The Almkvist-Zeilberger algorithm and multi-integration

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Outline

1. Multivariate AZ-algorithm
2. Mellin Transform of Holonomic Functions
3. Continuous Multivariate AZ-algorithm

Multivariate AZ-algorithm.

Multivariate Almkvist Zeilberger algorithm [Apagodu and Zeilberger, 2006]

We use the Almkvist Zeilberger algorithm to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

where $F(n; x_1, \dots, x_d)$ is a hyperexponential function i.e.,

$$F(n; x_1, \dots, x_d) = q(n; x_1, \dots, x_d) \cdot e^{\frac{a(x_1, \dots, x_d)}{b(x_1, \dots, x_d)}} \cdot \prod_{p=1}^{P} S_p(x_1, \dots, x_d)^{\alpha_p} \cdot \frac{s(x_1, \dots, x_d)^n}{t(x_1, \dots, x_d)^n},$$

with

- $q(n; x_1, \dots, x_d) \in \mathbb{K}[n, x_1, \dots, x_d]$;
- $a(x_1, \dots, x_d), s(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d]$;
- $b(x_1, \dots, x_d), t(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d] \setminus \{0\}$;
- $S_p(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d], \alpha_p \in \mathbb{K}$ for $1 \leq p \leq P$;
- With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

Multivariate Almkvist Zeilberger algorithm [Apagodu and Zeilberger, 2006]

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(n; x_1, \ldots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \ldots, x_d)$

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- compute a recurrence for the integrand $F(n; x_1, \ldots, x_d)$
- use the recurrence for the integrand to derive a recurrence for the integral
- compute initial values for the integral
- solve the recurrence

Recurrence for the Integrand

$$F(n; x_1, \ldots, x_d) = q(n; x_1, \ldots, x_d) H(n; x_1, \ldots, x_d),$$

where

$$H(n; x_1, \ldots, x_d) = e^{\frac{a(x_1, \ldots, x_d)}{b(x_1, \ldots, x_d)}} \cdot \prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \cdot \frac{s(x_1, \ldots, x_d)^n}{t(x_1, \ldots, x_d)^n}$$

- $q(n; x_1, \ldots, x_d) \in \mathbb{K}[n, x_1, \ldots, x_d]$;
- $a(x_1, \ldots, x_d), b(x_1, \ldots, x_d), s(x_1, \ldots, x_d), t(x_1, \ldots, x_d) \in \mathbb{K}[x_1, \ldots, x_d]$;
- $S_p(x_1, \ldots, x_d) \in \mathbb{K}[x_1, \ldots, x_d], \alpha_p \in \mathbb{K}$ for $1 \leq p \leq P$.

$$F(n; x_1, \ldots, x_d) = q(n; x_1, \ldots, x_d) H(n; x_1, \ldots, x_d),$$

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$$H(n; x_1, \ldots, x_d) = e^{\frac{a(x_1, \ldots, x_d)}{b(x_1, \ldots, x_d)}} \cdot \prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \cdot \frac{s(x_1, \ldots, x_d)^n}{t(x_1, \ldots, x_d)^n}$$

Then there exist

- $L \in \mathbb{N}$,
- $e_0(n), e_1(n), \ldots, e_L(n) \in \mathbb{K}[n]$, *not all zero,*
- $R_i(n; x_1, \ldots, x_d) \in \mathbb{K}(n, x_1, \ldots, x_d)$,

such that

$$G_i(n; x_1, \ldots, x_d) := R_i(n; x_1, \ldots, x_d) F(n; x_1, \ldots, x_d)$$

satisfy

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d).$$

$$\overline{H}(n; x_1, \ldots, x_d) = e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}} \cdot \prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \cdot \frac{s(x_1, \ldots, x_d)^n}{t(x_1, \ldots, x_d)^{n+L}}, \;\; 0 \leq L \in \mathbb{N}.$$

Then we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = h(x_1, \ldots, x_d)\overline{H}(n; x_1, \ldots, x_d).$$

where $h(x_1, \ldots, x_d)$ is a polynomial i.e.,

$$h(x_1, \ldots, x_d) := \sum_{i=1}^{L} e_i(n) q(n+i, x_1, \ldots, x_d) \frac{s(x_1, \ldots, x_d)^i}{t(x_1, \ldots, x_d)^{i-L}}$$

$$\overline{H}(n; x_1, \ldots, x_d) = e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}} \cdot \prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \cdot \frac{s(x_1, \ldots, x_d)^n}{t(x_1, \ldots, x_d)^{n+L}}, \quad 0 \leq L \in \mathbb{N}.$$

From the logarithmic derivatives

$$\frac{D_{x_i} \overline{H}(n; x_1, \ldots, x_d)}{\overline{H}(n; x_1, \ldots, x_d)} = \frac{q_i(x_1, \ldots, x_d)}{r_i(x_1, \ldots, x_d)}$$

we build the ansatz ($X_i(x_1, \ldots, x_d)$ are polynomials to be determined.)

$$G_i(n; x_1, \ldots, x_d) = \overline{H}(n; x_1, \ldots, x_d) \cdot r_i(x_1, \ldots, x_d) \cdot X_i(x_1, \ldots, x_d),$$

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

is equivalent to

$$\sum_{i=1}^{d} [D_{x_i} r_i(x_1, \ldots, x_d) + q_i(x_1, \ldots, x_d)] \cdot X_i(x_1, \ldots, x_d) + r_i(x_1, \ldots, x_d) \cdot D_{x_i} X_i(x_1, \ldots, x_d) = h(x_1, \ldots, x_d).$$

Ansatz:

$$\sum_{i=1}^{d}[D_{x_i} r_i(x_1,\dots,x_d) + q_i(x_1,\dots,x_d)] \cdot X_i(x_1,\dots,x_d)$$
$$+ r_i(x_1,\dots,x_d) \cdot D_{x_i} X_i(x_1,\dots,x_d) = h(x_1,\dots,x_d) \qquad (1)$$

Algorithm:

set $L = 0$

1. look for degree bounds for $X_i(x_1,\dots,x_d)$
2. try to find a solution of (1) by coefficient comparison
3. **if** there is no solution with not all $e_i(n)$'s equal to zero:
   increase L by one and go to 1
   else:
   return $\sum_{i=0}^{L} e_i(n) f(n+i)$ and $\underbrace{\overline{H}(n;x_1,\dots,x_d) r_i(x_1,\dots,x_d) X_i(x_1,\dots,x_d)}_{G_i(n;x_1,\dots,x_d)}$

Since according to [Apagodu and Zeilberger, 2006] the existence of a solution of (1) with not all the $e_i(n)$'s equal to zero is guaranteed for sufficiently large L, this process will eventually terminate.

In[1]:= << **Sigma.m**

In[2]:= << **HarmonicSums.m**

In[3]:= << **MultiIntegrate.m**

Sigma - A summation package by Carsten Schneider -©RISC- V 2.62 (December 18, 2019)

HarmonicSums by Jakob Ablinger -©RISC- Version 1.0 (19/02/20)

MultiIntegrate by Jakob Ablinger -©RISC- Version 1.0 (19/02/20)

In[4]:= $\mathbf{mAZ}\left[\frac{1}{(xy)^{n+1}(1-x-y+xy)^z}, n, \{x, y\}, f\right]$

Out[4]= $\left\{\left\{1, \frac{(-1+x)^{-z}(-1+y)^{-z}\left(\frac{1}{xy}\right)^n}{xy}\right\}, \left\{\frac{(-1+x)^{-z}(-1+y)^{-z}(xy)^{-2-n}}{xy}, \left\{\left\{nx^2y-nxy+x^2yz+x^2y-xyz-xy, x\right\}, \left\{ny^2-ny+2y^2-2y, y\right\}\right\}, \left(-1-2n-n^2-2z-2nz-z^2\right)f(1+n)+\left(4+4n+n^2\right)f(2+n)\right\}\right\}$

We now consider the integral

$$\mathcal{I}(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

Where for $F(n; x_1, \ldots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d).$$

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If

$$F(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) = F(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0,$$

we also have

$$G_i(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) = G_i(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0$$

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$$G_i(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) = G_i(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0$$

and hence $\mathcal{I}(n)$ satisfies the homogenous linear recurrence

$$\sum_{i=0}^{L} e_i(n) \mathcal{I}(n+i) = 0.$$

Example

Consider the integral

$$\mathcal{I}(n) := \int_a^1 \int_1^2 \frac{(a-x)(x-1)(y-2)(y-1)}{x^m y^n} dydx.$$

We find that the integrand satisfies the following differential equation

$$(-3+n)f(n) - 3(-1+n)f(1+n) + 2(1+n)f(2+n) = D_x 0 + D_y 0$$

Integration shows that $\mathcal{I}(n)$ satisfies

$$(-3+n)\mathcal{I}(n) - 3(-1+n)\mathcal{I}(1+n) + 2(1+n)\mathcal{I}(2+n) = 0.$$

Solving and comparing initial values leads to

$$\begin{aligned}\mathcal{I}(n) =& \frac{-5+15a+5m-5am+n-3an-mn+amn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\\ &+ \frac{2^{2-n}(1-3a-m+am+n-3an-mn+amn)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\\ &+ a^{-m}\left(\frac{-15a^2+5a^3+5a^2m-5a^3m+3a^2n-a^3n-a^2mn+a^3mn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\right.\\ &\left.+ \frac{2^{2-n}\left(3a^2-a^3-a^2m+a^3m+3a^2n-a^3n-a^2mn+a^3mn\right)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\right)\end{aligned}$$

In[5]:= **mAZIntegrate[$\frac{(a-x)(-1+x)(-2+y)(y-1)}{x^m y^n}$, n, {{x, a, 1}, {y, 1, 2}}]**

Out[5]=
$$\frac{-5+15a+5m-5am+n-3an-mn+amn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+\frac{2^{2-n}(1-3a-m+am+n-3an-mn+amn)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+a^{-m}\left(\frac{-15a^2+5a^3+5a^2m-5a^3m+3a^2n-a^3n-a^2mn+a^3mn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+\frac{2^{2-n}\left(3a^2-a^3-a^2m+a^3m+3a^2n-a^3n-a^2mn+a^3mn\right)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\right)$$

Recurrence for the Integral 2

We again consider the integral

$$\mathcal{I}(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds

Recurrence for the Integral 2

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- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \ldots, x_d)$ does not have to vanish at the bounds

Recurrence for the Integral 2

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- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \ldots, x_d)$ does not have to vanish at the bounds
- then force the G_i to vanish at the integration bounds by modifying the ansatz, and look for G_i of the form

$$G_i(n; x_1, \ldots, x_d) = \overline{G}_i(n; x_1, \ldots, x_d)(x_i - u_i)(x_i - o_i),$$

We again consider the integral

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- hence $\mathcal{I}(n)$ satisfies again a homogenous linear recurrence of the form

$$\sum_{i=0}^{L} e_i(n)\mathcal{I}(n+i) = 0.$$

Example

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} \, dx_1 \, dx_2 .$$

Note that the integrand does not vanish at the integration bounds.

