert z\rvert = 1$ Case with $p = q+1$:

$$k_0 := \sum_j b_j - \sum_i a_i$$

$\mathrm{Re}(k_0) \leq -1$	$-1 < \mathrm{Re}(k_0) \leq 0$	$0 < \mathrm{Re}(k_0)$
div. $\forall z$	div. $z = 1$	abs. conv. $\forall z$
	cond. conv. else	

Preparations

Def.: Given ${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right)$,

$$ {}_pP_q(x) := z\cdot\prod_{i=1}^{p}(x+a_i) - x\cdot\prod_{j=1}^{q}(x+b_j-1) $$

Note 1: Let $m := \max\{p, q+1\}$:

$$ \deg {}_pP_q(x) \begin{cases} \leq m-1, & \text{if } p=q+1 \;\&\; z=1 \\ = m, & \text{otherwise} \end{cases} $$

Preparations

Def.: Given ${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right)$,

$$ {}_pP_q(x) := z\cdot\prod_{i=1}^{p}(x+a_i) - x\cdot\prod_{j=1}^{q}(x+b_j-1) $$

Note 1: Let $m := \max\{p, q+1\}$:

$$ \deg {}_pP_q(x) \begin{cases} \leq m-1, & \text{if } p=q+1 \text{ \& } z=1 \\ = m, & \text{otherwise} \end{cases} $$

Note 2:

$$ {}_pP_q(zD_z)\ {}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right) = 0 $$

NOTATION. For $k \geq 0$,

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right)_k = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} $$

THEOREM 1

Case $z \neq 1$ OR $p \neq q+1$. $\to$ $d = q+1$ T2

Theorem Let $d = \deg {}_pP_q(x)$, then:

(1) $\exists\, c_\ell \in \mathbb{K}$, not all 0, and $\mathcal{C}(x) \in \mathbb{K}[x]$ s.t.

$$\sum_{\ell=0}^{d} c_\ell \cdot {}_pF_q\left(\begin{matrix} a_1+\alpha_1^{(\ell)}, \ldots, a_p+\alpha_p^{(\ell)} \\ b_1-\beta_1^{(\ell)}, \ldots, b_q-\beta_q^{(\ell)} \end{matrix}; z\right)_k$$

$$= \Delta_k\, \mathcal{C}(k)\; {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right)_k \quad ;$$

(2) $\mathcal{C}(0) = 0$, and if $\mathcal{C} \neq 0$,

THEOREM 1 (contd.)

$$\deg C(x) \le q + 1 - d + \max_{0 \le e \le d} \left\{ \alpha_1^{(e)} + \ldots + \alpha_p^{(e)} + \beta_1^{(e)} + \ldots + \beta_q^{(e)} \right\} =: M$$

Remark: $\alpha_i^{(e)}, \beta_j^{(e)} \in \mathbb{N}_0 = \{0, 1, \ldots\}$

THEOREM 1 (contd.)

$$\deg C(x) \le q + 1 - d + \underbrace{\max_{0 \le e \le d} \{ \alpha_1^{(e)} + \dots + \alpha_p^{(e)} + \beta_1^{(e)} + \dots + \beta_q^{(e)} \}}_{=: M}$$

Remark: $\alpha_i^{(e)}, \beta_j^{(e)} \in \mathbb{N}_0 = \{0, 1, \dots\}$

(3)

$$\lim_{k \to \infty} C(k)\, {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right)_k = 0$$

Example z.1.0: $z \neq 1$ and $(p, q) = (1, 0)$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a+\alpha^{(\ell)} \\ - \end{matrix}; z\right)_k = \Delta_k C(k)\, {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right)_k \tag{1}$$

$${}_1P_0(x) = z(x+a) - x\prod_{j=1}^{0}(x+b_j-1) = (z-1)x + za.$$

By THEOREM 1: $d = \deg({}_1P_0(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (1) holds with $2 = d+1$ summands on the left. Because of $C(0) = 0$ and $\lim_{k\to\infty} C(k)\, {}_1F_0(a; -; z)_k = 0$:

Example z.1.0: $z \neq 1$ and $(p, q) = (1, 0)$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a+\alpha^{(\ell)} \\ - \end{matrix}; z\right)_k = \Delta_k C(k)\, {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right)_k \qquad (1)$$

$${}_1P_0(x) = z(x+a) - x\prod_{j=1}^{0}(x+b_j-1) = (z-1)x + za.$$

By THEOREM 1: $d = \deg({}_1P_0(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (1) holds with $2 = d+1$ summands on the left. Because of $C(0) = 0$ and $\lim_{k\to\infty} C(k)\, {}_1F_0(a; -; z)_k = 0$:

$$c_0\, {}_1F_0\left(\begin{matrix} a+\alpha^{(0)} \\ - \end{matrix}; z\right) + c_1\, {}_1F_0\left(\begin{matrix} a+\alpha^{(1)} \\ - \end{matrix}; z\right) = 0.$$

