Empirical determinations of Feynman integrals using integer relation algorithms

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Antidifferentiation and the Calculation of Feynman Amplitudes

Integer relation algorithms empower quantum field theorists to turn numerical results into conjecturally exact evaluations of **Feynman intergrals**.

1. **1985**: **Periods** in the Dark Ages
2. **1995**: **PSLQ** in the Renaissance
3. **1999**: Improvements and **parallelization**
4. **2009**: Work on the **multiple zeta value** datamine
5. **2015**: Periods from **Erik Panzer** and **Oliver Schnetz**
6. **2017**: Periods and **quasi-periods** from **Stefano Laporta** in electrodynamics
7. **2018**: **Quadratic relations** for **all loops**, found with **David Roberts**
8. **2020**: **Quadratic relations** for **black holes**, found with **Kevin Acres**

1 Periods in the Dark Ages

Problem: Given numerical **approximations** to $n > 2$ real numbers, x_k, is there at least one **probable** relation

$$\sum_{k=1}^{n} z_k x_k = 0$$

with integers z_k, at least two of which are non-zero? If so, produce one.

Examples: I studied periods from 6-loop Feynman diagrams in 1985:

$$P_{6,1} = 168\zeta_9, \quad P_{6,2} = \frac{1063}{9}\zeta_9 + 8\zeta_3^3, \quad 16P_{6,3} + P_{6,4} = 1440\zeta_3\zeta_5$$

with Riemann zeta values $\zeta_a = \sum_{n>0} n^{-a}$. I had a strong intuition that $P_{6,3}$ and $P_{6,4}$ would involve ζ_8 and the **multiple zeta value** (MZV)

$$\zeta_{5,3} = \sum_{m>n>0} \frac{1}{m^5 n^3} = 0.037707672984847544011304782\ldots$$

but did not have enough digits for the **periods** to test this.

2 PSLQ in the Renaissance

In response to a request from **Dirk Kreimer**, I obtained $P_{6,3} = 256N_{3,5} + 72\zeta_3\zeta_5$ and $P_{6,4} = -4096N_{3,5} + 288\zeta_3\zeta_5$, with

$$N_{3,5} = \frac{27}{80}\zeta_{5,3} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8$$

found by PSLQ, after more digits were obtained for the periods.

We found $\zeta_{3,5,3}$, with weight 11 and depth 3, in some 7-loop periods.

Much experimenting with PSLQ led to the Broadhurst-Kreimer (BK) conjecture that the number $N(w,d)$ of independent **primitive** MZVs of **weight** w and **depth** d is generated by

$$\prod_{w>2}\prod_{d>0}(1 - x^w y^d)^{N(w,d)} = 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

with a final term inferred by relating MZVs to **alternating** sums.

2.1 PSLQ: Partial Sums, Lower triangular, orthogonal Quotient

PSLQ came from work by **Helaman Ferguson** and **Rodney Forcade** in 1977, implemented in **multiple-precision** `ForTran` by **David Bailey** in 1992, improved and **parallelized** in 1999. See Bailey and Broadhurst, *Parallel Integer Relation Detection: Techniques and Applications*, Math. Comp. 70 (2001), 1719–1736.

Initialization:

1. For $j := 1$ to n: for $i := 1$ to n: if $i = j$ then set $A_{ij} := 1$ and $B_{ij} := 1$ else set $A_{ij} := 0$ and $B_{ij} := 0$; endfor; endfor.

2. For $k := 1$ to n: set $s_k :=$ **sqrt** $\left(\sum_{j=k}^{n} x_j^2\right)$; endfor. Set $t = 1/s_1$.
   For $k := 1$ to n: set $y_k := tx_k$; $s_k := ts_k$; endfor.

3. For $j := 1$ to $n - 1$: for $i := 1$ to $j - 1$: set $H_{ij} := 0$; endfor;
   set $H_{jj} := s_{j+1}/s_j$; for $i := j + 1$ to n: set $H_{ij} := -y_i y_j/(s_j s_{j+1})$; endfor;
   endfor.

