A geometric framework for amplitude recursions: bridging between trees and loops

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Wolfgang-Pauli-Center Workshop

Antidifferentiation and the calculation of Feynman Amplitudes
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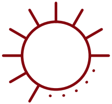
Introduction: Feynman integrals

Feynman rules:

lots of physics parameters, mathematically diffuse structure

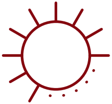


$$\rightarrow (\mu^2)^{\nu-\frac{\ell D}{2}} \int \prod_{r=1}^{\ell} \frac{\mathrm{d}^D k_r}{i\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{(-q_j{}^2 + m_j^2)^{\nu_j}}, \quad \nu = \sum_{j=1}^{n} \nu_j \,.$$

$\Rightarrow$ *complicated integral, formalize, replace D-dimensional integration*

Introduction: Feynman integrals

Feynman parametrization:

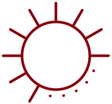


$$\rightarrow \int_{x_j \geq 0} \delta(1 - \sum_{j=1}^{n} x_j) \Big(\prod_{j=1}^{n} \mathrm{d}x_j \Big) \mathcal{I}(x_1, \ldots, x_n, D)$$

standard solution techniques:

- identify master integrals (IBP, partial fraction and more), write differential equation
  solve order by order in dimensional regularization parameter ε
- *most* Feynman integrals evaluate to (poly-)logarithms.
  some diagrams lead to *periods* beyond polylogarithms.
- beauty/brevity depends on choice of *letters, basis*
- (dimensional) regularization is a *feature* in solving the system

Introduction: (formal) Feynman integrals



$$\rightarrow \int_{x_j \geq 0} \delta\Big(1 - \sum_{j=1}^{n} x_j\Big)\Big(\prod_{j=1}^{n} \mathrm{d}x_j\Big)\,\mathcal{I}(x_1, \ldots, x_n, D)$$

This talk: structural/formalized approach

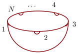
- *integration over simplex:* rewrite as *iterated integral.*
  (algebraic tools: shuffle, coproduct, functional relations)
- consider *singularity structure of integrand*
  (algebraic curve / Riemann surface (with boundaries) of suitable genus)
  (bounded orientable surface: keep notion of planar/non-planar)
- *recursive structure inherent*: employ canonical differential equations (KZ, KZB) to obtain solutions for generating functions at genus zero and one
- depart from actual integral, rather identify classes of integrals suitable for a particular singularity surface

⇒ ***„surface part" of string theory as a laboratory for quantum field theory.***

Outline

- **introduction**: from Feynman diagrams to period integrals
- **genus zero**: polylogs, multiple zeta values and the KZ equation
- **genus one**: elliptic iterated integrals, elliptic multiple zeta values and the KZB equation
- **more (toric) geometry**: multiple banana loops and the GKZ system
- **bonus**: geometrical interpretation of double-copy/single-valued projection?
- **outlook**

genus zero



iterated integrals:

$G(a_1,..,a_r;x) = \int_0^x \frac{\mathrm{d}t}{t-a_1} G(a_2,..,a_r;t)$

multiple zeta values: $\zeta(), \zeta^{\mathsf{sv}}()$

Selberg integrals:

$$\mathrm{S}[i_4,\ldots,i_L](x_1=0,x_2=1,x_3) =$$

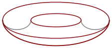
$$\int_{\mathcal{C}(x_3)} \prod_{i=4}^{L} \mathrm{d}x_i \prod_{k=4}^{L} \frac{1}{x_{ki_k}} \prod \exp(s_{ij}\log x_{ji})$$

formal recursion on $\mathcal{M}_{0,L}$*:*

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\begin{pmatrix}\mathrm{S}_1\\ \vdots \\ \mathrm{S}_r\end{pmatrix} = \left(\frac{e_0}{x_3} + \frac{e_1}{x_3-1}\right)\begin{pmatrix}\mathrm{S}_1\\ \vdots \\ \mathrm{S}_r\end{pmatrix}$$

$$\mathbf{C}_1 = \Phi(e_0,e_1)\,\mathbf{C}_0$$

genus one



elliptic iterated integrals:

$\tilde{\Gamma}\begin{pmatrix} n_1,..,a_r \\ a_1,..,a_r\end{pmatrix} = \int_0^z \mathrm{d}t\, g^{(n_1)}(t-a_1)\tilde{\Gamma}\begin{pmatrix} n_2,..,a_r \\ a_2,..,a_r\end{pmatrix}$

elliptic multiple zeta values: $\omega(), \omega^{\mathsf{sv}}()$

genus-one Selberg integrals:

$$\mathrm{S}^{\mathsf{E}}\left[\begin{matrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L\end{matrix}\right](z_1=0,z_2) =$$

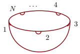
$$\int_{\mathcal{C}(z_2)} \prod_{i=3}^{L} \mathrm{d}z_i \prod_{k=3}^{L} g^{(n_k)}_{k,i_k} \prod \exp(s_{ij}\,\tilde{\Gamma}_{ji})$$

formal recursion on $\mathcal{M}_{1,L}$*:*

$$\frac{\mathrm{d}}{\mathrm{d}z_2}\begin{pmatrix}\mathrm{S}^{\mathsf{E}}_1\\ \vdots \\ \mathrm{S}^{\mathsf{E}}_r\end{pmatrix} = \left(\sum_n g^{(n)}(z_2)x^{(n)}\right)\begin{pmatrix}\mathrm{S}^{\mathsf{E}}_1\\ \vdots \\ \mathrm{S}^{\mathsf{E}}_r\end{pmatrix}$$

$$\mathbf{C}^{\mathsf{E}}_1 = \Phi^{\mathsf{E}}(x^{(n)})\;\mathbf{C}^{\mathsf{E}}_0$$

Iterated integrals



Integrate *rational functions* on the Riemann sphere. Obstruction: $\int \frac{\mathrm{d}x}{x} = \log x$

Multiple polylogarithms [...][Goncharov]

$$G(a_1, a_2, \ldots, a_r; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_r; t), \qquad G(;x) = 1\,.$$

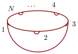
- multivalued functions: represent structure of unitary cuts.
- graded algebra, shuffle product/coproduct (Hopf algebra)
  $\to$ functional identities

Multiple zeta values: [...][Brown]

$$\zeta_{n_1, n_2, \ldots, n_r} = (-1)^r G(\underbrace{0, .., 1}_{n_r}, 0, 0, .., 1, \ldots, \underbrace{0, .., 1}_{n_1}; 1)$$

fast and stable numerical evaluation [Zagier et al. PARI][Weinzierl GINAC]

Selberg integrals



...generalize multiple polylogarithms:

- allow for more unintegrated values (fix symmetries) ($\mathcal{M}_{g,n}$)
- include Selberg seed in the empty integral
- expansion in $s_{ij} = \alpha'(p_i+p_j)^2$: multiple polylogarithms, polynomial coefficients

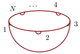
Selberg seed: $\log x_{ji} = G(0, x_{ji})$

$$\mathrm{S} = \mathrm{S}[](x_1, \ldots, x_L) = \prod_{0 \leq x_i < x_j \leq 1} \exp\left(s_{ij} \log x_{ji}\right), \qquad x_{ji} = x_j - x_i$$

Selberg integral: [Selberg]

$$\mathrm{S}[i_{k+1}, \ldots, i_L](x_1, \ldots, x_k) = \int_0^{x_k} \frac{\mathrm{d}x_{k+1}}{x_{k+1,i_{k+1}}} \mathrm{S}[i_{k+2}, \ldots, i_L](x_1, \ldots, x_{k+1}),$$

Recursion with extra marked point

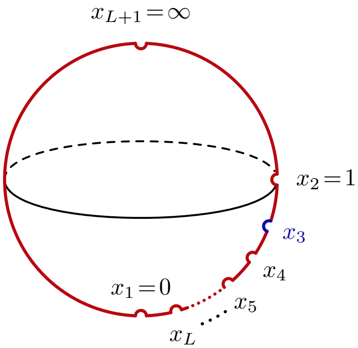


take Selberg integrals (fixed $\mathrm{SL}(2,\mathbb{C})$-symmetry)

$$\mathrm{S}[i_4,\ldots,i_L](x_1=0,x_2=1,x_3) =$$

$$\int_0^{x_3} \frac{\mathrm{d}x_4}{x_{4,i_4}}\,\mathrm{S}[i_5,\ldots,i_L](x_1=0,x_2=1,x_3,x_4)$$

with **extra marked point** x_3.