Example z.1.0: $z \neq 1$ and $(p,q) = (1,0)$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a+\alpha^{(\ell)} \\ - \end{matrix}; z\right)_k = \Delta_k C(k)\, {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right)_k \qquad (1)$$

$${}_1P_0(x) = z(x+a) - x\prod_{j=1}^{0}(x+b_j-1) = (z-1)x + za.$$

By THEOREM 1: $d = \deg({}_1P_0(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (1) holds with $2 = d+1$ summands on the left. Because of $C(0) = 0$ and $\lim_{k\to\infty} C(k)\,{}_1F_0(a; -; z)_k = 0$:

$$c_0 \underbrace{{}_1F_0\left(\begin{matrix} a+\alpha^{(0)} \\ - \end{matrix}; z\right)}_{(1-z)^{-(a+\alpha^{(0)})}} + c_1 \underbrace{{}_1F_0\left(\begin{matrix} a+\alpha^{(1)} \\ - \end{matrix}; z\right)}_{(1-z)^{-(a+\alpha^{(1)})}} = 0.$$

Example z.2.1: $z \neq 1$ and $(p,q) = (2,1)$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; z\right)_k = \Delta_k C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)_k \qquad (2)$$

$${}_2P_1(x) = z(x+a)(x+b) - x(x+c-1) = (z-1)x^2 + (az+bz-c+1)x + abz$$

By THEOREM 1: $d = \deg({}_1P_0(x)) = 2$ and $\exists c_0, c_1, c_2 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (2) holds with $3 = d+1$ summands on the left. Because of $C(0) = 0$ and $\lim_{k\to\infty} C(k)\,{}_2F_1(a;b;c;z)_k = 0$:

Example z.2.1: $z \neq 1$ and $(p,q) = (2,1)$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; z\right)_k = \Delta_k C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)_k \qquad (2)$$

$${}_2P_1(x) = z(x+a)(x+b) - x(x+c-1) = (z-1)x^2 + (az+bz-c+1)x + abz$$

By THEOREM 1: $d = \deg({}_1P_0(x)) = 2$ and $\exists c_0, c_1, c_2 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (2) holds with $3 = d+1$ summands on the left. Because of $C(0) = 0$ and $\lim_{k\to\infty} C(k)\, {}_2F_1(a;b;c;z)_k = 0$:

$$c_0\, {}_2F_1\left(\begin{matrix} a+\alpha^{(0)}, b+\beta^{(0)} \\ c-\gamma^{(0)} \end{matrix}; z\right) + c_1\, 2F_1\left(\begin{matrix} a+\alpha^{(1)}, b+\beta^{(1)} \\ c-\gamma^{(1)} \end{matrix}; z\right)$$
$$+ c_2\, 2F_1\left(\begin{matrix} a+\alpha^{(2)}, b+\beta^{(2)} \\ c-\gamma^{(2)} \end{matrix}; z\right) = 0. \qquad \text{(Gauß)}$$

THEOREM 2a for $z = 1$ and $p = q + 1$ and

Let $\sum_j b_j - \sum_i a_i - q \notin \mathbb{N}_0$.

Then for $\boxed{d = q}$ $\left(= \deg {}_{q+1}P_q(x)\right)$:

(1) $\exists\, c_\ell \in \mathbb{K}$, not all 0, and $C(x) \in \mathbb{K}[x]$ s.t.

$$\sum_{\ell=0}^{d} c_\ell \; {}_{q+1}F_q\left(\begin{matrix} a_1 + \alpha_1^{(\ell)}, \ldots, a_{q+1} + \alpha_{q+1}^{(\ell)} \\ b_1 - \beta_1^{(\ell)}, \ldots, b_q - \beta_q^{(\ell)} \end{matrix}; 1\right)_k$$

$$= \Delta_k C(k) \underbrace{{}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k}_{=:\, t(k)} ;$$

(2) $C(0) = 0$, and

$$\lim_{k \to \infty} C(k)\, t(k) = 0 .$$

Example 1.1.0.a: $z = 1$ and $(p, q) = (1, 0)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a + \alpha^{(\ell)} \\ - \end{matrix}; 1\right)_k = \Delta_k C(k) {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; 1\right)_k \quad (3)$$

$${}_1P_0(x) = 1(x+a) - x \prod_{j=1}^{0}(x+b_j-1) = a; \sum_j b_j - \sum_i a_i = -a \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2 a: $d = \deg({}_1P_0(x)) = q = 0$ and $\exists c_0 \in \mathbb{C} \setminus \{0\}$ and $C(x) \in \mathbb{C}[x]$ such that relation (3) holds with $1 = d + 1$ summand on the left. I.e., we have a TELESCOPING sum

Example 1.1.0.a: $z = 1$ and $(p, q) = (1, 0)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a + \alpha^{(\ell)} \\ - \end{matrix}; 1\right)_k = \Delta_k C(k) {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; 1\right)_k \tag{3}$$

$${}_1P_0(x) = 1(x+a) - x\prod_{j=1}^{0}(x+b_j-1) = a; \sum_j b_j - \sum_i a_i = -a \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2 a: $d = \deg({}_1P_0(x)) = q = 0$ and $\exists c_0 \in \mathbb{C} \setminus \{0\}$ and $C(x) \in \mathbb{C}[x]$ such that relation (3) holds with $1 = d + 1$ summand on the left. I.e., we have a TELESCOPING sum

$$\sum_{k=0}^{N} {}_1F_0\left(\begin{matrix} a + \alpha^{(0)} \\ - \end{matrix}; 1\right)_k = \sum_{k=0}^{N} (-1)^k \binom{-(a + \alpha^{(0)})}{k} = \mathrm{cf}(N).$$