4. For $i := 2$ to n: for $j := i - 1$ to 1 step -1: set $t :=$ **round**(H_{ij}/H_{jj});
   $y_j := y_j + ty_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - tH_{jk}$; endfor;
   for $k := 1$ to n: set $A_{ik} := A_{ik} - tA_{jk}$, $B_{kj} := B_{kj} + tB_{ki}$; endfor; endfor;
   endfor.

Iteration:

1. Select m such that $(\mathbf{4/3})^{i/2}|H_{ii}|$ is maximal when $i = m$. **Swap** the entries of y indexed m and $m+1$, the corresponding rows of A and H, and the corresponding columns of B.

2. If $m \le n-2$ then set $t_0 := \textbf{sqrt}(H_{mm}^2 + H_{m,m+1}^2)$, $t_1 := H_{mm}/t_0$ and $t_2 := H_{m,m+1}/t_0$; for $i := m$ to n: set $t_3 := H_{im}$, $t_4 := H_{i,m+1}$, $H_{im} := t_1 t_3 + t_2 t_4$ and $H_{i,m+1} := -t_2 t_3 + t_1 t_4$; endfor; endif.

3. For $i := m+1$ to n: for $j := \min(i-1, m+1)$ to 1 step -1: set $t := \textbf{round}(H_{ij}/H_{jj})$ and $y_j := y_j + t y_i$; for $k := 1$ to j: set $H_{ik} := H_{ik} - t H_{jk}$; endfor; for $k := 1$ to n: set $A_{ik} := A_{ik} - t A_{jk}$ and $B_{kj} := B_{kj} + t B_{ki}$; endfor; endfor; endfor.

4. If the largest entry of A exceeds the precision, then **fail**, else if a component of the y vector is very small, then output the **relation** from the corresponding column of B, else go back to Step 1.

For big problems, the **parallelization** of PSLQ has been vital, especially for the magnetic moment of the electron. For smaller problems, there is an alternative.

2.2 LLL

In 1982, Arjen Lenstra, Hendrik Lenstra and László Lovász gave the LLL algorithm for lattice reduction to a basis with short and almost orthogonal components. An extension of this underlies `lindep` in Pari-GP.

```
$ Z53=0.037707672984847544011304782;
$ P63=107.71102484102;
$ V=[P63,Z53,zeta(3)*zeta(5),zeta(8)];
$ for(d=10,16,U=lindep(V,d);U*=sign(U[1]);print([d,U~]));
[10, [12, 44, -936, -127]]
[11, [4, -827, -460, 173]]
[12, [4, -827, -460, 173]]
[13, [4, -827, -460, 173]]
[14, [5, -432, -1260, 1044]]
[15, [5, -432, -1260, 1044]]
[16, [196, 1652, -9701, -9045]]
```

3 Improvements and parallelization

Multi-level improvement: perform most operations at 64-bit precision, some at intermediate precision (we chose 125 digits) and only the bare **minimum** of the most delicate operations at **full** precision (more than 10000 digits, for some big problems).

Multi-pair improvement: swap up to $0.4n$ disjoint **pairs** of the n indices at each iteration. In this case, it is not proven that the algorithm will succeed, but it ain't yet been found to fail.

Parallelization: distribute the disjoint-pair jobs; for each pair, distribute the full-precision matrix multiplication in the outermost loop.

3.1 Fourth bifurcation of the logistic map

Working at **10000** digits, we found that the constant associated with the fourth bifurcation is the root of a polynomial of degree **240**.