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Note that the integrand does not vanish at the integration bounds.

- Using our ansatz we find

$$\begin{aligned} &2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\ &\ + (n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\ =&D_{x_1} F(n)(x_1 - 1)\, x_1\, (x_1 + 1)\, x_2\, ((n+3)x_1 x_2 + 2) \\ &\ + D_{x_2} F(n)(x_2 - 1)\, x_2\, \left(x_2^2 x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2 x_1^2 + (n+2)x_2 x_1 + 1\right). \end{aligned}$$

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- Integration of this recurrence yields

$$\begin{aligned} & 2(n+1)(\varepsilon - n - 2)\mathcal{I}(\varepsilon, n) - (n+2)(5\varepsilon - 5n - 13)\mathcal{I}(\varepsilon, n+1) \\ & + (n+3)(4\varepsilon - 4n - 13)\mathcal{I}(\varepsilon, n+2) - (n+4)(\varepsilon - n - 4)\mathcal{I}(\varepsilon, n+3) = 0. \end{aligned}$$

$$\mathcal{I}(\varepsilon,n)=\int_0^1\int_0^1 \underbrace{\frac{(1+x_1\cdot x_2)^n}{(1+x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds.

- Using our ansatz we find

$$\begin{aligned}&2(n+1)(\varepsilon-n-2)F(n)-(n+2)(5\varepsilon-5n-13)F(n+1)\\&+(n+3)(4\varepsilon-4n-13)F(n+2)-(n+4)(\varepsilon-n-4)F(n+3)\\=&D_{x_1}F(n)(x_1-1)\,x_1\,(x_1+1)\,x_2\,((n+3)x_1x_2+2)\\&+D_{x_2}F(n)(x_2-1)\,x_2\,\left(x_2^2x_1^3(-\varepsilon+n+4)-(\varepsilon-3)x_2x_1^2+(n+2)x_2x_1+1\right).\end{aligned}$$

- Integration of this recurrence yields

$$\begin{aligned}&2(n+1)(\varepsilon-n-2)\mathcal{I}(\varepsilon,n)-(n+2)(5\varepsilon-5n-13)\mathcal{I}(\varepsilon,n+1)\\&+(n+3)(4\varepsilon-4n-13)\mathcal{I}(\varepsilon,n+2)-(n+4)(\varepsilon-n-4)\mathcal{I}(\varepsilon,n+3)=0.\end{aligned}$$

- Solving the recurrence leads to

$$\mathcal{I}(\varepsilon,n)=\frac{1}{n+1}\left(\sum_{i=1}^n\frac{1}{-i+\varepsilon-1}-2^{1-\varepsilon}\sum_{i=1}^n\frac{2^i}{-i+\varepsilon-1}+\frac{2^{-\varepsilon}\left(2^\varepsilon-2\right)}{\varepsilon-1}\right)$$

In[6]:= **mAZDirectIntegrate[** $\dfrac{(1+x1*x2)^n}{(1+x1)^{\varepsilon}}$**, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[6]= $$\frac{\sum_{\iota_1=1}^{n}\frac{1}{\iota_1-\varepsilon+1}}{-n-1}-\frac{2\sum_{\iota_1=1}^{n}\frac{2^{\iota_1}}{-\iota_1+\varepsilon-1}}{(n+1)2^{\varepsilon}}+\frac{2^{\varepsilon}-2}{(n+1)(\varepsilon-1)2^{\varepsilon}}$$

In[7]:= **mAZDirectIntegrate[** $\dfrac{(1+x1+x2+x1*x2)^n}{(1+x1)^{\varepsilon}}$**, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[7]= $$\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$$

In[8]:= **mAZDirectIntegrate[** $\dfrac{(1+x1+x2+x1*x2)^n}{(1+x1)^{\varepsilon}}$**, n, {{x1, 0,** $\dfrac{1}{2}$**}, {x2, 0, 2}}]**

Out[8]= $$\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$$

We look again at the integral

$$\mathcal{I}(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

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$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\sum_{i=0}^{L} e_i(n) \mathcal{I}(n+i) =$$

$$\sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$
$$- \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

with

$$O_i(n) := G_i(n; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d)$$
$$U_i(n) := G_i(n; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d).$$

We look again at the integral

$$\mathcal{I}(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\sum_{i=0}^{L} e_i(n) \mathcal{I}(n+i) =$$

$$\sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

$$- \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

with

$$O_i(n) := G_i(n; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d)$$
$$U_i(n) := G_i(n; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d).$$

Note that there are $2 \cdot d$ integrals of dimension $d-1$, to compute.

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

- Applying the algorithm leads to

$$(n+2)F(n+1; x_1, x_2) - (n+1)F(n; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}.$$

Example

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

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- By integration we find

$$-(n+1)\mathcal{I}(\varepsilon, n) + (n+2)\mathcal{I}(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{\mathcal{I}_1(n)} - \int_0^1 0 dx_1.$$

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

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- By integration we find

$$-(n+1)\mathcal{I}(\varepsilon, n) + (n+2)\mathcal{I}(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{\mathcal{I}_1(n)} - \int_0^1 0 dx_1.$$

- Now we apply the algorithm to $\mathcal{I}_1(n)$; we find

$$\mathcal{I}_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

Example

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

- Plugging in yields the recurrence

$$-(n+1)\mathcal{I}(\varepsilon, n) + (n+2)\mathcal{I}(\varepsilon, n+1) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

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- For $n = 0$ we find the initial value

$$\mathcal{I}(\varepsilon, 0) = \frac{2^\epsilon - 2}{2^\epsilon(\epsilon - 1)}$$

Example

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n; x_1, x_2) :=} dx_1 dx_2,$$

- Plugging in yields the recurrence

$$-(n+1)\mathcal{I}(\varepsilon, n) + (n+2)\mathcal{I}(\varepsilon, n+1) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

- For $n = 0$ we find the initial value

$$\mathcal{I}(\varepsilon, 0) = \frac{2^\epsilon - 2}{2^\epsilon(\epsilon - 1)}$$

- Solving this recurrence yields

$$\mathcal{I}(\varepsilon, n) = \frac{1}{n+1}\left(\sum_{i=1}^{n} \frac{1}{-i + \varepsilon - 1} - 2^{1-\varepsilon} \sum_{i=1}^{n} \frac{2^i}{-i + \varepsilon - 1} + \frac{2^{-\varepsilon}(2^\varepsilon - 2)}{\varepsilon - 1}\right).$$

In[9]:= **mAZIntegrate[** $\dfrac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$ **, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[9]= $$\frac{\sum_{\iota_1=1}^{n}\frac{1}{\iota_1-\varepsilon+1}}{-n-1}-\frac{2\sum_{\iota_1=1}^{n}\frac{2^{\iota_1}}{-\iota_1+\varepsilon-1}}{(n+1)2^\varepsilon}+\frac{2^\varepsilon-2}{(n+1)(\varepsilon-1)2^\varepsilon}$$

In[10]:= **mAZIntegrate[** $\dfrac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$ **, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[10]= $$\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$$

In[11]:= **mAZIntegrate[** $\dfrac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$ **, n, {{x1, 0,** $\frac{1}{2}$**}, {x2, 0, 2}}]**

Out[11]= $$\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$$

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(n).$$

Laurent Series Expansion of the Integral

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$$\mathcal{I}(\varepsilon, n) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(n).$$

- find $I_{-K}(n), I_{-K+1}(n), \dots, I_u(n)$ in terms of indefinite nested product-sum expressions.

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$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

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$$\mathcal{I}(\varepsilon, n) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(n).$$

- find $I_{-K}(n), I_{-K+1}(n), \ldots, I_u(n)$ in terms of indefinite nested product-sum expressions.
- compute a recurrence for $\mathcal{I}(\varepsilon, n)$ in the form

$$a_0(\varepsilon, n) T(\varepsilon, n) + \cdots + a_d(\varepsilon, n) T(\varepsilon, n + d) = h_0(n) + \cdots + h_u(n)\varepsilon^u + O(\varepsilon^{u+1});$$

  by using one of the methods presented above

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(n).$$

- find $I_{-K}(n), I_{-K+1}(n), \ldots, I_u(n)$ in terms of indefinite nested product-sum expressions.
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  by using one of the methods presented above
- find a Laurent series expansion of the integral by using an algorithm implemented in `Sigma`.

Example

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

- Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}.$$

Example

$$\mathcal{I}(\varepsilon,n)=\int_0^1\int_0^1\underbrace{\frac{(1+x_1\cdot x_2)^n}{(1+x_1)^\varepsilon}}_{F(n;x_1,x_2):=}dx_1dx_2,$$

- Applying the algorithm leads to

$$-(n+1)F(n;x_1,x_2)+(n+2)F(n+1;x_1,x_2)=D_{x_1}0+D_{x_2}\frac{x_2(x_1\cdot x_2+1)^{n+1}}{(1+x_1)^\varepsilon}.$$

- Integration yields

$$-(n+1)\mathcal{I}(\varepsilon,n)+(n+2)\mathcal{I}(\varepsilon,n+1)=\underbrace{\int_0^1(x_1+1)^{n+1-\varepsilon}dx_1}_{\mathcal{I}_1(\varepsilon,n)}-\int_0^1 0dx_1.$$

$$\mathcal{I}(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

- Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}.$$

- Integration yields

$$-(n+1)\mathcal{I}(\varepsilon, n) + (n+2)\mathcal{I}(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{\mathcal{I}_1(\varepsilon,n)} - \int_0^1 0 dx_1.$$

- In the next step apply the method to $\mathcal{I}_1(\varepsilon, n)$; we find

$$\mathcal{I}_1(\varepsilon, n) = \frac{2^{n+2} - 1}{n+2} + \frac{\varepsilon \left(-2^{n+2} \left(\log(2)(n+2) - 1\right) - 1\right)}{(n+2)^2}$$
$$+ \frac{\varepsilon^2 \left(2^{n+1} \left(\log(2)^2(n+2)^2 - 2\log(2)(n+2) + 2\right) - 1\right)}{(n+2)^3} + O\left(\varepsilon^3\right).$$