Example 1.1.0.a: $z = 1$ and $(p, q) = (1, 0)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_1F_0\left(\begin{matrix} a + \alpha^{(\ell)} \\ - \end{matrix}; 1\right)_k = \Delta_k C(k)\, {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; 1\right)_k \qquad (3)$$

$${}_1P_0(x) = 1(x+a) - x\prod_{j=1}^{0}(x+b_j-1) = a; \sum_j b_j - \sum_i a_i = -a \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2 a: $d = \deg({}_1P_0(x)) = q = 0$ and $\exists c_0 \in \mathbb{C} \setminus \{0\}$ and $C(x) \in \mathbb{C}[x]$ such that relation (3) holds with $1 = d + 1$ summand on the left. I.e., we have a TELESCOPING sum

$$\sum_{k=0}^{N} {}_1F_0\left(\begin{matrix} a + \alpha^{(0)} \\ - \end{matrix}; 1\right)_k = \sum_{k=0}^{N} (-1)^k \binom{-(a + \alpha^{(0)})}{k} = \mathrm{cf}(N).$$

NOTE. ${}_1F_0\left(\begin{matrix} a+\alpha^{(0)} \\ - \end{matrix}; 1\right) = 0 = (1-1)^{-(a+\alpha^{(0)})}$, if convergent.

Example 1.2.1a: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (4)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2a: $d = \deg({}_2P_1(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (4) holds with $2 = d+1$ summands on the left. With parameterized telescoping one, e.g., computes:

Example 1.2.1a: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \tag{4}$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2a: $d = \deg({}_2P_1(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (4) holds with $2 = d + 1$ summands on the left. With parameterized telescoping one, e.g., computes:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{(c-1)(c-a-b-1)}{(c-1-a)(c-1-b)} {}_2F_1\left(\begin{matrix} a, b \\ c-1 \end{matrix}; 1\right)$$

Example 1.2.1a: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q \notin \mathbb{Z}_{\geq 0}$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (4)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 \notin \mathbb{Z}_{\geq 0}$$

By THEOREM 2a: $d = \deg({}_2P_1(x)) = 1$ and $\exists c_0, c_1 \in \mathbb{C}$, not all 0, and $C(x) \in \mathbb{C}[x]$ such that relation (4) holds with $2 = d+1$ summands on the left. With parameterized telescoping one, e.g., computes:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{(c-1)(c-1-a-b-1)}{(c-1-a)(c-1-b)}\, {}_2F_1\left(\begin{matrix} a, b \\ c-1 \end{matrix}; 1\right)$$

$$= \frac{(c-1)(c-1-a-b)}{(c-1-a)(c-1-b)} \frac{(c-2)(c-2-a-b)}{(c-2-a)(c-2-b)}\, {}_2F_1\left(\begin{matrix} a, b \\ c-2 \end{matrix}; 1\right) \quad \text{(Gauß)}$$

THEOREM 2b for $z = 1$ and $p = q + 1$ and

Let $\sum_j b_j - \sum_i a_i - q \in \mathbb{N}$. $(=\{1,2,\ldots\})$

Then for $\boxed{d = q-1}^{(*)}$: $\left(= \deg {}_{q+1}P_q(x) - 1\right)$

$(*)$ $d = 0$ if $q = 0$.

(1) $\exists\, c_\ell \in \mathbb{K}$, not all 0, and $C(x) \in \mathbb{K}[x]$ s.t.

$$\sum_{\ell=0}^{d} c_\ell \; {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(\ell)}, \ldots, a_{q+1}+\alpha_{q+1}^{(\ell)} \\ b_1-\beta_1^{(\ell)}, \ldots, b_q-\beta_q^{(\ell)} \end{matrix}; 1\right)_k$$

$$= \Delta_k C(k) \underbrace{{}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k}_{=:\, t(k)};$$

(2) $C(0) = 0$, and

$$\lim_{k\to\infty} C(k)\, t(k) = \begin{cases} 0 \\ \text{OR} \\ \operatorname{lc}(C(x)) \cdot \dfrac{\prod_j \Gamma(b_j)}{\prod_i \Gamma(a_i)} \end{cases}$$

Example 1.2.1b: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q \in \mathbb{Z}_{\geq 1}$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (5)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 \in \mathbb{Z}_{\geq 1}$$

By THEOREM 2b: $d = q - 1 = 0$ and $\exists c_0 \in \mathbb{C}$, not 0, and $C(x) \in \mathbb{C}[x]$ such that relation (5) holds with $1 = d + 1$ summand on the left. With Gosper one, e.g., computes:

Example 1.2.1b: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q \in \mathbb{Z}_{\geq 1}$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (5)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 \in \mathbb{Z}_{\geq 1}$$

By THEOREM 2b: $d = q - 1 = 0$ and $\exists c_0 \in \mathbb{C}$, not 0, and $C(x) \in \mathbb{C}[x]$ such that relation (5) holds with $1 = d + 1$ summand on the left. With Gosper one, e.g., computes:

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+2)_k} = \frac{(ab+a+b+N+1)}{(a+1)\,(b+1)} \frac{(a+1)_N (b+1)_N}{N!(a+b+2)_N}.$$

NOTE. In THEOREM 2b the choice 2 is minimal.