3.2 Alternating sums

We tested my conjecture on alternating sums defined by

$$\zeta\begin{pmatrix} s_1, & s_2 & \cdots & s_r \\ \sigma_1, & \sigma_2 & \cdots & \sigma_r \end{pmatrix} = \sum_{k_1>k_2>\cdots>k_r>0} \frac{\sigma_1^{k_1}}{k_1^{s_1}} \frac{\sigma_2^{k_2}}{k_2^{s_2}} \cdots \frac{\sigma_r^{k_r}}{k_r^{s_r}}$$

where $\sigma_j = \pm 1$ are signs and $s_j > 0$ are integers, namely that at weight $w = \sum_j s_j$ every alternating sum is a rational linear combination of elements of a basis of size $F_{w+1} = F_w + F_{w-1}$, i.e. the **Fibonacci** number with index $w+1$. At $w = 11$, integer relations of size $n = F_{12} + 1 = \mathbf{145}$ were found, at **5000**-digit precision.

3.3 Inverse binomial sums

Noting that $S(4) = \frac{17}{36}\zeta_4$, I conjectured that

$$S(w) = \sum_{n=1}^{\infty} \frac{1}{n^w \binom{2n}{n}}$$

is reducible to weigth w nested sums that involve **sixth roots of unity**, i.e. with $\sigma_j^6 = 1$, above. This was confirmed for all weights $w \leq 20$, with `525990827847624469523748125835264000`$\times S(20)$ given by **106** terms.

4 Work on the multiple zeta value datamine

The BK conjecture was a rash leap based on a PSLQ dicovery:

$$\begin{aligned}
2^5 \cdot 3^3 \zeta_{4,4,2,2} - 2^{14} \sum_{m>n>0} \frac{(-\mathbf{1})^{m+n}}{(m^3 n)^3} = \\
2^5 \cdot 3^2\, \zeta_3^4 + 2^6 \cdot 3^3 \cdot 5 \cdot 13\, \zeta_9\, \zeta_3 + 2^6 \cdot 3^3 \cdot 7 \cdot 13\, \zeta_7\, \zeta_5 \\
+ 2^7 \cdot 3^5\, \zeta_7\, \zeta_3\, \zeta_2 + 2^6 \cdot 3^5\, \zeta_5^2\, \zeta_2 - 2^6 \cdot 3^3 \cdot 5 \cdot 7\, \zeta_5\, \zeta_4\, \zeta_3 \\
- 2^8 \cdot 3^2\, \zeta_6\, \zeta_3^2 - \frac{13177 \cdot 15991}{691}\, \zeta_{12} \\
+ 2^4 \cdot 3^3 \cdot 5 \cdot 7\, \zeta_{6,2}\, \zeta_4 - 2^7 \cdot 3^3\, \zeta_{8,2}\, \zeta_2 - 2^6 \cdot 3^2 \cdot 11^2\, \zeta_{10,2}
\end{aligned}$$

is reducible to MZVs of depth $d \leq 2$ and their products. It means that $\zeta_{4,4,2,2}$ is **pushed down** to depth $d = 2$, if we allow **alternating** sums in the MZV basis. When constructing the MZV datamine, **Johannes Blümlein** and **Jos Vermaseren** and I were able to **prove** this by massive use of computer algebra. It is harder to prove my discovery of pushdown at weight 21 and depth 7, where

$$81\zeta_{6,2,3,3,5,1,1} + 326 \sum_{j>k>l>m>n>0} \frac{(-\mathbf{1})^{k+m}}{(jk^2 lm^2 n)^3}$$

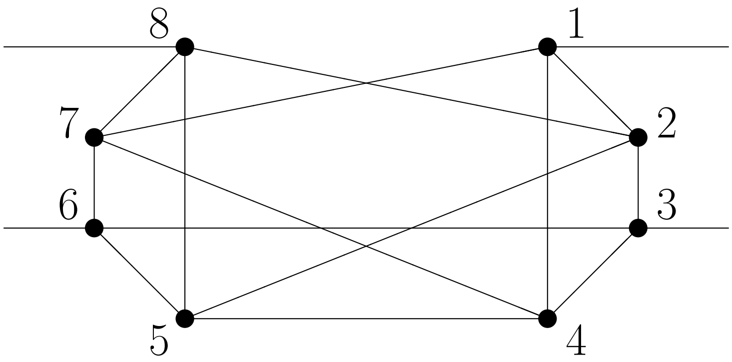
is empirically reducible to **150** terms containing MZVs of depths $d \leq 5$.