Consider derivative of a (basis) vector of Selberg integrals (length L).
Apply partial fraction/integration by parts to find: [Aomoto Terasoma]

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\mathbf{S}(x_3) = \left(\frac{e_0}{x_3}+\frac{e_1}{x_3-1}\right)\mathbf{S}(x_3)\,.$$

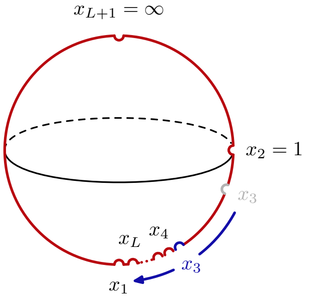
e_0, e_1: representations of generators of a (free) Lie algebra, matrices *linear* in parameters s_{ij}.

$\Rightarrow$ ***Knizhnik–Zamolodchikov equation*** [Knizhnik Zamolodchikov]

Knizhnik–Zamolodchikov equation and Drinfeld associator

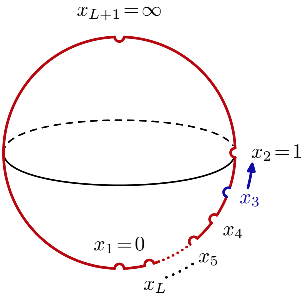
[Drinfeld]

$$\mathbf{C}_0 = \lim_{x_3 \to 0} x_3^{-e_0}\, \mathbf{S}(x_3)$$



contains
$(N-1)$-point amplitude

$$\mathbf{C}_1 = \lim_{x_3 \to 1} (1 - x_3)^{-e_1}\, \mathbf{S}(x_3)$$



contains
N-point amplitude

$$\boxed{\Phi(e_0, e_1)\; \mathbf{C}_0 = \mathbf{C}_1.}$$

Drinfeld associator: [Le, Murakami][Drinfeld]

generating series of multiple zeta values parametrized by algebra generators e_0, e_1.

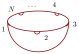
$$\Phi(e_0,e_1) = \sum_{w\in\{e_0,e_1\}^\times} w\,\zeta_w = 1 - \zeta_2[e_0,e_1] - \zeta_3[e_0+e_1,[e_0,e_1]] + \zeta_4\Big([e_1,[e_1,[e_1,e_0]]] + \tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]] - [e_0,[e_0,[e_0,e_1]]] + \tfrac{5}{4}[e_0,e_1]^2\Big) + \ldots \quad ,$$

⇒ ***all-order α'-expansion of open-string tree-level amplitudes***
„Parke–Taylor"-formula for open strings

[Drummond, Ragoucy][Broedel, Schlotterer, Stieberger, Terasoma]

⇒ ***„surface" implementation of Berends-Giele recursion,***
BG-combinatorics hidden in matrix representation

genus zero



iterated integrals:

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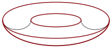

elliptic iterated integrals:

$$\tilde\Gamma\left(\begin{smallmatrix}n_1,..,a_r\\ a_1,..,a_r\end{smallmatrix}\right) = \int_0^z \mathrm{d}t\, g^{(n_1)}(t-a_1)\tilde\Gamma\left(\begin{smallmatrix}n_2,..,a_r\\ a_2,..,a_r\end{smallmatrix}\right)$$

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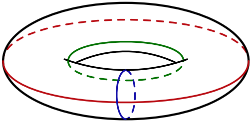
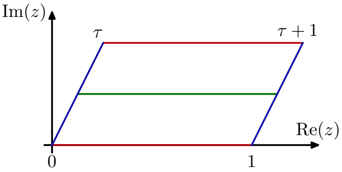
$$\mathbf{C}^{\mathsf{E}}_1 = \Phi^{\mathsf{E}}(x^{(n)})\;\mathbf{C}^{\mathsf{E}}_0$$