- Plugging in and solving the resulting recurrence by means of `Sigma` and combining the solution with the initial values of the integral yields:

$$\begin{aligned}
\mathcal{I}(\varepsilon,n) =& \frac{\mathrm{S}_1(2;n)}{n+1} - \frac{\mathrm{S}_1(n)}{n+1} + \frac{1}{n+1} + \frac{2(2^n-n-1)}{(n+1)^2} + \frac{n}{(n+1)^2} \\
&+ \varepsilon\Bigg(\log(2)\left(\frac{-2^{n+2}(n+1)-2^{n+2}n(n+1)}{2(n+1)^4} - \frac{\mathrm{S}_1(2;n)}{n+1}\right) \\
&+ \frac{\left(2(n+1)n^2+4(n+1)n+2(n+1)\right)\mathrm{S}_2(2;n)}{2(n+1)^4} - \frac{\mathrm{S}_2(n)}{n+1} \\
&+ \frac{2^{n+2}(n+1)-2(n+1)}{2(n+1)^4}\Bigg) + \varepsilon^2\Bigg(\frac{\mathrm{S}_3(2;n)}{n+1} - \frac{\mathrm{S}_3(n)}{n+1} \\
&+ \frac{2^{n+2}-2}{2(n+1)^4} + \log(2)\left(\frac{-2^{n+2}n-2^{n+2}}{2(n+1)^4} - \frac{\mathrm{S}_2(2;n)}{n+1}\right) \\
&+ \log(2)^2\left(\frac{\mathrm{S}_1(2;n)}{2(n+1)} + \frac{2^n(n^2+2n+1)}{(n+1)^4}\right)\Bigg) + O\left(\varepsilon^3\right).
\end{aligned}$$

In[12]:= << **EvaluateMultiSums.m**

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In[13]:= **mAZExpandedIntegrate[$\frac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$, n, {ε, 0, 2}, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[13]=

$$\Big\{\Big\{\frac{\text{S}[1,\{2\},n]}{n+1}-\frac{\text{S}[1,n]}{n+1}+\frac{2^{n+1}-1}{(n+1)^2},-\frac{\log(2)\text{S}[1,\{2\},n]}{n+1}+\frac{\text{S}[2,\{2\},n]}{n+1}-\frac{\text{S}[2,n]}{n+1}+\frac{\log(2)\left(-2^{n+1}\right)(n+1)+2^{n+1}-1}{(n+1)^3},\frac{\log(2)^2\text{S}[1,\{2\},n]}{2n+2}-\frac{\log(2)\text{S}[2,\{2\},n]}{n+1}+\frac{\text{S}[3,\{2\},n]}{n+1}-\frac{\text{S}[3,n]}{n+1}+\frac{\log(2)^2 2^n(n+1)^2-\log(2)2^{n+1}(n+1)+2^{n+1}-1}{(n+1)^4}\Big\},0,2\Big\}$$

In[12]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider -©RISC- V 2.62 (December 18, 2019)

In[14]:= $\mathbf{mAZExpandedIntegrate}\left[\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^{\varepsilon}}, n, \{\varepsilon, 0, 1\}, \{\{x1, 0, 1\}, \{x2, 0, 1\}\}\right]$

Out[14]= $$\left\{\left\{\frac{\left(2^{n+1}-1\right)^2}{(n+1)^2}, -\frac{\left(2^{n+1}-1\right)\left(-2^{n+1}+2^{n+1}n\log(2)+2^{n+1}\log(2)+1\right)}{(n+1)^3}\right\}, 0, 1\right\}$$

In[12]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider -©RISC- V 2.62 (December 18, 2019)

In[15]:= $\mathbf{mAZExpandedIntegrate}\left[\frac{(1+x1+x2+x1*x2)^{n}}{(1+x1)^{\varepsilon}}, n, \{\varepsilon, 0, 1\}, \{\{x1, 0, \frac{1}{2}\}, \{x2, 0, 2\}\}\right]$

Out[15]= $\left\{\left\{\frac{\left(3^{n+1}-2^{n+1}\right)\left(3^{n+1}-1\right)}{2^{n+1}(n+1)^{2}}, \frac{\left(3^{n+1}-1\right)\left(2^{n+1}-3^{n+1}+3^{n+1}\log\left(\frac{3}{2}\right)(n+1)\right)}{2^{n+1}(n+1)^{3}}\right\}, 0, 1\right\}$

A bigger example: b56729a [A. et al., 2014]

$$\int_0^1\int_0^1\int_0^1 \frac{(xyz(1-xz)(1-yz))^{\frac{\varepsilon}{2}}(x+y-1)^N z^{N+1}\Gamma\left(\frac{-\varepsilon}{2}\right)^2\Gamma(1+N)}{(1-z)^{2+\frac{3\varepsilon}{2}}\Gamma\left(2+\frac{\varepsilon}{2}+N\right)}\,dz\,dy\,dx$$

Initial values:

	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ^{1}	ϵ^{2}	ϵ^{3}
n=2	$\frac{20}{27}$	$-\frac{40}{27}$	$\frac{5\zeta(2)}{18}+\frac{1393}{486}$	$-\frac{5\zeta(2)}{9}+\frac{49\zeta(3)}{54}-\frac{9601}{1944}$	.	.	.
n=3	$\frac{1}{6}$	$-\frac{11}{48}$	$\frac{\zeta(2)}{16}+\frac{703}{3456}$	$-\frac{11\zeta(2)}{128}+\frac{41\zeta(3)}{48}-\frac{9773}{9216}$	.	.	.
n=4	$\frac{64}{225}$	$-\frac{1748}{3375}$	$\frac{8\zeta(2)}{75}+\frac{102181}{101250}$	$-\frac{437\zeta(2)}{2250}+\frac{196\zeta(3)}{225}-\frac{5738207}{2430000}$	.	.	.
n=5	$\frac{2}{27}$	$-\frac{17}{162}$	$\frac{\zeta(2)}{36}+\frac{7583}{97200}$	$-\frac{17\zeta(2)}{432}+\frac{113\zeta(3)}{108}-\frac{1666837}{1296000}$	.	.	.
n=6	$\frac{22}{147}$	$-\frac{535}{2058}$	$\frac{11\zeta(2)}{196}+\frac{630043}{1234800}$	$-\frac{535\zeta(2)}{5488}+\frac{775\zeta(3)}{588}-\frac{1949958721}{871274880}$	.	.	.
n=7	$\frac{1}{24}$	$-\frac{23}{384}$	$\frac{\zeta(2)}{64}+\frac{2699659}{67737600}$	$-\frac{23\zeta(2)}{1024}+\frac{115\zeta(3)}{64}-\frac{49626263807}{22759833600}$	.	.	.
n=8	$\frac{112}{1215}$	$-\frac{2836}{18225}$	$\frac{14\zeta(2)}{405}+\frac{74074613}{241116750}$	$-\frac{709\zeta(2)}{12150}+\frac{3082\zeta(3)}{1215}-\frac{1392921564301}{405076140000}$	.	.	.

A bigger example: B56729a

```
In[1]:= << Sigma.m
        << HarmonicSums.m
        << EvaluateMultiSums.m
        << MultiIntegrate.m
```

Sigma – A summation package by Carsten Schneider – © RISC – V 2.62 (December 18, 2019) Help

HarmonicSums by Jakob Ablinger – © RISC – Version 1.0 (19 / 02 / 19) Help

EvaluateMultiSums by Carsten Schneider – © RISC – V 0.99 (December 18, 2019)

MultiIntegrate by Jakob Ablinger – © RISC – Version 1.0 (19 / 02 / 16)

```
In[5]:= B56729aInit = << B56729aInit;

In[6]:= B56729a = mAZExpandedDirectIntegrate[x^(ϵ/2) * y^(ϵ/2) * z^(1 + ϵ/2) * (1 - z)^(-2 - 3/2 * ϵ) * (1 - x * z)^(ϵ/2) *
          (1 - z * y)^(ϵ/2) * (x + y - 1)^N * z^N * Gamma[-ϵ/2]^2 * Gamma[N + 1] / Gamma[N + 2 + ϵ/2], N, {ϵ, -3, 2},
        {{z, 0, 1}, {y, 0, 1}, {x, 0, 1}}, MIPrint → 4, RaiseDegreeBounds → 0, Test → False, PrintRecurrence → True,
        StartingValue → 2, InitValues -> B56729aInit, TestToFile → "B56729a"]
```

```
Computing the recurrence of the 3-dimensional integral...
Looking for a recurrence of order 0 ...
Number of solutions: 0
Looking for a recurrence of order 1 ...
Number of solutions: 0
Looking for a recurrence of order 2 ...
Number of solutions: 0
Looking for a recurrence of order 3 ...
Number of solutions: 0
Looking for a recurrence of order 4 ...
Number of solutions: 0
```

A bigger example: b56729a

```
Looking for a recurrence of order 5 ...
Number of solutions: 4
339
340
Solving of the system...
Variables in the system: {ep, N}
MakeTriangularMatrix...
Time for line 1: 0.093921. Time: 0.093921.
Time for line 2: 0.103560. Time: 0.197481.
Time for line 3: 0.094854. Time: 0.29234.
Time for line 4: 0.107270. Time: 0.39961.
Time for line 5: 0.118178. Time: 0.51778.
Time for line 6: 0.083011. Time: 0.60079.
Time for line 7: 0.099021. Time: 0.69982.
Time for line 8: 0.101248. Time: 0.80106.
Time for line 9: 0.105792. Time: 0.90686.
Time for line 10: 0.176128. Time: 1.08298.
Time for line 11: 0.107279. Time: 1.19026.
Time for line 12: 0.112057. Time: 1.30232.
Time for line 13: 0.114501. Time: 1.41682.
Time for line 14: 0.119833. Time: 1.53665.
Time for line 15: 0.106854. Time: 1.64351.
Time for line 16: 0.110722. Time: 1.75423.
Time for line 17: 0.139883. Time: 1.89411.
Time for line 18: 0.132837. Time: 2.02695.
Time for line 19: 0.177293. Time: 2.20424.
Time for line 20: 0.073420. Time: 2.27766.
```