5 Periods from Panzer and Schnetz

I found empirical reductions to MZVs for a pair of 7-loop periods

$$
\begin{aligned}
P_{7,8} &= \frac{22383}{20}\zeta_{11} + \frac{4572}{5}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - 700\zeta_3^2\zeta_5 \\
&\quad + 1792\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right) \\
P_{7,9} &= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{5,3}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 \\
&\quad + 896\zeta_3\left(\frac{9}{320}\left(12\zeta_{5,3} - 29\zeta_8\right) + \frac{45}{64}\zeta_5\zeta_3\right)
\end{aligned}
$$

that had been expected to involve alternating sums. These results were later proven, one by **Erik Panzer** and the other by **Oliver Schnetz**. They obtained complicated combinations of **alternating** sums which then gave my MZV formulas by use of proven results in the datamine.



The period from this **7-loop** diagram is called $P_{7,11}$ in the census of Schnetz. All other periods up to 7 loops reduce to MZVs; only $P_{7,11}$ requires nested sums with **sixth roots of unity**. Panzer evaluated $\sqrt{3}P_{7,11}$ in terms of 4589 such sums, each of which he evaluated to 5000 digits. Then he found an empirical reduction to a 72-dimensional basis. The rational coefficient of π^{11} in his result was

$$C_{11} = -\frac{964259961464176555529722140887}{2733669078108291387021448260000}$$

whose **denominator** contains 8 primes greater than 11, namely 19, 31, 37, 43, 71, 73, **50909** and **121577**.

I built an empirical datamine to enable substantial simplification.

Let $A=\mathrm{d}\log(x)$, $B=-\mathrm{d}\log(1-x)$ and $D=-\mathrm{d}\log(1-\exp(2\pi\mathrm{i}/6)x)$ be **letters**, forming **words** W that define **iterated integrals** $Z(W)$. Let

$$W_{m,n}=\sum_{k=0}^{n-1}\frac{\zeta_3^k}{k!}A^{m-2k}D^{n-k}$$

$P_n=(\pi/3)^n/n!$, $I_n=\mathrm{Cl}_n(2\pi/3)$ and $I_{a,b}=\Im Z(A^{b-a-1}DA^{2a-1}B)$. Then

$$\begin{aligned}
\sqrt{3}P_{7,11} &= -10080\Im Z(W_{7,4}+W_{7,2}P_2)+50400\zeta_3\zeta_5P_3\\
&+ \left(35280\Re Z(W_{8,2})+\frac{46130}{9}\zeta_3\zeta_7+17640\zeta_5^2\right)P_1\\
&- 13277952T_{2,9}-7799049T_{3,8}+\frac{6765337}{2}I_{4,7}-\frac{583765}{6}I_{5,6}\\
&- \frac{121905}{4}\zeta_3I_8-93555\zeta_5I_6-102060\zeta_7I_4-141120\zeta_9I_2\\
&+ \frac{42452687872649}{6}P_{11}
\end{aligned}$$

with the datamine transformations

$$\begin{aligned}
I_{2,9} &= 91(11T_{2,9})-898T_{3,8}+11I_{4,7}-292P_{11}\\
I_{3,8} &= 24(11T_{2,9})+841T_{3,8}-190I_{4,7}-255P_{11}
\end{aligned}$$

removing denominator primes greater than 3.