Elliptic iterated integrals





Canonical generating series for differential forms (inverse prime form): [Enriquez][Brown Levin][Levin]

$$\eta F(z,\eta,\tau)\ \mathrm{d}z = \sum_{n=0}^{\infty} \eta^n g^{(n)}(z,\tau)\ \mathrm{d}z$$

Elliptic iterated integrals: [Brown Levin]

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 \dots n_r \\ a_1 \dots a_r \end{smallmatrix}; z,\tau\right) = \int_0^z \mathrm{d}t\, g^{(n_1)}(t-a_1,\tau)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 \dots n_r \\ a_2 \dots a_r \end{smallmatrix}; t,\tau\right), \quad \widetilde{\Gamma}(\ ; z,\tau) = 1$$

Elliptic multiple zeta values:

$$\omega(n_r,\dots,n_1;\tau) = \widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ 0 & \dots & 0 \end{smallmatrix}; 1,\tau\right)$$

genus-one Selberg integrals

repeat generalization:

- add genus-one Selberg seed in „empty" integral

genus-one Selberg seed:

$$\mathrm{S}^{\mathsf{E}}(\tau) = \mathrm{S}^{\mathsf{E}}\left[\,\right](z_1,\ldots,z_N,\tau) = \prod_{0=z_1\leq z_i<z_j\leq z_2} \exp\left(s_{ij}\widetilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};z_{ji},\tau\right)\right).$$

$$z_{ji} = z_j - z_i$$

genus-one Selberg integral: [Broedel Kaderli]

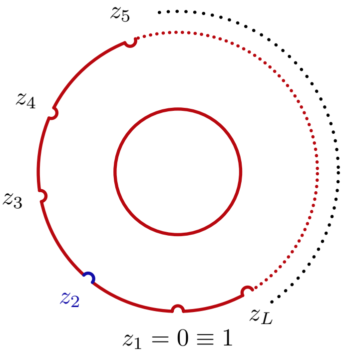
$$\mathrm{S}^{\mathsf{E}}\left[\begin{smallmatrix} n_{k+1}, & \ldots, & n_L \\ i_{k+1}, & \ldots, & i_L \end{smallmatrix}\right](z_1,\ldots,z_k) = \int_0^{z_k} \mathrm{d}z_{k+1}\, g^{(n_{k+1})}_{k+1,i_{k+1}}\, \mathrm{S}^{\mathsf{E}}\left[\begin{smallmatrix} n_{k+2}, & \ldots, & n_L \\ i_{k+2}, & \ldots, & i_L \end{smallmatrix}\right](z_1,\ldots,z_{k+1}),$$

Recursion with an extra marked point

start with Selberg integrals (fixed torus symmetry)

$$\mathrm{S}^{\mathsf{E}}\left[\begin{smallmatrix} n_3, & \ldots, & n_L \\ i_3, & \ldots, & i_L \end{smallmatrix}\right](z_1 = 0, z_2)$$

$$= \int_0^{z_2} \mathrm{d}z_3\, g_{3,i_3}^{(n_3)}\, \mathrm{S}^{\mathsf{E}}\left[\begin{smallmatrix} n_4, & \ldots, & n_L \\ i_4, & \ldots, & i_L \end{smallmatrix}\right](z_1 = 0, z_2, z_3)$$



with **extra marked point** z_2

consider derivative of an (infinite) vector of generalized Selberg integrals :

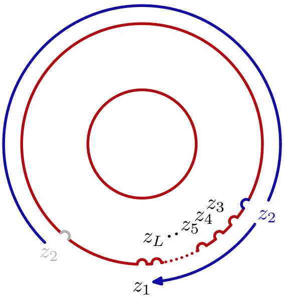
$$\frac{\mathrm{d}}{\mathrm{d}z_2}\,\mathbf{S}^{\mathsf{E}}(z_2) = \sum_{n\geq 0} g_{21}^{(n)} x^{(n)}\, \mathbf{S}^{\mathsf{E}}(z_2)\,,$$

$x^{(n)}$: representations of generators of a (free) Lie algebra, matrices *linear* in parameters s_{ij}.