⋮

$$
\begin{aligned}
&4\,(1+N)\,(2+N)\,(-2-2N+\epsilon)\,(4+2N+\epsilon)^2\\
&\quad\Big(16\,(3+N)^2\,(760+946N+425N^2+83N^3+6N^4)-16\,(11856+17500N+10258N^2+2988N^3+433N^4+25N^5)\,\epsilon+4\,(16483+20643N+10088N^2+2394N^3+274N^4+12N^5)\,\epsilon^2-\\
&\quad 4\,(2212+3009N+1484N^2+321N^3+26N^4)\,\epsilon^3-(1489+861N+132N^2+4N^3)\,\epsilon^4+(491+353N+60N^2)\,\epsilon^5+22\,(3+N)\,\epsilon^6+2\,\epsilon^7\Big)\,G[N]+\\
&(2+N)\,\Big(256\,(2+N)^2\,(3+N)^3\,(5544+11702N+8675N^2+2968N^3+481N^4+30N^5)-256\,(3+N)^2\,(166392+437060N+480046N^2+286009N^3+99910N^4+20482N^5+2284N^6+107N^7)\,\epsilon+\\
&\quad 128\,(961146+2676363N+3291506N^2+2357710N^3+1089697N^4+338845N^5+71254N^6+9812N^7+805N^8+30N^9)\,\epsilon^2-\\
&\quad 64\,(-379338-704959N-447044N^2-37256N^3+96299N^4+55870N^5+14305N^6+1823N^7+94N^8)\,\epsilon^3-16\,(560287+1340455N+1382074N^2+802660N^3+284762N^4+61778N^5+7578N^6+404N^7)\,\epsilon^4+\\
&\quad 16\,(6100+108259N+165290N^2+104918N^3+33569N^4+5348N^5+336N^6)\,\epsilon^5+8\,(116092+226821N+174692N^2+67246N^3+12961N^4+994N^5)\,\epsilon^6+4\,(15987+14351N+6065N^2+1857N^3+266N^4)\,\epsilon^7-\\
&\quad (54229+63597N+22936N^2+2548N^3)\,\epsilon^8-(11745+8611N+1500N^2)\,\epsilon^9-2\,(435+157N)\,\epsilon^{10}-22\,\epsilon^{11}\Big)\,G[1+N]+\\
&2\,\Big(-256\,(3+N)^3\,(8+6N+N^2)^2\,(655+1114N+557N^2+104N^3+6N^4)+128\,(3+N)^2\,(709216+1880936N+2103708N^2+1293326N^3+476321N^4+106931N^5+14125N^6+981N^7+26N^8)\,\epsilon-\\
&\quad 64\,(3829812+10327886N+12174609N^2+8305127N^3+3671693N^4+1122454N^5+248042N^6+40451N^7+4724N^8+350N^9+12N^{10})\,\epsilon^2+\\
&\quad 32\,(-1217770-2362944N-1551017N^2-105413N^3+395334N^4+236215N^5+63510N^6+8651N^7+506N^8+4N^9)\,\epsilon^3+\\
&\quad 16\,(1100084+2304248N+2161699N^2+1240186N^3+493839N^4+139926N^5+26550N^6+2940N^7+140N^8)\,\epsilon^4+8\,(-23458-191528N-206516N^2-55105N^3+25532N^4+19137N^5+4314N^6+340N^7)\,\epsilon^5-\\
&\quad 8\,(275664+563176N+486841N^2+227386N^3+59749N^4+8210N^5+450N^6)\,\epsilon^6-2\,(185290+337198N+269119N^2+113313N^3+24056N^4+1996N^5)\,\epsilon^7+2\,(50850+69119N+30942N^2+4983N^3+172N^4)\,\epsilon^8+\\
&\quad (38907+41901N+14190N^2+1504N^3)\,\epsilon^9+(4795+3553N+620N^2)\,\epsilon^{10}+6\,(45+17N)\,\epsilon^{11}+6\,\epsilon^{12}\Big)\,G[2+N]-\\
&4\,\Big(128\,(24+26N+9N^2+N^3)^2\,(720+2014N+2009N^2+872N^3+169N^4+12N^5)-64\,(3284496+13784028N+24797006N^2+25409054N^3+16516984N^4+7144405N^5+2088122N^6+407980N^7+51066N^8+3701N^9+118N^{10})\,\epsilon+\\
&\quad 32\,(8824032+27875896N+38244797N^2+29964399N^3+14824351N^4+4834279N^5+1053620N^6+152630N^7+14362N^8+830N^9+24N^{10})\,\epsilon^2-\\
&\quad 16\,(2031600+4272517N+3714620N^2+1854572N^3+723136N^4+287700N^5+99545N^6+22500N^7+2768N^8+140N^9)\,\epsilon^3-\\
&\quad 8\,(3878602+9621283N+10076450N^2+5813984N^3+2025336N^4+439922N^5+59527N^6+4802N^7+188N^8)\,\epsilon^4+4\,(2113696+5071913N+5349388N^2+3218059N^3+1186719N^4+265889N^5+33154N^6+1756N^7)\,\epsilon^5+\\
&\quad (649744+823518N+349802N^2+54816N^3+8384N^4+4100N^5+640N^6)\,\epsilon^6-4\,(221769+412258N+304963N^2+112365N^3+20460N^4+1456N^5)\,\epsilon^7-(84655+112966N+63837N^2+17868N^3+1948N^4)\,\epsilon^8+\\
&\quad (34305+39746N+13839N^2+1444N^3)\,\epsilon^9+2\,(3896+2948N+519N^2)\,\epsilon^{10}+(596+226N)\,\epsilon^{11}+16\,\epsilon^{12}\Big)\,G[3+N]+\\
&8\,\Big(64\,(6+5N+N^2)^2\,(51720+131134N+129703N^2+67054N^3+19864N^4+3400N^5+313N^6+12N^7)-\\
&\quad 32\,(13147272+44902452N+67518826N^2+58900035N^3+33018349N^4+12426820N^5+3178478N^6+545125N^7+59929N^8+3808N^9+106N^{10})\,\epsilon+\\
&\quad 16\,(20544948+58573022N+72551278N^2+51298261N^3+22873183N^4+6710299N^5+1314686N^6+172000N^7+14897N^8+826N^9+24N^{10})\,\epsilon^2-\\
&\quad 8\,(3111104+6120858N+4687014N^2+1847484N^3+530980N^4+208191N^5+82966N^6+19923N^7+2424N^8+116N^9)\,\epsilon^3-\\
&\quad 4\,(9745752+22066744N+21401558N^2+11588223N^3+3836854N^4+802421N^5+105800N^6+8356N^7+316N^8)\,\epsilon^4+\\
&\quad 2\,(5260048+11433064N+10664390N^2+5604391N^3+1801014N^4+353301N^5+38868N^6+1828N^7)\,\epsilon^5+2\,(694370+952507N+536911N^2+169650N^3+36275N^4+5507N^5+420N^6)\,\epsilon^6-\\
&\quad (858544+1440105N+948722N^2+311957N^3+51138N^4+3304N^5)\,\epsilon^7-(94650+103981N+47258N^2+11117N^3+1084N^4)\,\epsilon^8+(22488+26188N+8924N^2+906N^3)\,\epsilon^9+(4902+3538N+594N^2)\,\epsilon^{10}+\\
&\quad 24\,(14+5N)\,\epsilon^{11}+8\,\epsilon^{12}\Big)\,G[4+N]-\\
&16\,(5+N)\,(-4-N+\epsilon)\,(6+N+\epsilon)\,(-3-N+2\epsilon)\,(7+N+2\epsilon)\\
&\quad\Big(16\,(2+N)^2\,(162+321N+212N^2+59N^3+6N^4)-16\,(2034+4341N+3642N^2+1506N^3+308N^4+25N^5)\,\epsilon+4\,(3796+6613N+4430N^2+1418N^3+214N^4+12N^5)\,\epsilon^2-4\,(392+900N+677N^2+217N^3+26N^4)\,\epsilon^3-\\
&\quad (756+609N+120N^2+4N^3)\,\epsilon^4+(198+233N+60N^2)\,\epsilon^5+22\,(2+N)\,\epsilon^6+2\,\epsilon^7\Big)\,G[5+N]
\end{aligned}
$$

⋮

```
Computing the expansion of the recurrence of the 3-dimensional integral...

Computing the expansion of the 3-dimensional integral...
```

Out[6]=

$$\Big\{\Big\{\frac{8\left(3+(-1)^N+2N+(-1)^N N\right)}{3(1+N)^2(2+N)},\ -\frac{4\left(26+\langle\langle 6\rangle\rangle+3\left(\langle\langle 1\rangle\rangle\ \langle\langle 1\rangle\rangle^N N^3\right)\right)}{3(1+N)^3(2+N)^2},\ \langle\langle 4\rangle\rangle,$$

$$\frac{\langle\langle 1\rangle\rangle}{\langle\langle 1\rangle\rangle}+\langle\langle 773\rangle\rangle+\frac{\langle\langle 1\rangle\rangle}{\langle\langle 1\rangle\rangle},\ \frac{4641664-7489152(-1)^N+\langle\langle 59\rangle\rangle}{48(1+N)^9(2+N)^8}+\frac{\left(\langle\langle 1\rangle\rangle\right)z2}{32(1+N)^7(2+N)^6}+\frac{1}{640(1+N)^5(2+N)^4}$$

$$\big(253736+54984(-1)^N-1467\times 2^{8+N}+329428N+173652(-1)^N N-4401\times 2^{7+N}N-104338N^2+263726(-1)^N N^2-$$

$$1773\times 2^{6+N}N^2-427193N^3+\langle\langle 5\rangle\rangle+549\times 2^{8+N}N^4-125904N^5+48372(-1)^N N^5+459\times 2^{6+N}N^5-$$

$$26316N^6+9396(-1)^N N^6+81\times 2^{5+N}N^6-2322N^7+783(-1)^N N^7\big)\,z2^2+$$

$$\frac{\left(998794+354114(-1)^N-41877\times 2^{6+N}+272379N+\langle\langle 6\rangle\rangle+3807\times 2^{5+N}N^2-64698N^3+59019(-1)^N N^3\right)z2^3}{26880(1+N)^3(2+N)^2}+\frac{\left(\langle\langle 1\rangle\rangle\right)z3}{48\,\langle\langle 1\rangle\rangle^6\,\langle\langle 1\rangle\rangle^5}+$$

$$\langle\langle 2485\rangle\rangle+\frac{2^{4+N}S\left[1,2,1,1,1,1,\left\{\frac{1}{2},-1,1,1,1,1\right\},N\right]}{(1+N)^2(2+N)}+\frac{2^{3+N}S\left[1,2,1,1,1,1,\left\{\frac{1}{2},1,1,1,1,1\right\},N\right]}{(1+N)^2(2+N)}\Big\},\ \langle\langle 1\rangle\rangle,\ 4\Big\}$$

large output | **show less** | **show more** | **show all** | **set size limit...**