6 Periods and quasi-periods from Laporta

The **magnetic moment** of the electron, in Bohr magnetons, has electrodynamic contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L=4$ loops by

$$\begin{aligned}
a_0 &= \quad 1 \qquad [\textbf{Dirac}, 1928] \\
a_1 &= \quad 0.5 \quad [\textbf{Schwinger}, 1947] \\
a_2 &= -0.32847896557919378458217281696489239241111929867962\ldots \\
a_3 &= \quad 1.18124145658720000627475398221287785336878939093213\ldots \\
a_4 &= -1.91224576492644557415264716743983005406087339065872\ldots
\end{aligned}$$

In 1957, corrections by **Petermann** and **Sommerfield** resulted in

$$a_2 = \frac{197}{144} + \frac{\zeta_2}{2} + \frac{3\zeta_3 - 2\pi^2 \log 2}{4}\,.$$

In 1996, **Laporta** and **Remiddi** [hep-ph/9602417] gave us

$$\begin{aligned}
a_3 \;=\; & \frac{28259}{5184} + \frac{17101\zeta_2}{135} + \frac{139\zeta_3 - 596\pi^2 \log 2}{18} \\
& - \frac{39\zeta_4 + 400U_{3,1}}{24} - \frac{215\zeta_5 - 166\zeta_3\zeta_2}{24}\,.
\end{aligned}$$

The 3-loop contribution contains a weight-4 depth-2 **polylogarithm**

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{\zeta_4}{2} + \frac{(\pi^2 - \log^2 2)\log^2 2}{12} - 2\sum_{n>0} \frac{1}{2^n n^4}$$

encountered in my study of **alternating** sums [arXiv:hep-th/9611004].

Equally fascinating is the **Bessel** moment B, at weight 4, in the breath-taking evaluation by **Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals.
U comes from 6 light-by-light integrals, still under investigation.

The weight-4 **non-polylogarithm** at 4 loops has $N = 6$ Bessel functions:

$$B = -\int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t)K_0^5(t)\mathrm{d}t.$$

6.1 Bessel moments and modular forms

Gauss noted on 30 May 1799 that the **lemniscate** constant

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}} = \frac{(\Gamma(1/4))^2}{4\sqrt{2\pi}} = \frac{\pi/2}{\mathbf{agm}(1,\sqrt{2})}$$

is given by the reciprocal of an **arithmetic-geometric mean**. This is an example of the Chowla-Selberg formula (1949) at the **first** singular value. In 1939, **Watson** encountered the **sixth** singular value, in work on integrals from condensed matter physics. Here, $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{6}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{24})$ with $k=1,5,7,11$, as observed by **Glasser and Zucker** in 1977. In 2007, I identified a **Feynman** period at the **fifteenth** singular value, where $\left(\sum_{n\in\mathbf{Z}}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4$ gives the product of $\Gamma(k/\mathbf{15})$ with $k=1,2,4,8$.

With $N=a+b$ **Bessel** functions and $c\ge 0$, I define **moments**

$$M(a,b,c)=\int_0^\infty I_0^a(t)K_0^b(t)t^c\mathrm{d}t$$

that converge for $b>a>0$. Then the 5-Bessel matrix is

$$\left[\begin{array}{cc} M(1,4,1) & M(1,4,3)\\ M(2,3,1) & M(2,3,3)\end{array}\right] = \left[\begin{array}{cc} \pi^2 C & \pi^2\left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right)\\ \frac{\sqrt{15}\pi}{2}C & \frac{\sqrt{15}\pi}{2}\left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right)\end{array}\right].$$

The **determinant** $2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C = \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty}\exp(-\sqrt{\mathbf{15}}\pi n^2)\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{\mathbf{15}}\right).$$

The **L-series** for $N=5$ Bessel functions comes from a **modular form** of weight **3** and level **15** [arXiv:1604.03057]:

$$\begin{aligned}
\eta_n &= q^{n/24}\prod_{k>0}(1-q^{nk}), \quad q=\exp(2\pi \mathrm{i}\tau),\\
f_{3,15}(\tau) &= (\eta_3\eta_5)^3+(\eta_1\eta_{15})^3=\sum_{n>0}A_5(n)q^n\\
L_5(s) &= \sum_{n>0}\frac{A_5(n)}{n^s} \quad \text{for } s>2\\
L_5(1) &= \sum_{n>0}\frac{A_5(n)}{n}\left(2+\frac{\sqrt{\mathbf{15}}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{\mathbf{15}}}\right)\\
&= 5C = \frac{5}{\pi^2}\int_0^\infty I_0(t)K_0^4(t)t\mathrm{d}t\,.
\end{aligned}$$