THEOREM 2c for $z = 1$ and $p = q + 1$ and

Let $\sum_j b_j - \sum_i a_i - q = 0$.

Then for $d = \max\{M, \deg_{q+1} P_q(x)\}$ $(< q)$

(1) $\exists\, c_\ell \in \mathbb{K}$, not all 0, and $C(x) \in \mathbb{K}[x]$ s.t.

$$\sum_{\ell=0}^{d} c_\ell \; {}_{q+1}F_q\left(\begin{matrix} a_1+\alpha_1^{(\ell)}, \ldots, a_{q+1}+\alpha_{q+1}^{(\ell)} \\ b_1-\beta_1^{(\ell)}, \ldots, b_q-\beta_q^{(\ell)} \end{matrix}; 1\right)_k$$

$$= \Delta_k C(k) \; \underbrace{{}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; 1\right)_k}_{=: t(k)};$$

(2) $C(0) = 0$, and

$$\lim_{k\to\infty} C(k)\, t(k) = \begin{cases} 0 \\ \text{OR} \\ \operatorname{lcf}(C(x)) \cdot \dfrac{\prod_j \Gamma(b_j)}{\prod_i \Gamma(a_i)} \end{cases}$$

Example 1.2.1c: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q = 0$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k) {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (6)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 = 0 \quad \text{and}$$

$$d = \max\{\alpha^{(0)} + \beta^{(0)} + \gamma^{(0)}, \ldots, \alpha^{(d)} + \beta^{(d)} + \gamma^{(d)}, \underbrace{\deg({}_2P_1(x))}_{=0}\}$$

By THEOREM 2c: choosing $d = 0$ and $\alpha^{(0)} = \beta^{(0)} = \gamma^{(0)} = 0$, $\exists c_0 \in \mathbb{C}$, not 0, and $C(x) \in \mathbb{C}[x]$ such that relation (6) holds with $1 = d + 1$ summand on the left. With Gosper one computes:

Example 1.2.1c: $z = 1$ and $(p, q) = (2, 1)$ and $k_0 - q = 0$

$$\sum_{\ell=0}^{d} {}_2F_1\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)} \\ c-\gamma^{(\ell)} \end{matrix}; 1\right)_k = \Delta_k C(k)\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)_k \qquad (6)$$

$${}_2P_1(x) = (x+a)(x+b) - x(x+c-1) = (-c+a+b+1)x + ab;$$

$$\sum_j b_j - \sum_i a_i - q = c - a - b - 1 = 0 \quad \text{and}$$

$$d = \max\{\alpha^{(0)} + \beta^{(0)} + \gamma^{(0)}, \ldots, \alpha^{(d)} + \beta^{(d)} + \gamma^{(d)}, \underbrace{\deg({}_2P_1(x))}_{=0}\}$$

By THEOREM 2c: choosing $d = 0$ and $\alpha^{(0)} = \beta^{(0)} = \gamma^{(0)} = 0$, $\exists c_0 \in \mathbb{C}$, not 0, and $C(x) \in \mathbb{C}[x]$ such that relation (6) holds with $1 = d + 1$ summand on the left. With Gosper one computes:

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+1)_k} = \frac{(a+1)_N (b+1)_N}{N!(a+b+1)_N}.$$

REMARKS. (1) E.g., with the choice $\alpha^{(0)} = 1$ and $\beta^{(0)} = \gamma^{(0)} = 0$,

$$\sum_{k=0}^{N} \frac{(a+1)_k (b)_k}{k!(a+b+1)_k} \text{ is not telescoping.}$$

REMARKS. (1) E.g., with the choice $\alpha^{(0)} = 1$ and $\beta^{(0)} = \gamma^{(0)} = 0$,

$$\sum_{k=0}^{N} \frac{(a+1)_k (b)_k}{k!(a+b+1)_k} \text{ is not telescoping.}$$

(2) E.g., with the choice $\alpha^{(0)} = \beta^{(0)} = 0$ and $\gamma^{(0)} = 1$,

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+1-1)_k} \text{ is not telescoping.}$$

REMARKS. (1) E.g., with the choice $\alpha^{(0)} = 1$ and $\beta^{(0)} = \gamma^{(0)} = 0$,

$$\sum_{k=0}^{N} \frac{(a+1)_k (b)_k}{k!(a+b+1)_k} \quad \text{is not telescoping.}$$

(2) E.g., with the choice $\alpha^{(0)} = \beta^{(0)} = 0$ and $\gamma^{(0)} = 1$,

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+1-1)_k} \quad \text{is not telescoping.}$$

(3) The cases (1) and (2) require the extension to 2 $(= d+1)$ which is covered by THEOREM 2a; see the Example 1.2.1a.