In[7]:= `TimeUsed[]`

Out[7]= 28 203.7

In[8]:= `MaxMemoryUsed[]`

Out[8]= 1 576 165 800

A BIGGER EXAMPLE: B56729A

$$\epsilon^{-3}:\ \frac{8\left(3+2N+(-1)^N+N(-1)^N\right)}{3(1+N)^2(2+N)}$$

$$\epsilon^{-2}:\ -\frac{4\left(26+51N+32N^2+6N^3+18(-1)^N+31N(-1)^N+18N^2(-1)^N+3N^3(-1)^N\right)}{3(1+N)^3(2+N)^2}$$

$$\epsilon^{-1}:\ \frac{1}{3(1+N)^4(2+N)^3}\Big(264+1072N+1510N^2+980N^3+300N^4+36N^5+36\,z2+132N\,z2+189N^2\,z2+132N^3\,z2+45N^4\,z2+6N^5\,z2+408(-1)^N+1000N(-1)^N+1066N^2(-1)^N+590N^3(-1)^N+162N^4(-1)^N+18N^5(-1)^N+12\,z2(-1)^N+48N\,z2(-1)^N+75N^2\,z2(-1)^N+57N^3\,z2(-1)^N+21N^4\,z2(-1)^N+3N^5\,z2(-1)^N\Big)+\frac{4\left(-9-N+(-1)^N+N(-1)^N\right)S[-2,N]}{3(1+N)^2(2+N)}+\frac{4\left(-1+(-1)^N+N(-1)^N\right)S[2,N]}{(1+N)^2(2+N)}$$

$$\epsilon^{0}:\ \frac{1}{6(1+N)^5(2+N)^4}\Big(-656-7624N-19340N^2-23014N^3-15272N^4-5856N^5-1224N^6-108N^7-312\,z2-1548N\,z2-3234N^2\,z2-3681N^3\,z2-2460N^4\,z2-963N^5\,z2-204N^6\,z2-18N^7\,z2-48\,z3-152N\,z3-108N^2\,z3+150N^3\,z3+306N^4\,z3+210N^5\,z3+66N^6\,z3+8N^7\,z3-3792(-1)^N-11976N(-1)^N-18188N^2(-1)^N-16710N^3(-1)^N-9588N^4(-1)^N-3336N^5(-1)^N-648N^6(-1)^N-54N^7(-1)^N-216\,z2(-1)^N-1020N\,z2(-1)^N-2034N^2\,z2(-1)^N-2217N^3\,z2(-1)^N-1422N^4\,z2(-1)^N-534N^5\,z2(-1)^N-108N^6\,z2(-1)^N-9N^7\,z2(-1)^N-16\,z3(-1)^N-88N\,z3(-1)^N-204N^2\,z3(-1)^N-258N^3\,z3(-1)^N-192N^4\,z3(-1)^N-84N^5\,z3(-1)^N-20N^6\,z3(-1)^N-2N^7\,z3(-1)^N+384\,z3\,2^N+1728N\,z3\,2^N+3168N^2\,z3\,2^N+3024N^3\,z3\,2^N+1584N^4\,z3\,2^N+432N^5\,z3\,2^N+48N^6\,z3\,2^N\Big)-\frac{4\left(N+2(-1)^N+2N(-1)^N\right)S[-3,N]}{3(1+N)^2(2+N)}+\frac{4S[1,N]\times S[2,N]}{(1+N)^2(2+N)}+\frac{2\left(18+N+2(-1)^N+2N(-1)^N\right)S[3,N]}{3(1+N)^2(2+N)}-\frac{8S[-2,1,N]}{(1+N)^2(2+N)}+S[2,N]\left(-\frac{2\left(-14-9N+18(-1)^N+31N(-1)^N+18N^2(-1)^N+3N^3(-1)^N\right)}{(1+N)^3(2+N)^2}+\frac{8\times 2^N S\left[1,\left\{\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}\right)+S[-2,N]\left(-\frac{2\left(-118-93N-16N^2-3N^3+18(-1)^N+31N(-1)^N+18N^2(-1)^N+3N^3(-1)^N\right)}{3(1+N)^3(2+N)^2}+\frac{8S[1,N]}{(1+N)^2(2+N)}+\frac{24\times 2^N S\left[1,\left\{\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}\right)-\frac{4S[2,1,N]}{(1+N)^2(2+N)}+\frac{24\times 2^N S\left[3,\left\{-\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}+\frac{8\times 2^N S\left[3,\left\{\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}-\frac{24\times 2^N S\left[2,1,\left\{-1,\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}-\frac{8\times 2^N S\left[2,1,\left\{1,\frac{1}{2}\right\},N\right]}{(1+N)^2(2+N)}$$

Detour to Mellin Transforms and Holonomic Functions.

- A sequence $(f_n)_{n\geq 0}$ is called *holonomic* if there exist polynomials $p_i(n)$ (not all $p_i = 0$) such that

$$p_d(n)f_{n+d} + \cdots + p_1(n)f_{n+1} + p_0(n)f_n = 0.$$

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- A function $f = f(x)$ is called *holonomic* if there exist polynomials $p_i(x)$ (not all $p_i = 0$) such that

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$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0.$$
- Let $(f_n)_{n\geq 0}$ be a holonomic sequence, then the generating function
$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$
is a holonomic function.
- Let $f(x)$ be a holonomic function such that the Mellin transform
$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x)dx$$
exists, then $\mathbf{M}[\mathrm{f(x)}](\mathrm{n})$ is a holonomic sequence.

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

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Method:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geq 0}$.
2. Use the recurrence to derive a holonomic differential equation for $f(x)$.
3. Compute initial values.
4. Solve the differential equation to get a closed form representation for $f(x)$.

Note: This is implemented in `HarmonicSums`.

Given

$$g(i) := \frac{1}{i^2\binom{2i}{i}},$$

we want to compute a closed form for

$$f(x) = \sum_{i=1}^{\infty} x^i g(i).$$

- recurrence satisfied by $g(i)$:

$$(1+i)^2 g(i) - 2(2+i)(3+2i)g(1+i) = 0$$

- from the recurrence we can derive a differential equation for $f(x)$

$$f(x) + (-12+7x)f'(x) + 6x(-3+x)f''(x) + x^2(-4+x)f'''(x) = 0$$

- initial values

$$f(0) = 0,\ f'(0) = \frac{1}{2},\ f''(0) = \frac{1}{12}.$$

- solving the differential equation and combing with initial values we find:

$$f(x) = \frac{(16x - 20x^2 + 8x^3 - x^4)}{32} + \frac{(x-2)\sqrt{(4-x)x}}{8}\int_0^x \sqrt{4-\tau}\sqrt{\tau}d\tau + \frac{1}{8}\left(\int_0^x \sqrt{4-\tau}\sqrt{\tau}d\tau\right)^2$$

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

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2. Use the differential equation to derive a holonomic recurrence for $(f_n)_{n\geq 0}$.
3. Compute initial values $f_0, f_1, \ldots$.
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Note: This is implemented in `HarmonicSums` using the recurrence solver of `Sigma`.

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

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1. Compute a holonomic differential equation for $f(x)$.
2. Use the differential equation to derive a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
3. Compute initial values $\mathbf{M}[f(x)](0), \mathbf{M}[f(x)](1), \ldots$
4. Solve the recurrence to get a closed form representation for $\mathbf{M}[f(x)](n)$.

Note: This is implemented in `HarmonicSums` using the recurrence solver of `Sigma`.

Given a holonomic sequence $F(n)$.
Find an expression $f(x)$ given as a linear combination of indefinite iterated integrals such that for all $n \in \mathbb{N}$ (from a certain point on) we have

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Inverse Mellin transform

Given a holonomic sequence $F(n)$.
Find an expression $f(x)$ given as a linear combination of indefinite iterated integrals such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) = F(n).$$

Method:

1. Compute a holonomic recurrence for $F(n)$.
2. Use the recurrence to derive a differential equation for $f(x)$.
3. Compute a linear independent set of solutions of the differential equation.
4. Compute initial values $F(0), F(1), \ldots$
5. Combine the initial values and the solutions to get a closed form representation for $f(x)$.

Note: This is implemented in `HarmonicSums`.

SOLVING DIFFERENTIAL EQS.: HYPEREXPONENTIAL SOLUTIONS [PETKOVŠEK, 1992]

Consider the linear differential equation ($p_i(x) \in \mathbb{K}[x]$)

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0. \tag{2}$$

Solving differential eqs.: Hyperexponential solutions [Petkovšek, 1992]

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Consider the linear differential equation ($p_i \in \mathbb{K}[x]$; $d > 1$)

$$\left(p_d(x)D_x^d + \cdots + p_1(x)D_x + p_0(x)\right) f(x) = 0. \tag{3}$$

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- compute a hyperexponential solution $r(x)$ of (3)
- $r(x)$ is a solution of $v'(x) - \frac{r'(x)}{r(x)} v(x) = 0$
- compute (by division) $q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x)$, such that

$$\begin{aligned}(q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x))\left(D_x - \frac{r'(x)}{r(x)}\right)\\ = p_d(x)D_x^d + \cdots + p_1(x)D_x + p_0(x)\end{aligned}$$

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$$(q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x)) \left(D_x - \frac{r'(x)}{r(x)}\right) = p_d(x)D_x^d + \cdots + p_1(x)D_x + p_0(x)$$

- compute d'Alembert solutions $u_1(x), \ldots, u_k(x)$ of

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- compute (by division) $q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x)$, such that

$$(q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x)) \left(D_x - \frac{r'(x)}{r(x)}\right) = p_d(x)D_x^d + \cdots + p_1(x)D_x + p_0(x)$$

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$$\left(q_{d-1}(x)D_x^{d-1} + \cdots + q_1(x)D_x + q_0(x)\right) f(x) = 0$$

- $r(x) \int \frac{u_1(x)}{r(x)} dx, \ldots, r(x) \int \frac{u_k(x)}{r(x)} dx, r(x)$ are solutions of (3)

Solving differential equations: Kovacic's algorithm [Kovacic, 1986]

Consider the linear differential equation ($p_i(x) \in \mathbb{C}[x]$)

$$p_2(x)f''(x) + p_1(x)f'(x) + p_0(x)f(x) = 0. \quad (4)$$

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- all solutions are algebraic over $\mathbb{C}(x)$ and the previous cases do not hold;

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- has no Liouvillian solution

and finds the solutions if they exist.

Kovacic's algorithm is implemented in `HarmonicSums`.

Continuous multivariate AZ-algorithm.

We use the continuous Almkvist Zeilberger algorithm to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(x; y_1, \ldots, y_d)$ is of the form

$$F(x; x_1, \ldots, x_d) = q(x; x_1, \ldots, x_d) \cdot e^{\frac{a(x, x_1, \ldots, x_d)}{b(x, x_1, \ldots, x_d)}} \cdot \prod_{p=1}^{P} S_p(x, x_1, \ldots, x_d)^{\alpha_p},$$

with

- $a(x, x_1, \ldots, x_d), q(x; x_1, \ldots, x_d) \in \mathbb{K}[x, x_1, \ldots, x_d]$;
- $b(x, x_1, \ldots, x_d) \in \mathbb{K}[x, x_1, \ldots, x_d] \setminus \{0\}$;
- $S_p(x, x_1, \ldots, x_d) \in \mathbb{K}[x, x_1, \ldots, x_d]$ for $1 \leq p \leq P$;
- α_p commuting indeterminates for $1 \leq p \leq P$;
- With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

Continuous multivariate AZ-algorithm [Apagodu and Zeilberger, 2006]

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(x; x_1, \ldots, x_d)$ is a hyperexponential function is:

- compute a linear differential equation for the integrand $F(x; x_1, \ldots, x_d)$

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- use the differential equation for the integrand to derive a differential equation for the integral
- compute initial values for the integral
- solve the differential equation

Differential equation for the Integrand

$$F(x; x_1, \dots, x_d) = q(x; x_1, \dots, x_d) H(x; x_1, \dots, x_d),$$

where

$$H(x; x_1, \dots, x_d) = e^{\frac{a(x, x_1, \dots, x_d)}{b(x, x_1, \dots, x_d)}} \cdot \prod_{p=1}^{P} S_p(x, x_1, \dots, x_d)^{\alpha_p}$$