6.2 Periods and quasi-periods for the Laporta problem

Laporta's work engages the first row of the **6-Bessel determinant**

$$\det \begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{5\zeta_4}{32}$$

associated to a **modular form** $f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2$ with weight **4** and level **6**. At top left we have $M(1,5,1)$, from the on-shell **4-loop sunrise** diagram, in two spacetime dimensions. Below it, $M(2,4,1)$ comes from **cutting** an internal line. The **second column** comes from **differentiating** the first, with respect to the external momentum, to produce **quasi-periods** associated with a **weakly holomorphic** modular form

$$\widehat{f}_{4,6}(\tau) = \mu f_{4,6}(\tau), \quad \mu = \frac{1}{32}\left(w + \frac{3}{w}\right)^4 - \frac{9}{16}\left(w + \frac{3}{w}\right)^2, \quad w = \frac{3\eta_3^4\eta_2^2}{\eta_1^4\eta_6^2}.$$

With $s = 1, 2$, I computed compute 10,000 digits of the **Eichler lintegrals**

$$\frac{\Omega_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} f_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy, \quad \frac{\widehat{\Omega}_s}{(2\pi)^s} = \int_{1/\sqrt{3}}^{\infty} \widehat{f}_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) y^{s-1} dy.$$

6.3 Laporta's intersection number

LLL readily gave me 4 **linear relations**

$$\frac{2}{\pi^2}\begin{bmatrix} 4M_{0,0}(1) & \frac{36}{5}\left(M_{0,0}(1)+M_{0,1}(1)\right) \\ \frac{5}{3}M_{1,0}(1) & 3\left(M_{1,0}(1)+M_{1,1}(1)\right) \end{bmatrix} = \begin{bmatrix} -\Omega_2 & \widehat{\Omega}_2 \\ -\Omega_1 & \widehat{\Omega}_1 \end{bmatrix}$$

between **Feynman integrals**, the **periods** $\Omega_{1,2}$ and the **quasi-periods** $\widehat{\Omega}_{1,2}$.

The **intersection number** is the **determinant** of this matrix, namely 1/12.

David Roberts and I converted this into a **quadratic relation** between **hypergeometeric series**:

$$\begin{array}{rcl} F_a & = & {}_4F_3(\ \ 1/2,\ \ 2/3,\ \ 2/3,\ 5/6;\ \ 7/6,\ 7/6,\ 4/3;\ 1) \\ F_b & = & {}_4F_3(\ -1/2,\ \ 1/6,\ \ 1/3,\ 4/3;\ -1/6,\ 5/6,\ 5/3;\ 1) \\ F_c & = & {}_4F_3(\ \ 1/6,\ \ 1/3,\ \ 1/3,\ 1/2;\ \ 2/3,\ 5/6,\ 5/6;\ 1) \\ F_d & = & {}_4F_3(\ -7/6,\ -1/2,\ -1/3,\ 2/3;\ -5/6,\ 1/6,\ 1/3;\ 1) \end{array}$$

namely

$$7F_aF_b + 10F_cF_d = 40.$$

7 Quadratic relations for all loops

Conjecture: [Broadhurst and Roberts] *With the Feynman, de Rham and Betti matrices below, we conjecture that*

$$F_N D_N F_N^{\mathtt{tr}} = B_N.$$

The elements of the **Feynman** matrices F_N are the Bessel moments

$$\begin{aligned} F_{2k+1}(u,a) &= \frac{(-1)^{a-1}}{\pi^u} M(k+1-u, k+u, 2a-1) \\ F_{2k+2}(u,a) &= \frac{(-1)^{a-1}}{\pi^{u+1/2}} M(k+1-u, k+1+u, 2a-1) \end{aligned}$$