$\Rightarrow$ ***elliptic KZB equation*** [Bernard, Knizhnik Zamolodchikov]

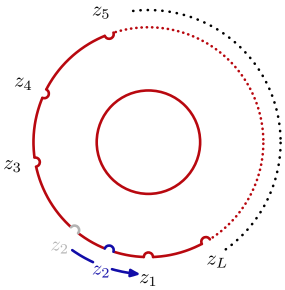
Elliptic KZB equation and the KZB associator

$$\mathbf{C}_0^{\mathsf{E}} = \lim_{z_2 \to 0} (2\pi i z_2)^{-x^{(1)}} \, \mathbf{S}^{\mathsf{E}}(z_2)$$



contains
$(N+2)$-point open-string
genus-zero integral

$$\mathbf{C}_1^{\mathsf{E}} = \lim_{z_2 \to 1} (2\pi i (1-z_2))^{-x^{(1)}} \, \mathbf{S}^{\mathsf{E}}(z_2) \, .$$



contains
N-point open-string
genus-one integral

$$\boxed{\Phi^{\mathsf{E}}(x^{(0)}, x^{(1)}, \ldots) \; \mathbf{C}_0^{\mathsf{E}} = \mathbf{C}_1^{\mathsf{E}}}$$

KZB associator:

generating function for elliptic multiple zeta values

$$\begin{aligned}\Phi^{\mathsf{E}}(x^{(n)},\tau) &= \sum_{w\in X} w\,\omega(w^t;\tau)\\ &= 1 + x^{(0)} - 2\zeta_2 x^{(2)}\\ &\quad + \frac{1}{2}x^{(0)}x^{(0)} - [x^{(0)},x^{(1)}]\,\omega(0,1;\tau) - \zeta_2\{x^{(0)},x^{(2)}\}\\ &\quad + [x^{(1)},x^{(2)}]\big(\omega(0,3;\tau) - 2\zeta_2\omega(0,1;\tau)\big) + 5\zeta_4 x^{(2)}x^{(2)} + \cdots\end{aligned}$$

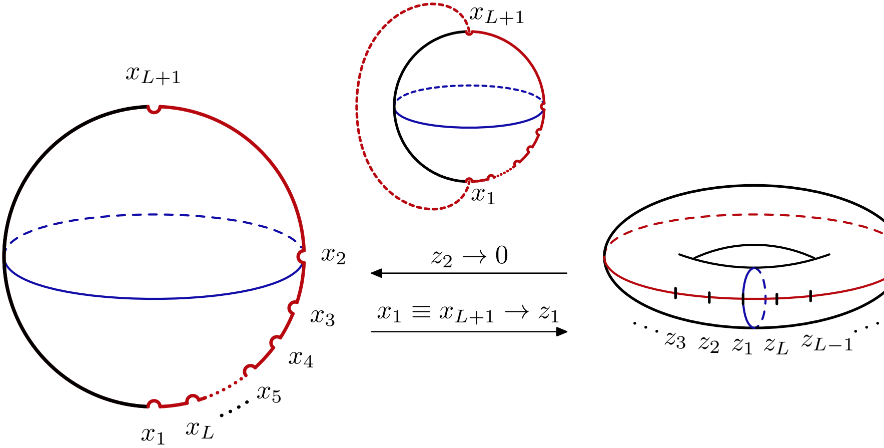
$\Rightarrow$ ***all-order in α' open-string one-loop scattering amplitudes***

[Broedel Kaderli]

$\Rightarrow$ ***surface implementation of***
„cutting a one-loop graph in all possible ways"

complements/explains empiric recursion for one-loop open strings [Mafra Schlotterer]

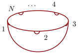
Geometric interpretation



- fundamental domain of torus is squeezed ($\tau \to i\infty$) in the limit $z_2 \to 0$
- red cycle appears to be infinitely long,
  red cycle is cut open to create two points: 0 and ∞ on Riemann sphere

$\Rightarrow$ ***representation of torus degeneration in terms of Selberg integrals***

genus zero



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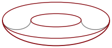
genus one



elliptic iterated integrals:

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$$\mathbf{C}_1^{\mathsf{E}} = \Phi^{\mathsf{E}}\left(x^{(n)}\right)\,\mathbf{C}_0^{\mathsf{E}}$$

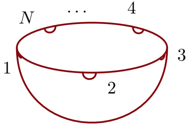
Geometric interpretation of the single-valued projection?