- $a(x, x_1, \dots, x_d), b(x, x_1, \dots, x_d), q(x; x_1, \dots, x_d) \in \mathbb{K}[x, x_1, \dots, x_d]$;
- $S_p(x, x_1, \dots, x_d) \in \mathbb{K}[x, x_1, \dots, x_d]$ for $1 \leq p \leq P$;
- α_p commuting indeterminates for $1 \leq p \leq P$;

$$F(x;x_1,\dots,x_d) = q(x;x_1,\dots,x_d)H(x;x_1,\dots,x_d),$$

where

$$H(x;x_1,\dots,x_d) = e^{\frac{a(x,x_1,\dots,x_d)}{b(x,x_1,\dots,x_d)}} \cdot \prod_{p=1}^{P} S_p(x,x_1,\dots,x_d)^{\alpha_p}$$

Then there exist

- $L \in \mathbb{N}$,
- $e_0(x), e_1(x), \dots, e_L(x) \in \mathbb{K}[x]$, *not all zero,*
- $R_i(x;x_1,\dots,x_d) \in \mathbb{K}(x,x_1,\dots,x_d)$,

such that

$$G_i(x;x_1,\dots,x_d) := R_i(x;x_1,\dots,x_d)F(x;x_1,\dots,x_d)$$

satisfy

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x;x_1,\dots,x_d) = \sum_{i=1}^{d} D_{x_i} G_i(x;x_1,\dots,x_d).$$

$$\overline{H}(x;x_1,\dots,x_d)=\frac{e^{\frac{a(x,x_1,\dots,x_d)}{b(x,x_1,\dots,x_d)}}}{b(x,x_1,\dots,x_d)^{2L}}\cdot\prod_{p=1}^{P}S_p(x,x_1,\dots,x_d)^{\alpha_p},0\leq L\in\mathbb{N}$$

Then we have

$$\sum_{i=0}^{L}e_i(x){D_x}^iF(x;x_1,\dots,x_d)=h(x,x_1,\dots,x_d)\overline{H}(x;x_1,\dots,x_d).$$

for some polynomial $h(x,x_1,\dots,x_d)$, that can be determined.

Differential equation for the Integrand

$$\overline{H}(x;x_1,\ldots,x_d) = \frac{e^{\frac{a(x,x_1,\ldots,x_d)}{b(x,x_1,\ldots,x_d)}}}{b(x,x_1,\ldots,x_d)^{2L}} \cdot \prod_{p=1}^{P} S_p(x,x_1,\ldots,x_d)^{\alpha_p}, 0 \le L \in \mathbb{N}$$

From the logarithmic derivatives

$$\frac{D_{x_i}\overline{H}(x;x_1,\ldots,x_d)}{\overline{H}(x;x_1,\ldots,x_d)} = \frac{q_i(x,x_1,\ldots,x_d)}{r_i(x,x_1,\ldots,x_d)}$$

we build the ansatz (where $X_i(x_1,\ldots,x_d)$ are polynomials to be determined)

$$G_i(x;x_1,\ldots,x_d) = \overline{H}(x;x_1,\ldots,x_d) \cdot r_i(x,x_1,\ldots,x_d) \cdot X_i(x_1,\ldots,x_d).$$

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x;x_1,\ldots,x_d) = \sum_{i=1}^{d} D_{x_i} G_i(x;x_1,\ldots,x_d)$$

is equivalent to

$$\sum_{i=1}^{d} [D_{x_i} r_i(x,x_1,\ldots,x_d) + q_i(x,x_1,\ldots,x_d)] \cdot X_i(x_1,\ldots,x_d)$$
$$+ r_i(x,x_1,\ldots,x_d) \cdot D_{x_i} X_i(x_1,\ldots,x_d) = h(x,x_1,\ldots,x_d).$$

Differential equation for the Integrand

Ansatz:

$$\sum_{i=1}^{d} [D_{x_i} r_i(x_1,\ldots,x_d) + q_i(x_1,\ldots,x_d)] \cdot X_i(x_1,\ldots,x_d)$$
$$+ r_i(x_1,\ldots,x_d) \cdot D_{x_i} X_i(x_1,\ldots,x_d) = h(x_1,\ldots,x_d) \qquad (5)$$

Algorithm:

set $L = 0$

1. look for degree bounds for $X_i(x_1,\ldots,x_d)$
2. try to find a solution of (5) by coefficient comparison
3. **if** there is no solution with not all $e_i(x)$'s equal to zero:
   increase L by one and go to 1
   else:
   return $\sum_{i=0}^{L} e_i(x) D_x^i$ and $\underbrace{\overline{H}(x; x_1,\ldots,x_d) r_i(x_1,\ldots,x_d) X_i(x_1,\ldots,x_d)}_{G_i(x;x_1,\ldots,x_d)}$

Since according to [Apagodu and Zeilberger, 2006] the existence of a solution of (5) with not all the $e_i(x)$'s equal to zero is guaranteed for sufficiently large L, this process will eventually terminate.

Differential equation for the Integral 1

We now consider the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

Where for $F(n; x_1, \ldots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(x; x_1, \ldots, x_d).$$

Differential equation for the Integral 1

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If

$$F(x; \dots, x_{i-1}, u_i, x_{i+1}, \dots) = F(x; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0,$$

we also have

$$G_i(x; \dots, x_{i-1}, u_i, x_{i+1}, \dots) = G_i(x; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0$$

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$$G_i(x; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) = G_i(x; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0$$

and hence $\mathcal{I}(x)$ satisfies the homogenous linear differential equation

$$\sum_{i=0}^{L} e_i(x) D_x^i \mathcal{I}(x) = 0.$$

Consider the integral

$$\mathcal{I}(x) := \int_{-1}^{1}\int_{-1}^{1} e^{-x\ x_1 x_2}\left(x_1^2-1\right)\left(x_2^2-1\right) dx_2 dx_1.$$

We find that the integrand satisfies the following differential equation

$$\begin{aligned} -x\, f(x) + \left(16-x^2\right)\ f'(x) + 9x\, f''(x) + x^2\, f'''(x) = \\ D_{x_1}\left(e^{-xx_1x_2}\left(x_1^2-1\right)^2 x_2(xx_1x_2-4)\left(+x_2^2-1\right)\right) \\ D_{x_2}\left(e^{-xx_1x_2} x\left(x_1^2-1\right) x_2\left(x_2^2-1\right)^2\right) \end{aligned}$$

Since the integrand vanishes at the integration bounds $\mathcal{I}(x)$ satisfies

$$-x\,\mathcal{I}(x) + \left(16-x^2\right)\ \mathcal{I}'(x) + 9x\,\mathcal{I}''(x) + x^2\,\mathcal{I}'''(x) = 0.$$

Solving and comparing initial values leads to

$$\begin{aligned} \mathcal{I}(x) =& \frac{2+x^2}{x^3} - \frac{e^{-x}\left(-2+2e^x+2e^{2x}+2x+2e^{2x}x+e^x x^2\right)}{x^3} \\ & - \frac{2\left(2+x^2\right) \mathrm{G}\left(\frac{e^{-\tau}}{\tau}, x\right)}{x^3} + \frac{2\left(2+x^2\right) \mathrm{G}\left(\frac{e^{\tau}}{\tau}, x\right)}{x^3} \end{aligned}$$

In[16]:= **cmAZIntegrate[$e^{-x x_1 x_2}(x_1^2-1)(x_2^2-1)$, x, {{$x_1$, -1, 1}, {$x_2$, -1, 1}}]**

Out[16]= $$\frac{2+x^2}{x^3}-\frac{e^{-x}\left(-2+2e^x+2e^{2x}+2x+2e^{2x}x+e^x x^2\right)}{x^3}-\frac{2\left(2+x^2\right)\mathrm{G}\left(\frac{e^{-\tau}}{\tau},x\right)}{x^3}+\frac{2\left(2+x^2\right)\mathrm{G}\left(\frac{e^{\tau}}{\tau},x\right)}{x^3}$$

Differential equation for the Integral 2

We again consider the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(x; x_1, \ldots, x_d)$ does not vanish at the bounds

We again consider the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(x; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(x; x_1, \ldots, x_d)$ does not have to vanish at the bounds

Differential equation for the Integral 2

We again consider the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(x; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(x; x_1, \ldots, x_d)$ does not have to vanish at the bounds
- then force the G_i to vanish at the integration bounds by modifying the ansatz, and look for G_i of the form

$$G_i(x; x_1, \ldots, x_d) = \overline{G}_i(x; x_1, \ldots, x_d)(x_i - u_i)(x_i - o_i),$$

Differential equation for the Integral 2

We again consider the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(x; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(x; x_1, \ldots, x_d)$ does not have to vanish at the bounds
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$$G_i(x; x_1, \ldots, x_d) = \overline{G}_i(x; x_1, \ldots, x_d)(x_i - u_i)(x_i - o_i),$$

- hence $\mathcal{I}(x)$ satisfies again a homogenous linear differential equation of the form

$$\sum_{i=0}^{L} e_i(x) D_x^i \mathcal{I}(x) = 0.$$

Consider the integral

$$\mathcal{I}(x) := \int_{-1}^{1}\int_{-1}^{1} e^{-x\,x_1 x_2}\, dx_2 dx_1.$$

We find that the integrand satisfies the following differential equation

$$\begin{aligned}-x\,f(x) - \left(-4+x^2\right)\,f'(x) + 5x\,f''(x) + x^2\,f'''(x) = \\ D_{x_1}\left(e^{-xx_1x_2}\,(-1+x_1)\,(1+x_1)\,x_2\,(-2+xx_1x_2)\right) \\ + D_{x_2}\left(e^{-xx_1x_2}\,x\,(-1+x_2)\,x_2\,(1+x_2)\right)\end{aligned}$$

After integrating we find

$$-x\,\mathcal{I}(x) - \left(-4+x^2\right)\,\mathcal{I}'(x) + 5x\,\mathcal{I}''(x) + x^2\,\mathcal{I}'''(x) = 0.$$

Solving and comparing initial values leads to

$$\mathcal{I}(x) = -\frac{2\left(\mathrm{G}\left(\frac{e^{-\tau}}{\tau};x\right) - \mathrm{G}\left(\frac{e^{\tau}}{\tau};x\right)\right)}{x}$$

In[17]:= **cmAZDirectIntegrate[$e^{-x x_1 x_2}$, x, {{x_1, -1, 1}, {x_2, -1, 1}}]**

Out[17]= $$-\frac{2\left(G\left(\frac{e^{-\tau}}{\tau};x\right)-G\left(\frac{e^{\tau}}{\tau};x\right)\right)}{x}$$

We look again at the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(x; x_1, \ldots, x_d)$$

We look again at the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

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By integration with respect to $x_1, \ldots, x_d$ we get

$$\sum_{i=0}^{L} e_i(x) D_x^i \mathcal{I}(x) =$$

$$\sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(x) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

$$- \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(x) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

with

$$O_i(x) := G_i(x; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d)$$
$$U_i(x) := G_i(x; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d).$$

Differential equation for the Integral 3

We look again at the integral

$$\mathcal{I}(x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(x) D_x^i F(x; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(x; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\sum_{i=0}^{L} e_i(x) D_x^i \mathcal{I}(x) =$$

$$\sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(x) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

$$- \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(x) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

with

$$O_i(x) := G_i(x; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d)$$
$$U_i(x) := G_i(x; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d).$$

Note that there are $2 \cdot d$ integrals of dimension $d - 1$, to compute.