with u and a, as well as later indices v and b, running from 1 to k. $F_N^{\mathtt{tr}}$ is the **transpose** of F_N. The **Betti** matrices B_N have rational elements given by

$$\begin{aligned} B_{2k+1}(u,v) &= (-1)^{u+k} 2^{-2k-2} (k+u)!(k+v)! Z(u+v) \\ B_{2k+2}(u,v) &= (-1)^{u+k} 2^{-2k-3} (k+u+1)!(k+v+1)! Z(u+v+1) \\ Z(m) &= \frac{1+(-1)^m}{(2\pi)^m} \zeta(m). \end{aligned}$$

For the **de Rham** matrices D_N, let v_k and w_k be the rationals generated by

$$\frac{J_0^2(t)}{C(t)} = \sum_{k\geq 0}\frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{17t^2}{54}+\frac{3781t^4}{186624}+\dots$$
$$\frac{2J_0(t)J_1(t)}{tC(t)} = \sum_{k\geq 0}\frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{41t^2}{216}+\frac{325t^4}{186624}+\dots$$

where $J_0(t)=I_0(\mathrm{i}t)$, $J_1(t)=-J_0'(t)$ and

$$C(t)=\frac{32(1-J_0^2(t)-tJ_0(t)J_1(t))}{3t^4}=1-\frac{5t^2}{27}+\frac{35t^4}{2304}-\frac{7t^6}{9600}+\dots$$

Construct rational bivariate polynomials $H_s=H_s(y,z)$ by the recursion

$$H_s=zH_{s-1}-(s-1)yH_{s-2}-\sum_{k=1}^{s-1}\binom{s-1}{k}(v_kH_{s-k}-w_kzH_{s-k-1})$$

for $s>0$, with $H_0=1$ and $H_{-1}=0$. Use these to define

$$d_s(N,c)=\frac{H_s(3c/2,N+2-2c)}{4^s s!}.$$

Finally, construct de Rham matrices with the **rational** elements

$$D_N(a,b) = \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}.$$

7.1 Remarks

1. The discovery of this formula for the coefficients of these quadratic relations involved intensive use of LLL, at high numerical precision. At **20 loops**, there are 100 Feynman integrals to consider. We claim to have found **all** of the quadratic relations between their **5050 products**.
2. **Javier Fresán**, **Claude Sabbah** and **Jeng-Daw Yu** have verified that our formulas hold up to 20 loops, after which they ran out of computing power.
3. They encountered subtleties when N is divisible 4. These are entirely avoided by our uniform formula.
4. Last month, at *Elliptics20*, **Roman Lee** announced that he is able to generate our de Rham matrices iteratively and check our claim up to some modest number of loops that is limited by his computing power.

8 Quadratic relations for black holes

Last December, Philip **Candelas**, Xenia **de la Ossa**, Mohamed **Elmi** and Duco **van Straten** announced a remarkable discovery of *A One Parameter Family of Calabi-Yau Manifolds with Attractor Points of Rank Two* [arXiv:1912.06146].

They compactified a 10-dimensional **supergravity** theory on a **Calabi-Yau** three-fold with complex structure, to obtain 4-dimensional **black holes**, with event horizons whose **areas** are determined by their electric and magnetic charges and by ratios of **periods** of **modular forms** of weight 4 and **levels 14 or 34**.

Hearing of this on a visit to Oxford, in November, I observed that their Calabi-Yau periods come from solutions to a **homogeneous** differential equation associated with **4 loop sunrise integrals**, namely

$$\begin{aligned} M_{m,n}(z) &= \int_0^\infty I_0(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+1}\mathrm{d}x \\ N_{m,n}(z) &= z\int_0^\infty I_1(xz)[I_0(x)]^m[K_0(x)]^{5-m}x^{2n+2}\mathrm{d}x \end{aligned}$$

with $m \in \{0,1,2\}$, integers $n \geq 0$ and real $z^2 < (5-2m)^2$. The **uncut** diagram gives $M_{0,0}(z)$ and satisfies an **inhomogeneous** differential equation.