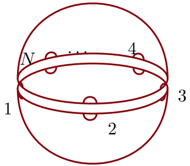
$\Rightarrow$ degeneration of surfaces leads to recursive structure for iterated integrals on surfaces with boundary. *What about sewing?*

genus-zero KLT relations: combinatorical sum over $(N-3)!$ terms:
\# of ways to traverse $N-3$ points on the Riemann sphere.

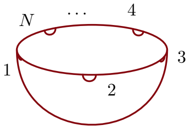
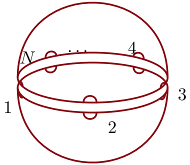
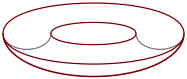
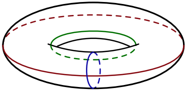
single-valued projection at genus zero [Kawai, Lewellen, Tye][Stieberger][Brown][Brown, Dupont]

	open	closed
genus zero	$\beta(s,t) = \int_0^1 \mathrm{d}x\, x^{s-1}(1-x)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$	$\beta_{\mathbb{C}}(s,t) = \frac{-1}{2\pi i}\int_{\mathbb{CP}^1} \mathrm{d}z\wedge\mathrm{d}\bar{z}\,\lvert z\rvert^{2(s-1)}\lvert 1-z\rvert^{2(t-1)} = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s-t)}{\Gamma(s+t)\Gamma(1-s)\Gamma(1-t)}$
	$\zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_2\zeta_3, \ldots, \zeta_{5,3}, \ldots$	$\zeta^{\mathrm{sv}}_{2k} = 0,\ \zeta^{\mathrm{sv}}_{2k+1} = 2\,\zeta_{2k+1}, \zeta^{\mathrm{sv}}_{5,3} = 14\,\zeta_3\,\zeta_5$





Generalization to genus one?

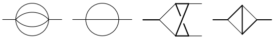
	open	**closed**
genus zero	 multiple zeta values	 single-valued zeta values
genus one	 elliptic multiple zeta values	 modular graph functions ?

GKZ system for multiloop banana graphs

- „Analytic structure of all loop banana amplitudes” [Bönisch, Fischbach Klemm, Nega, Safari]
  „The l-loop banana amp. from GKZ system and relative CY periods” [Klemm, Nega Safari]
- Gelfand, Kapranov, Zelevinsky: many integrals in QFT are residue integrals of rational functions on toric variety.
- for l-loop banana graphs:
  - can write down the particular toric variety,
  - use associated GKZ system,
  - identify Picard-Fuchs differential ideal (possible in this simple case)
  - and solve: obtain contribution to the maximal cut

Recommendation: Albrecht Klemms talk at *Elliptics and Beyond 2020*
`https://www.youtube.com/watch?v=0aoYkUSlfoMfeature=youtu.be`

Summary

- generalized iterated integrals and associated periods from genus zero to genus one: elliptic polylogarithms and elliptic multiple zeta values. [Lewin][Brown, Lewin]
  string laboratory: [Broedel, Mafra, Matthes, Schlotterer][Broedel, Kaderli]
  elliptic language for Feynman diagrams: [Broedel, Duhr, Dulat, Penante, Tancredi]
  Feynman diagrams:  [Bogner, Weinzierl][Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi]

- Selberg integrals: generating functions for period integrals/scattering amps. S-matrix theory for open-string tree-level and one-loop amplitudes available through KZ and KZB equation. [Drinfeld][Bernard, Knizhnik, Zamolodchikov][Aomoto, Terasoma][Broedel, Schlotterer, Stieberger, Terasoma][Broedel, Kaderli]

- any connection to *topological recursion*? [Borot, Eynard]

- recursion for multi-Regge amplitudes [Broedel, Sprenger][Broedel, Sprenger, Torres-Orjuela][del Duca, Druc, Duhr, Drummond, Dulat]

- geometric interpretation of double-copy/single-valued relation at genus zero and genus one [Work in progress]

Structure to be investigated in capable setup / string laboratory
To be applied in field theory context for phenomenologically interesting problems.

Thanks!