In[18]:= **cmAZIntegrate[$e^{-x(w_1*w_2+w_3*w_4)}$, x, {{w_1, -1, 1}, {w_2, -1, 1},{w_3, -1, 1}, {w_4, -1, 1}}]**

Out[18]= $$\frac{8\left(G\left(\frac{e^{-\tau}}{\tau},\frac{e^{-\tau}}{\tau};x\right)-G\left(\frac{e^{-\tau}}{\tau},\frac{e^{\tau}}{\tau};x\right)-G\left(\frac{e^{\tau}}{\tau},\frac{e^{-\tau}}{\tau};x\right)+G\left(\frac{e^{\tau}}{\tau},\frac{e^{\tau}}{\tau};x\right)\right)}{x^2}$$

Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon, x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- we look again at the integral

$$\mathcal{I}(\varepsilon, x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d .$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, x) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(x).$$

Laurent Series Expansion of the Integral

- we look again at the integral

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- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, x) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(x).$$

- find $I_{-K}(x), I_{-K+1}(x), \ldots, I_u(x)$ in terms of iterated integral expressions.

- we look again at the integral

$$\mathcal{I}(\varepsilon, x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, x) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(x).$$

- find $I_{-K}(x), I_{-K+1}(x), \ldots, I_u(x)$ in terms of iterated integral expressions.
- compute a differential equation for $\mathcal{I}(\varepsilon, x)$ in the form

$$a_0(\varepsilon, x) T(\varepsilon, x) + \cdots + a_d(\varepsilon, x) D_x^d T(\varepsilon, x) = h_0(x) + \cdots + h_u(x)\varepsilon^u + O(\varepsilon^{u+1});$$

  by using one of the methods presented above

Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon,x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

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$$\mathcal{I}(\varepsilon,x) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(x).$$

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- compute a differential equation for $\mathcal{I}(\varepsilon,x)$ in the form

$$a_0(\varepsilon,x)T(\varepsilon,x) + \cdots + a_d(\varepsilon,x)D_x^d T(\varepsilon,x) = h_0(x) + \cdots + h_u(x)\varepsilon^u + O(\varepsilon^{u+1});$$

  by using one of the methods presented above
- compute initial values

Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon, x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, x) = \sum_{k=-K}^{\infty} \varepsilon^k I_k(x).$$

- find $I_{-K}(x), I_{-K+1}(x), \ldots, I_u(x)$ in terms of iterated integral expressions.
- compute a differential equation for $\mathcal{I}(\varepsilon, x)$ in the form

$$a_0(\varepsilon, x) T(\varepsilon, x) + \cdots + a_d(\varepsilon, x) D_x^d T(\varepsilon, x) = h_0(x) + \cdots + h_u(x)\varepsilon^u + O(\varepsilon^{u+1});$$

  by using one of the methods presented above
- compute initial values
- use the following algorithm to compute a Laurent series expansion of the solution of the differential equation

$$
\begin{aligned}
&a_0(\varepsilon,x)\Big[I(\varepsilon,x)\Big] \\
&+ a_1(\varepsilon,x)\Big[D_x I(\varepsilon,x)\Big] \\
&+ \\
&\vdots \\
&+ a_d(\varepsilon,x)\Big[D_x^d I(\varepsilon,x)\Big] \\
&\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^0 + h_1(x)\varepsilon^1 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon, x)\left[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^{0} + \cdots\right] \\
& + a_1(\varepsilon, x)\left[D_x I(\varepsilon, x)\right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\left[D_x^d I(\varepsilon, x)\right] \\
& \qquad\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^{0} + h_1(x)\varepsilon^{1} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon, x)\Big[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + \cdots\Big] \\
& + a_1(\varepsilon, x)\Big[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + \cdots\Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\Big[D_x^d I(\varepsilon, x)\Big] \\
& \qquad\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^0 + h_1(x)\varepsilon^1 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon, x)\Big[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + \cdots\Big] \\
& + a_1(\varepsilon, x)\Big[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + \cdots\Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\Big[D_x^d I_{-2}(x)\varepsilon^{-2} + D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + \cdots\Big] \\
& \qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^0 + h_1(x)\varepsilon^1 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon, x)\Big[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + \cdots\Big] \\
& + a_1(\varepsilon, x)\Big[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + \cdots\Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\Big[D_x^d I_{-2}(x)\varepsilon^{-2} + D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + \cdots\Big] \\
& \qquad\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^0 + h_1(x)\varepsilon^1 + \cdots
\end{aligned}
$$

$\Downarrow$ lowest order terms must agree

$$
a_0(0, x)I_{-2}(x) + a_1(0, x)D_x I_{-2}(x) + \cdots + a_d(0, x)D_x^d I_{-2}(x) = h_{-2}(x)
$$

$$
\begin{aligned}
& a_0(\varepsilon, x)\left[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^{0} + \cdots\right] \\
& + a_1(\varepsilon, x)\left[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^{0} + \cdots\right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\left[D_x^d I_{-2}(x)\varepsilon^{-2} + D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^{0} + \cdots\right] \\
& \qquad\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^{0} + h_1(x)\varepsilon^{1} + \cdots
\end{aligned}
$$

$$\Big\Downarrow \text{ lowest order terms must agree}$$

$$a_0(0, x) I_{-2}(x) + a_1(0, x) D_x I_{-2}(x) + \cdots + a_d(0, x) D_x^d I_{-2}(x) = h_{-2}(x)$$

Use the differential equation solver to determine $I_{-2}(x)$.

$$
\begin{aligned}
&a_0(\varepsilon,x)\Big[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^{0} + \cdots\Big]\\
&+ a_1(\varepsilon,x)\Big[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^{0} + \cdots\Big]\\
&+\\
&\vdots\\
&+ a_d(\varepsilon,x)\Big[D_x^d I_{-2}(x)\varepsilon^{-2} + D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^{0} + \cdots\Big]\\
&\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^{0} + h_1(x)\varepsilon^{1} + \cdots
\end{aligned}
$$

Plugging in $I_{-2}(x)$ in

$$a_0(\varepsilon,x)I_{-2}(x) + a_1(\varepsilon,x)D_x I_{-2}(x) + \cdots + a_d(\varepsilon,x)D_x^d I_{-2}(x)$$

yields

$$h_{-2}(x)\varepsilon^{-2} + g_{-1}(x)\varepsilon^{-2} + g_0(x)\varepsilon^{0} + g_1(x)\varepsilon^{1} + \cdots .$$

$$
\begin{aligned}
& a_0(\varepsilon,x)\Big[I_{-2}(x)\varepsilon^{-2} + I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^{0} + \cdots\Big] \\
& + a_1(\varepsilon,x)\Big[D_x I_{-2}(x)\varepsilon^{-2} + D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^{0} + \cdots\Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon,x)\Big[D_x^d I_{-2}(x)\varepsilon^{-2} + D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^{0} + \cdots\Big] \\
& \qquad\qquad = h_{-2}(x)\varepsilon^{-2} + h_{-1}(x)\varepsilon^{-1} + h_0(x)\varepsilon^{0} + h_1(x)\varepsilon^{1} + \cdots
\end{aligned}
$$

Plugging in $I_{-2}(x)$ in

$$a_0(\varepsilon,x)I_{-2}(x) + a_1(\varepsilon,x)D_x I_{-2}(x) + \cdots + a_d(\varepsilon,x)D_x^d I_{-2}(x)$$

yields

$$h_{-2}(x)\varepsilon^{-2} + g_{-1}(x)\varepsilon^{-2} + g_0(x)\varepsilon^{0} + g_1(x)\varepsilon^{1} + \cdots .$$

Now we can update the equation!

$$
\begin{aligned}
& a_0(\varepsilon,x)\left[I_{-1}(x)\varepsilon^{-1}+I_0(x)\varepsilon^{0}+I_1(x)\varepsilon^{1}+\cdots\right] \\
& +a_1(\varepsilon,x)\left[D_x I_{-1}(x)\varepsilon^{-1}+D_x I_0(x)\varepsilon^{0}+D_x I_1(x)\varepsilon^{1}+\cdots\right] \\
& + \\
& \vdots \\
& +a_d(\varepsilon,x)\left[D_x^d I_{-1}(x)\varepsilon^{-1}+D_x^d I_0(x)\varepsilon^{0}+D_x^d I_1(x)\varepsilon^{1}+\cdots\right] \\
& \qquad = \bar{h}_{-1}(x)\varepsilon^{-1}+\bar{h}_0(x)\varepsilon^{0}+\bar{h}_1(x)\varepsilon^{1}+\bar{h}_2(x)\varepsilon^{2}+\cdots
\end{aligned}
$$

Where $\bar{h}_i(x) = h_i(x) - g_i(x)$.

$$
\begin{aligned}
& a_0(\varepsilon, x)\left[I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + \cdots\right] \\
& + a_1(\varepsilon, x)\left[D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + \cdots\right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\left[D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + \cdots\right] \\
& \qquad = \bar{h}_{-1}(x)\varepsilon^{-1} + \bar{h}_0(x)\varepsilon^0 + \bar{h}_1(x)\varepsilon^1 + \bar{h}_2(x)\varepsilon^2 + \cdots
\end{aligned}
$$

Where $\bar{h}_i(x) = h_i(x) - g_i(x)$.