REMARKS. (1) E.g., with the choice $\alpha^{(0)} = 1$ and $\beta^{(0)} = \gamma^{(0)} = 0$,

$$\sum_{k=0}^{N} \frac{(a+1)_k (b)_k}{k!(a+b+1)_k} \text{ is not telescoping.}$$

(2) E.g., with the choice $\alpha^{(0)} = \beta^{(0)} = 0$ and $\gamma^{(0)} = 1$,

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+1-1)_k} \text{ is not telescoping.}$$

(3) The cases (1) and (2) require the extension to 2 $(= d+1)$ which is covered by THEOREM 2a; see the Example 1.2.1a.

(4) The telescoping sum

$$\sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!(a+b+1)_k} = \frac{(a+1)_N (b+1)_N}{N!(a+b+1)_N}.$$

is the special case $e = -N$ of Pfaff-Saalschütz:

$$S(N) = \sum_{k=0}^{N} \frac{(-N)_k (a)_k (b)_k}{k!(e)_k (1+a+b-e-N)_k} = \frac{(e-a)_N (e-b)_N}{(e)_N (e-a-b)_N}.$$

Example 1.3.2c: $z = 1$ and $(p, q) = (3, 2)$ and $k_0 - q = 0$

$$\sum_{\ell=0}^{d} {}_3F_2\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)}, c+\gamma^{(0)}\} \\ e-\epsilon^{(\ell)}, f-\delta^{(\ell)} \end{matrix}; 1\right)_k \overset{(S)}{=} \Delta_k C(k)\, {}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix}; 1\right)_k$$

$${}_3P_2 = (2+a+b+c-e-f)x^2 + (-1+ab+ac+bc+e+f-ef)x + abc;$$

$$\sum_j b_j - \sum_i a_i - q = e+f-a-b-c-2 = 0 \quad \text{and}$$

$$d = \max\{\alpha^{(0)}+\beta^{(0)}+\gamma^{(0)}+\epsilon^{(0)}+\delta^{(0)}, \dots, \\ \alpha^{(d)}+\beta^{(d)}+\gamma^{(d)}+\epsilon^{(d)}+\delta^{(d)}, \underbrace{\deg({}_3P_2)}_{=1}\}$$

By THEOREM 2c, choosing $d = 1$, one, e.g., can compute this relation:

Example 1.3.2c: $z = 1$ and $(p, q) = (3, 2)$ and $k_0 - q = 0$

$$\sum_{\ell=0}^{d} {}_3F_2\left(\begin{matrix} a+\alpha^{(\ell)}, b+\beta^{(\ell)}, c+\gamma^{(0)}\} \\ e-\epsilon^{(\ell)}, f-\delta^{(\ell)} \end{matrix}; 1\right)_k \stackrel{(S)}{=} \Delta_k C(k)\, {}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix}; 1\right)_k$$

$${}_3P_2 = (2+a+b+c-e-f)x^2 + (-1+ab+ac+bc+e+f-ef)x + abc;$$

$$\sum_j b_j - \sum_i a_i - q = e+f-a-b-c-2 = 0 \quad \text{and}$$

$$d = \max\{\alpha^{(0)}+\beta^{(0)}+\gamma^{(0)}+\epsilon^{(0)}+\delta^{(0)}, \dots, \alpha^{(d)}+\beta^{(d)}+\gamma^{(d)}+\epsilon^{(d)}+\delta^{(d)}, \underbrace{\deg({}_3P_2)}_{=1}\}$$

By THEOREM 2c, choosing $d = 1$, one, e.g., can compute this relation:

$$c_0 \cdot {}_3F_2\left(\begin{matrix} a, b, c \\ e, f-1 \end{matrix}; 1\right) + c_1 \cdot {}_3F_2\left(\begin{matrix} a+1, b, c \\ e, f \end{matrix}; 1\right) = 0 \text{ if } f = a+b+c-e+2$$

Example 1.3.2c (cont'd.): $z = 1$ and $(p, q) = (3, 2)$ and $k_0 - q = 0$
By THEOREM 2c, choosing $d = 1$, one, e.g., can compute this relation:

$$c_0 \cdot {}_3F_2\left(\begin{matrix} a, b, c \\ e, f-1 \end{matrix}; 1\right) + c_1 \cdot {}_3F_2\left(\begin{matrix} a, b, c+1 \\ e, f \end{matrix}; 1\right) = 0 \text{ if } f = a+b+c-e+2$$

Concretely, relation (S) becomes:

$$\begin{aligned} &- c(1+c-e)(1+a+b+c-e)\frac{(a)_k(b)_k(c)_k}{(e)_k(a+b+c-e+1)_k} \\ &+ c(1+a+c-e)(1+b+c-e)\frac{(a)_k(b)_k(c+1)_k}{(e)_k(a+b+c-e+2)_k} \\ &= \Delta_k(e+k-1)(a+b+c-e+k+1)k\frac{(a)_k(b)_k(c)_k}{(e)_k(a+b+c-e+2)_k} \end{aligned}$$

Example 1.3.2c (cont'd.): $z=1$ and $(p,q)=(3,2)$ and $k_0-q=0$
By THEOREM 2c, choosing $d=1$, one, e.g., can compute this relation:

$$c_0\cdot {}_3F_2\left(\begin{matrix}a,b,c\\ e,f-1\end{matrix};1\right)+c_1\cdot {}_3F_2\left(\begin{matrix}a,b,c+1\\ e,f\end{matrix};1\right)=0 \text{ if } f=a+b+c-e+2$$