The **external mass** is z. At $z = 1$ we obtain Laporta's **on-shell** periods, for the **magnetic moment of the electron** at 4 loops, coming from the modular form $f_{4,6}(\tau) = (\eta_1\eta_2\eta_3\eta_6)^2$ with **level 6**. With mass $z = \sqrt{\mathbf{17}} - 4$, I obtained **level 34** periods. At the **space-like** point $z = \sqrt{-\mathbf{7}}$, I obtained **level 14** periods.

Candelas et al. were unable to identify all of the 16 Calabi-Yau periods. At each of the levels 14 and 34, I found that are given by 8 Feynman integrals, satisfying two **quadratic relations**. These 8 integrals determine a pair of **periods** and a pair of **quasi-periods** at each of the weights 2 and 4.

Here I indicate the situation at **level 14**, where I identified

$$f_{4,14}(\tau) = \frac{(\eta_2\eta_7)^6}{(\eta_1\eta_{14})^2} - 4(\eta_1\eta_2\eta_7\eta_{14})^2 + \frac{(\eta_1\eta_{14})^6}{(\eta_2\eta_7)^2}$$

as the relevant modular form of **weight 4**. Its **periods** are **critical values** of the L-function $L(f_{4,14}, s) = ((2\pi)^s/\Gamma(s)) \int_0^\infty f_{4,14}(\mathrm{i}y)y^{s-1}\mathrm{d}y$, with

$$\begin{aligned} L(f_{4,14}, 3) &= M_{1,0}(\sqrt{-7}) = \int_0^\infty J_0(\sqrt{7}x)I_0(x)K_0^4(x)x\mathrm{d}x = \frac{\pi^2}{7}L(f_{4,14}, 1) \\ \tfrac{1}{2}L(f_{4,14}, 2) &= M_{2,0}(\sqrt{-7}) = \int_0^\infty J_0(\sqrt{7}x)I_0^2(x)K_0^3(x)x\mathrm{d}x. \end{aligned}$$

There is also a modular form of **weight 2** to consider, $f_{2,14}(\tau) = \eta_1\eta_2\eta_7\eta_{14}$. This provides a modular parametrization of a **quartic elliptic curve**, namely

$$\begin{aligned} d^2 &= (1+h)(1+8h)(1+5h+8h^2), \\ h &= \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7}\right)^3 = q + 3q^2 + 6q^3 + 13q^4 + O(q^5), \\ d &= \frac{q}{f_{2,14}}\frac{\mathrm{d}h}{\mathrm{d}q} = 1 + 7q + 27q^2 + 92q^3 + 259q^4 + O(q^5). \end{aligned}$$

At weight 2, we obtain **complete elliptic integrals**.

From my work with **Kevin Acres** on **Rademacher sums**, I was able to determine a **weakly holomorphic** form that gives the **weight-4 quasi-periods**. The space of cuspforms is **4-dimensional** and we had to solve a $\mathbf{4 \times 10}$ matrix problem, with each of the 4 associated weakly holomorphic forms obtained by multiplying $f_{2,14}^2$ by a polynomial that is **linear** in d and **quartic** in h.

The published chapter will give details of a more demanding problem solved by LLL, at **weight 6** and **level 24**, where the space of cuspforms is **16-dimensional**.

Summary

1. PSLQ and LLL have enlivened quests for analytical results.
2. PSLQ led to the Broadhurst-Kreimer conjecture.
3. PSLQ has been parallelized.
4. PSLQ and LLL have provided strong tests on conjectures.
5. PSLQ and LLL have condensed huge expressions.
6. Parallel PSLQ was of the essence in Laporta's work in electrodynamics.
7. LLL led to a conjecture on quadratic relations for all loops.
8. LLL led to exact results for the black hole problem.

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