Repeat the process to compute $I_{-1}(x)$.

$$
\begin{aligned}
& a_0(\varepsilon, x)\Big[I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + \cdots\Big] \\
& + a_1(\varepsilon, x)\Big[D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + \cdots\Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\Big[D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + \cdots\Big] \\
& \qquad = \bar{h}_{-1}(x)\varepsilon^{-1} + \bar{h}_0(x)\varepsilon^0 + \bar{h}_1(x)\varepsilon^1 + \bar{h}_2(x)\varepsilon^2 + \cdots
\end{aligned}
$$

$\Downarrow$ lowest order terms must agree

$$
a_0(0, x) I_{-1}(x) + a_1(0, x) D_x I_{-1}(x) + \cdots + a_d(0, x) D_x^d I_{-1}(x) = \bar{h}_{-1}(x)
$$

$$
\begin{aligned}
& a_0(\varepsilon,x)\left[I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + \cdots\right] \\
& + a_1(\varepsilon,x)\left[D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + \cdots\right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon,x)\left[D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + \cdots\right] \\
& \qquad = \bar{h}_{-1}(x)\varepsilon^{-1} + \bar{h}_0(x)\varepsilon^0 + \bar{h}_1(x)\varepsilon^1 + \bar{h}_2(x)\varepsilon^2 + \cdots
\end{aligned}
$$

$\Downarrow$ lowest order terms must agree

$$
a_0(0,x)I_{-1}(x) + a_1(0,x)D_x I_{-1}(x) + \cdots + a_d(0,x)D_x^d I_{-1}(x) = \bar{h}_{-1}(x)
$$

Use the differential equation solver to determine $I_{-1}(x)$.

$$
\begin{aligned}
& a_0(\varepsilon,x)\left[I_{-1}(x)\varepsilon^{-1} + I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + \cdots\right] \\
& + a_1(\varepsilon,x)\left[D_x I_{-1}(x)\varepsilon^{-1} + D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + \cdots\right] \\
& + \\
& \vdots \\
& + a_d(\varepsilon,x)\left[D_x^d I_{-1}(x)\varepsilon^{-1} + D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + \cdots\right] \\
& \qquad\qquad = \bar{h}_{-1}(x)\varepsilon^{-1} + \bar{h}_0(x)\varepsilon^0 + \bar{h}_1(x)\varepsilon^1 + \bar{h}_2(x)\varepsilon^2 + \cdots
\end{aligned}
$$

Plugging in $I_{-1}(x)$ in

$$a_0(\varepsilon,x)I_{-1}(x) + a_1(\varepsilon,x)D_x I_{-1}(x) + \cdots + a_d(\varepsilon,x)D_x^d I_{-1}(x)$$

yields

$$\bar{h}_{-1}(x)\varepsilon^{-1} + \bar{g}_0(x)\varepsilon^0 + \bar{g}_1(x)\varepsilon^1 + \bar{g}_2(x)\varepsilon^2 + \cdots .$$

$$\begin{aligned}
&a_0(\varepsilon,x)\Big[I_{-1}(x)\varepsilon^{-1}+I_0(x)\varepsilon^0+I_1(x)\varepsilon^1+\cdots\Big]\\
&+a_1(\varepsilon,x)\Big[D_xI_{-1}(x)\varepsilon^{-1}+D_xI_0(x)\varepsilon^0+D_xI_1(x)\varepsilon^1+\cdots\Big]\\
&+\\
&\vdots\\
&+a_d(\varepsilon,x)\Big[D_x^dI_{-1}(x)\varepsilon^{-1}+D_x^dI_0(x)\varepsilon^0+D_x^dI_1(x)\varepsilon^1+\cdots\Big]\\
&\qquad=\bar{h}_{-1}(x)\varepsilon^{-1}+\bar{h}_0(x)\varepsilon^0+\bar{h}_1(x)\varepsilon^1+\bar{h}_2(x)\varepsilon^2+\cdots
\end{aligned}$$

Plugging in $I_{-1}(x)$ in

$$a_0(\varepsilon,x)I_{-1}(x)+a_1(\varepsilon,x)D_xI_{-1}(x)+\cdots+a_d(\varepsilon,x)D_x^dI_{-1}(x)$$

yields

$$\bar{h}_{-1}(x)\varepsilon^{-1}+\bar{g}_0(x)\varepsilon^0+\bar{g}_1(x)\varepsilon^1+\bar{g}_2(x)\varepsilon^2+\cdots.$$

Now we can update the equation!

$$a_0(\varepsilon, x)\left[I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + I_2(x)\varepsilon^2 \cdots\right]$$

$$+ a_1(\varepsilon, x) D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + D_x I_2(x)\varepsilon^2 + \cdots\Big]$$

$$+$$

$$\vdots$$

$$+ a_d(\varepsilon, x)\left[D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + D_x^d I_2(x)\varepsilon^2 + \cdots\right]$$

$$= \bar{\bar{h}}_0(x)\varepsilon^0 + \bar{\bar{h}}_1(x)\varepsilon^1 + \bar{\bar{h}}_2(x)\varepsilon^2 + \bar{\bar{h}}_3(x)\varepsilon^3 + \cdots$$

Where $\bar{\bar{h}}_i(x) = \bar{h}_i(x) - \bar{g}_i(x)$.

$$
\begin{aligned}
& a_0(\varepsilon, x)\left[I_0(x)\varepsilon^0 + I_1(x)\varepsilon^1 + I_2(x)\varepsilon^2 \cdots \right] \\
& + a_1(\varepsilon, x) D_x I_0(x)\varepsilon^0 + D_x I_1(x)\varepsilon^1 + D_x I_2(x)\varepsilon^2 + \cdots \Big] \\
& + \\
& \vdots \\
& + a_d(\varepsilon, x)\left[D_x^d I_0(x)\varepsilon^0 + D_x^d I_1(x)\varepsilon^1 + D_x^d I_2(x)\varepsilon^2 + \cdots \right] \\
& \qquad\qquad = \bar{\bar{h}}_0(x)\varepsilon^0 + \bar{\bar{h}}_1(x)\varepsilon^1 + \bar{\bar{h}}_2(x)\varepsilon^2 + \bar{\bar{h}}_3(x)\varepsilon^3 + \cdots
\end{aligned}
$$

Where $\bar{\bar{h}}_i(x) = \bar{h}_i(x) - \bar{g}_i(x)$.

Repeat the process to compute $I_0(x)$.

- we look at the integral

$$\mathcal{I}(\varepsilon, x) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(x; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- compute a differential equation for $\mathcal{I}(\varepsilon, x)$ in the form

$$a_0(\varepsilon, x) T(\varepsilon, x) + \cdots + a_d(\varepsilon, x) D_x^d T(\varepsilon, x) = h_0(x) + \cdots + h_u(x)\varepsilon^u + O(\varepsilon^{u+1});$$

  by using one of the methods presented above
- compute initial values
- use the method presented before to compute a Laurent series s.t.

$$\mathcal{I}(\varepsilon, x) = \sum_{k=-K}^{m} \varepsilon^k I_k(x) + O(\varepsilon^{m+1}).$$

Do you have integrals to try?

In[19]:= $f = e^{xyw}((1-w)x(1-y))^{\frac{\varepsilon}{2}}((1-w)y(1-x)z(1-z));$

In[20]:= $init =$

$$\frac{8}{3(2+\varepsilon)^2(4+\varepsilon)^2} - \frac{4w(28+\varepsilon(12+\varepsilon))}{3(2+\varepsilon)(4+\varepsilon)^2(6+\varepsilon)^2} + \frac{w^2(-1664+\varepsilon(12+\varepsilon(12+\varepsilon))(72+\varepsilon(16+\varepsilon)))}{3(2+\varepsilon)(4+\varepsilon)^2(6+\varepsilon)^2(8+\varepsilon)^2};$$

Looking for one homogenous differential equation:

In[21]:= **cmAZExpandedDirectIntegrate[f, w,{ε, 0, 1}, {{x, 0, 1}, {y, 0, 1}, {z, 0, 1}},InitValues → init]**

Out[21]=

$$\Bigg\{\Bigg\{\frac{1}{6} - \frac{1}{6w} - \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{6w^2} + \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{6w}, -\frac{1}{12w^2} + \frac{\varepsilon^w}{12w^2} + \frac{1}{12w} - \frac{\varepsilon^w}{12w} - \frac{1}{12}\mathrm{G}\left(\frac{1}{1-\tau}; w\right) + \frac{\mathrm{G}\left(\frac{1}{1-\tau}; w\right)}{12w} - \frac{1}{12}\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}; w\right) - \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{12w^2} + \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{6w} + \frac{\mathrm{G}\left(\frac{1}{1-\tau}, \frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{12w^2} - \frac{\mathrm{G}\left(\frac{1}{1-\tau}, \frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{12w} + \frac{\mathrm{G}\left(\frac{1}{\tau}, -\frac{1-\varepsilon^{-\tau}}{\tau}; w\right)}{12w^2} - \frac{\mathrm{G}\left(\frac{1}{\tau}, -\frac{1-\varepsilon^{-\tau}}{\tau}; w\right)}{12w} + \frac{\mathrm{G}\left(\frac{1}{\tau}, \frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{12w^2} - \frac{\mathrm{G}\left(\frac{1}{\tau}, \frac{1-\varepsilon^{\tau}}{\tau}; w\right)}{12w} + \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}, \frac{1}{1-\tau}; w\right)}{12w^2} - \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}, \frac{1}{1-\tau}; w\right)}{12w} - \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}, -\frac{1-\varepsilon^{-\tau}}{\tau}; w\right)}{12w^2} + \frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau}, -\frac{1-\varepsilon^{-\tau}}{\tau}; w\right)}{12w}\Bigg\}, \{0, 1\}\Bigg\}$$

Using the recursive method:

In[22]:= **cmAZExpandedIntegrate[f, w,{ε, 0, 1}, {{x, 0, 1}, {y, 0, 1}, {z, 0, 1}},Assumptions → 0 < ε < 1]**

Out[22]=

$$\left\{\left\{\frac{1}{6}-\frac{1}{6w}-\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{6w^2}+\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{6w},-\frac{1}{12w^2}+\frac{\varepsilon^{w}}{12w^2}+\frac{1}{12w}-\frac{\varepsilon^{w}}{12w}-\right.\right.$$

$$\frac{1}{12}\mathrm{G}\left(\frac{1}{1-\tau};w\right)+\frac{\mathrm{G}\left(\frac{1}{1-\tau};w\right)}{12w}-\frac{1}{12}\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau};w\right)-\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{12w^2}+\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{6w}+$$

$$\frac{\mathrm{G}\left(\frac{1}{1-\tau},\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{12w^2}-\frac{\mathrm{G}\left(\frac{1}{1-\tau},\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{12w}+\frac{\mathrm{G}\left(\frac{1}{\tau},-\frac{1-\varepsilon^{-\tau}}{\tau};w\right)}{12w^2}-\frac{\mathrm{G}\left(\frac{1}{\tau},-\frac{1-\varepsilon^{-\tau}}{\tau};w\right)}{12w}+$$

$$\frac{\mathrm{G}\left(\frac{1}{\tau},\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{12w^2}-\frac{\mathrm{G}\left(\frac{1}{\tau},\frac{1-\varepsilon^{\tau}}{\tau};w\right)}{12w}+\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau},\frac{1}{1-\tau};w\right)}{12w^2}-\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau},\frac{1}{1-\tau};w\right)}{12w}-$$

$$\left.\left.\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau},-\frac{1-\varepsilon^{-\tau}}{\tau};w\right)}{12w^2}+\frac{\mathrm{G}\left(\frac{1-\varepsilon^{\tau}}{\tau},-\frac{1-\varepsilon^{-\tau}}{\tau};w\right)}{12w}\right\},\{0,1\}\right\}$$

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