Concretely, relation (S) becomes:

$$\begin{aligned}&-c(1+c-e)(1+a+b+c-e)\frac{(a)_k(b)_k(c)_k}{(e)_k(a+b+c-e+1)_k}\\&+c(1+a+c-e)(1+b+c-e)\frac{(a)_k(b)_k(c+1)_k}{(e)_k(a+b+c-e+2)_k}\\&=\Delta_k(e+k-1)(a+b+c-e+k+1)k\frac{(a)_k(b)_k(c)_k}{(e)_k(a+b+c-e+2)_k}\end{aligned}$$

E.g., choosing $c=-n$, we "rediscover" Pfaff-Saalschütz:

$$\underbrace{{}_3F_2\left(\begin{matrix}a,b,-n\\ e,a+b-e-n+1\end{matrix};1\right)}_{=S(n)}=\frac{(1+a-e-n)(1+b-e-n)}{(1-e-n)(1+a+b-e-n)}S(n-1).$$

Example 1.4.3: $z = 1$ and $(p, q) = (4, 3)$

THEOREM 2a, 2b, 2c reproduce celebrated 3-term contiguous relations between (balanced) ${}_4F_3$-series; e.g., in the context of classical orthogonal polynomials from the "Askey Scheme".

BUT this algorithmic theory also enables to discover NEW relations!

Example 1.4.3: $z = 1$ and $(p, q) = (4, 3)$

THEOREM 2a, 2b, 2c reproduce celebrated 3-term contiguous relations between (balanced) ${}_4F_3$-series; e.g., in the context of classical orthogonal polynomials from the "Askey Scheme".

BUT this algorithmic theory also enables to discover NEW relations!

Theorem. [Wilson (1978); see also Andrews-Askey-Roy (3.7.5)]
If $e + f + g - a - b - c - d - 3 = -2 \, (= k_0 - q)$ then

$$f\,g\;{}_4F_3\left(\begin{matrix}a, b, c, d\\ e, f, g\end{matrix}; 1\right)$$

$$-(f-a)(g-a)\;{}_4F_3\left(\begin{matrix}a, b+1, c+1, d+1\\ e+1, f+1, g+1\end{matrix}; 1\right)$$

$$+\frac{a(e-b)(e-c)(e-d)}{e(e+1)}\;{}_4F_3\left(\begin{matrix}a+1, b+1, c+1, d+1\\ e+2, f+1, g+1\end{matrix}; 1\right)$$

$= 0$, if one of the top parameters is a negative integer.

(J. A. Wilson; see [AAR])

THEOREM 2a together with parameterized telescoping delivers the following generalization:

$$= \frac{\Gamma(e+1)\,\Gamma(f+1)\,\Gamma(g+1)}{\Gamma(a)\,\Gamma(b+1)\,\Gamma(c+1)\,\Gamma(d+1)}$$

NOTE. For Wilson's condition this gives zero.

Why does Z's algorithm sometimes miss the minimal recurrence?

(Ex) For Bailey's ${}_4F_3$ summation

$$S(n,a,b) = \sum_{k=0}^{n} \frac{(a)_{2k}(b+n)_k(-n)_k}{(b)_{2k}(a+1)_k\, k!} = \frac{(b-a)_n}{(b)_n}$$

it returns

$$c_0\, S(n,a,b) + c_1\, S(n+1,a,b) + c_2\, S(n+2,a,b) = 0$$

instead of

$$(b-a+n)\, S(n,a,b) + (b+n)\, S(n+1,a,b) = 0.$$

Why does Z's algorithm sometimes miss the minimal recurrence?

(Ex) For Bailey's ${}_4F_3$ summation

$$S(n,a,b)=\sum_{k=0}^{n}\frac{(a)_{2k}(b+n)_k(-n)_k}{(b)_{2k}(a+1)_k\,k!}=\frac{(b-a)_n}{(b)_n}$$

it returns

$$c_0\,S(n,a,b)+c_1\,S(n+1,a,b)+c_2\,S(n+2,a,b)=0$$

instead of

$$(b-a+n)\,S(n,a,b)+(b+n)\,S(n+1,a,b)=0.$$

$$S(n,a,b)=\sum_{k=0}^{n}{}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix};1\right)_k=\frac{(b-a)_n}{(b)_n}$$

$$a_1=\frac{a}{2},\ a_2=\frac{a+1}{2},\ a_3=b+n,\ a_4=-n,\ b_1=\frac{b}{2},\ b_2=\frac{b+1}{2},\ b_3=a+1$$

$b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - a_4 - 3 = -2$, hence by THEOREM 2a:

$$\exists\ c_0\ {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k + c_1\ {}_4F_3\left(\begin{matrix} a_1, a_2, a_3-1, a_4+1 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

$$+ c_2\ {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4+1 \\ b_1, b_2, b_3-1 \end{matrix}; 1\right)_k \overset{*}{=} \Delta_k C(k)\ {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

This relation translates into:

$b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - a_4 - 3 = -2$, hence by THEOREM 2a:

$$\exists \; c_0 \, {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k + c_1 \, {}_4F_3\left(\begin{matrix} a_1, a_2, a_3-1, a_4+1 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

$$+ c_2 \, {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4+1 \\ b_1, b_2, b_3-1 \end{matrix}; 1\right)_k \overset{*}{=} \Delta_k C(k) \, {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

This relation translates into:

$$c_0 \, S(n, a, b) + c_1 \, S(n-1, a, b) + c_2 \, S(n-1, a-1, b) = 0.$$

$b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - a_4 - 3 = -2$, hence by THEOREM 2a:

$$\exists \; c_0 \, {}_4F_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k + c_1 \, {}_4F_3\left(\begin{matrix} a_1, a_2, a_3-1, a_4+1 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

$$+ c_2 \, {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4+1 \\ b_1, b_2, b_3-1 \end{matrix}; 1\right)_k \overset{*}{=} \Delta_k C(k) \, {}_4F_3\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

This relation translates into:

$$c_0 \, S(n, a, b) + c_1 \, S(n-1, a, b) + c_2 \, S(n-1, a-1, b) = 0.$$

NOTE 1. A new proof of this identity.

$b_1 + b_2 + b_3 - a_1 - a_2 - a_3 - a_4 - 3 = -2$, hence by THEOREM 2a:

$$\exists\; c_0\, {}_4F_3\!\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k + c_1\, {}_4F_3\!\left(\begin{matrix} a_1, a_2, a_3-1, a_4+1 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

$$+\, c_2\, {}_4F_3\!\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4+1 \\ b_1, b_2, b_3-1 \end{matrix}; 1\right)_k \overset{*}{=} \Delta_k\, C(k)\, {}_4F_3\!\left(\begin{matrix} a_1, a_2-1, a_3-1, a_4 \\ b_1, b_2, b_3 \end{matrix}; 1\right)_k$$

This relation translates into:

$$c_0\, S(n,a,b) + c_1\, S(n-1,a,b) + c_2\, S(n-1,a-1,b) = 0.$$

NOTE 1. A new proof of this identity.

NOTE 2. Optimal recurrence depth w.r.t. n.

An example of different type:

Ex $S(n) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{3k}{n} = (-3)^n$;

$T(n) := (-1)^n \binom{3n}{n}^{-1} \cdot S(3n)$

For both sums Z's algo returns an order 2 rec.!

An example of different type:

(Ex) $$S(n) := \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{3k}{n} = (-3)^n;$$

$$T(n) := (-1)^n \binom{3n}{n}^{-1} \cdot S(3n)$$

For both sums Z's algo returns an order 2 rec.!

EXPLANATION. THEOREM 2a finds an optimal 3-term telescoping relation which implies:

$$c_0 P(n) + c_1 T(n) + c_2 T(n+1) = 0$$

with

$$P(n) = {}_3F_2\left(\begin{matrix} -2n, n+\frac{1}{3}, n+\frac{2}{3} \\ \frac{1}{3}, \frac{5}{3} \end{matrix}; 1\right)$$

where $\mathbf{P(n) = 0}$ for $n \geq 1$ by Pfaff-Saalschütz.

A LAST EXAMPLE ("creative symmetrizing"):

$$S(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^2 \binom{n}{k-1}}_{f(n,k)}$$

$S(2n) = $ cf [Carlitz]

$T(n) := S(2n+1)$

} In both cases Z-rec is of order 2

A LAST EXAMPLE ("creative symmetrizing"):

$$S(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^2 \binom{n}{k-1}}_{f(n,k)}$$

$S(2n) =$ cf [Carlitz] — In both cases

$T(n) := S(2n+1)$ — Z-rec is of order 2

$$T(n) = \frac{1}{2} \sum_{k=0}^{2n+1} \left(f(2n+1,k) + f(2n+1, 2n+1-k) \right)$$

$$= \sum_{k=0}^{2n+1} \underbrace{(-1)^k \frac{n+1}{2(2n-k+2)} \binom{2n+1}{k}^2 \binom{2n+1}{k-1}}_{g(n,k)}$$

Miracle: For this summand Z's algo. returns an ORDER 1 recurrence!

EXPLANATION.

$$T(n) = \sum_{k=0}^{2n+1} f(2n+1, k)$$

$$= -(2n+1)^2 \; {}_3F_2\left(\begin{matrix} -(2n+1), -2n, -2n \\ 2, 2 \end{matrix}; 1\right)$$

∃ only 3-term relation $(d = q = 2)$

BUT

$$T(n) = \sum_{k=0}^{2n+1} g(n, k)$$

$$= -(n+1)(2n+1) \; {}_3F_2\left(\begin{matrix} -2n, -(2n+1), -(2n+1) \\ 2, 2 \end{matrix}; 1\right)$$

WELL POISED !

EXPLANATION.

$$T(n) = \sum_{k=0}^{2n+1} f(2n+1,k)$$

$$= -(2n+1)^2 \; {}_3F_2\left(\begin{matrix}-(2n+1), -2n, -2n \\ 2, 2\end{matrix}; 1\right)$$

∃ only 3-term relation $(d=q=2)$

BUT

$$T(n) = \sum_{k=0}^{2n+1} g(n,k)$$

$$= -(n+1)(2n+1) \; {}_3F_2\left(\begin{matrix}-2n, -(2n+1), -(2n+1) \\ 2, 2\end{matrix}; 1\right)$$

WELL POISED!

In fact, the well-poised ${}_3F_2$ is the special case

$$a = -2n, b = c = -(2n+1)$$

of Dixon's summation formula:

Dixon (1902): For $\mathrm{Re}(1+\frac{a}{2}-b-c)>0$,

$$ {}_3F_2\left({a,b,c \atop a+1-b,a+1-c};1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)}. $$

Dixon (1902): For $\mathrm{Re}(1+\frac{a}{2}-b-c)>0$,

$$
{}_3F_2\left(\begin{matrix} a,b,c \\ a+1-b,a+1-c \end{matrix};1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)}.
$$

THEOREM 2a and parameterized telescoping gives,

$$
c_0\cdot {}_3F_2\left(\begin{matrix} a,b,c \\ a+1-b,a+1-c \end{matrix};1\right)_k + c_1\cdot {}_3F_2\left(\begin{matrix} a+1,b,c \\ a+2-b,a+2-c \end{matrix};1\right)_k + c_2\cdot {}_3F_2\left(\begin{matrix} a+2,b,c \\ a+3-b,a+3-c \end{matrix};1\right)_k = \Delta_k C(k)\,{}_3F_2\left(\begin{matrix} a,b,c \\ a+1-b,a+1-c \end{matrix};1\right)_k,
$$

where

Dixon (1902): For $\mathrm{Re}(1+\frac{a}{2}-b-c)>0$,

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ a+1-b, a+1-c \end{matrix}; 1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)}.
$$

THEOREM 2a and parameterized telescoping gives,

$$
c_0 \cdot {}_3F_2\left(\begin{matrix} a, b, c \\ a+1-b, a+1-c \end{matrix}; 1\right)_k + c_1 \cdot {}_3F_2\left(\begin{matrix} a+1, b, c \\ a+2-b, a+2-c \end{matrix}; 1\right)_k + c_2 \cdot {}_3F_2\left(\begin{matrix} a+2, b, c \\ a+3-b, a+3-c \end{matrix}; 1\right)_k = \Delta_k C(k) {}_3F_2\left(\begin{matrix} a, b, c \\ a+1-b, a+1-c \end{matrix}; 1\right)_k,
$$

where

$$
\begin{aligned}
c_0 &= -a(1+a-b)(2+a-b)(2+a-2b-2c)(1+a-c)(2+a-c), \\
c_1 &= 0, \\
c_2 &= a(1+a)(2+a-2b)(2+a-2c)(1+a-b-c)(2+a-b-c),
\end{aligned}
$$

and

$$C(k) = p(k)\frac{(-2-a+b)(-1-a+b)(-2-a+c)(-1-a+c)k}{(1+a-b+k)(1+a-c+k)}$$

with

$$\begin{aligned}p(k) = {} & -2 - a + 3a^2 + 2a^3 + 4b + ab - 2a^2b - 2b^2 + 4c + ac - 2a^2c - 6bc - abc \\ & + 2b^2c - 2c^2 + 2bc^2 + 3ak + 3a^2k - 2abk - 2ack + ak^2.\end{aligned}$$

$$C(k) = p(k)\frac{(-2-a+b)(-1-a+b)(-2-a+c)(-1-a+c)k}{(1+a-b+k)(1+a-c+k)}$$

with

$$p(k) = -2 - a + 3a^2 + 2a^3 + 4b + ab - 2a^2b - 2b^2 + 4c + ac - 2a^2c - 6bc - abc + 2b^2c - 2c^2 + 2bc^2 + 3ak + 3a^2k - 2abk - 2ack + ak^2.$$

NOTE. In Dixon's identity $c \to \infty$ gives Kummer's summation theorem:

$${}_2F_1\left(\begin{matrix} a, b \\ a+1-b \end{matrix}; -1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)}{\Gamma(1+a))\Gamma(1+\frac{a}{2}-b)}.$$

$$C(k) = p(k)\frac{(-2-a+b)(-1-a+b)(-2-a+c)(-1-a+c)k}{(1+a-b+k)(1+a-c+k)}$$

with

$$p(k) = -2 - a + 3a^2 + 2a^3 + 4b + ab - 2a^2b - 2b^2 + 4c + ac - 2a^2c - 6bc - abc + 2b^2c - 2c^2 + 2bc^2 + 3ak + 3a^2k - 2abk - 2ack + ak^2.$$

NOTE. In Dixon's identity $c \to \infty$ gives Kummer's summation theorem:

$$_2F_1\left(\begin{matrix} a, b \\ a+1-b \end{matrix}; -1\right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)}{\Gamma(1+a))\Gamma(1+\frac{a}{2}-b)}.$$

NOTE. Alternatively Kummer's identity can be derived by setting $z = -1$ in Kummer's quadratic transformation:

$$_2F_1\left(\begin{matrix} a, b \\ a+1-b \end{matrix}; z\right)$$
$$= (1-z)^{-a}\,{}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} - b \\ a+1-b \end{matrix}; -\frac{4z}{(1-z)^2}\right).$$

Conclusion