# Elliptic Integrals

## Iterated integrals related to Feynman integrals associated to elliptic curves

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Section 1

# Background from Mathematics

# Algebraic curves

- Ground field $\mathbb{C}$
- **Algebraic curve** in $\mathbb{C}^2$ **defined by** a **polynomial** $P(x,y)$:

$$P(x,y) = 0$$

- Projective space $\mathbb{CP}^2$ with homogeneous coordinates $[x:y:z]$:
  Algebraic curve in $\mathbb{CP}^2$ defined by a homogeneous polynomial $P(x,y,z)$:

$$P(x,y,z) = 0$$

  We usually work in the chart $z = 1$.

# Elliptic curves

## Definition (Elliptic curve over $\mathbb{C}$)

An algebraic curve in $\mathbb{CP}^2$ of genus one with one marked point.

## Example (Weierstrass normal form)

In the chart $z = 1$:

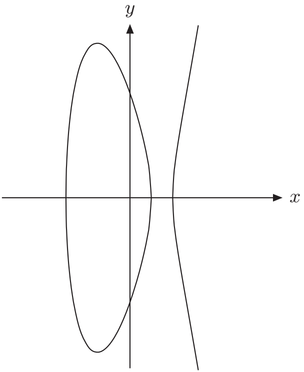
$$y^2 = 4x^3 - g_2 x - g_3$$

## Example (Quartic form)

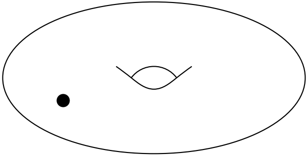
In the chart $z = 1$:

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

One complex dimension corresponds to two real dimensions.




Weierstrass normal form
$y^2 = 4x^3 - g_2 x - g_3$



Real Riemann surface of genus one with one marked point

# Periodic functions

Let us consider a non-constant meromorphic function $f$ of a complex variable $z$.

A **period** $\omega$ of the function $f$ is a constant such that for all $z$:

$$f(z+\omega) = f(z)$$

The set of all periods of $f$ forms a **lattice**, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Double periodic functions are called **elliptic functions**.

# Examples of periodic functions

- Singly periodic function: **Exponential function**

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: **Weierstrass's $\wp$-function**

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \backslash \{0\}} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{ n_1 \omega_1 + n_2 \omega_2 | n_1, n_2 \in \mathbb{Z} \},$$

$$\mathrm{Im}(\omega_2 / \omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods $\omega_1$ and $\omega_2$.

# Inverse functions

The corresponding inverse functions are in general **multivalued functions**.

- For the exponential function $x = \exp(z)$ the inverse function is the **logarithm**

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an **elliptic integral**

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

# Elliptic integrals

## Complete elliptic integrals

- First kind:

$$K(x) = \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(x) = \int\limits_0^1 dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

$$\Pi(v,x) = \int\limits_0^1 \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

## Incomplete elliptic integrals

- First kind:

$$F(z,x) = \int\limits_0^z \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

- Second kind:

$$E(z,x) = \int\limits_0^z dt \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}$$

- Third kind:

$$\Pi(v,z,x) = \int\limits_0^z \frac{dt}{(1-vt^2)\sqrt{(1-t^2)(1-x^2t^2)}}$$

# Abelian differentials

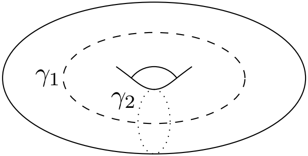
- Abelian differential of the first kind:
  **holomorphic**

- Abelian differential of the second kind:
  **meromorphic** with **all residues vanishing**

- Abelian differential of the third kind:
  **meromorphic** with **non-zero residues**

# Periods of an elliptic curve

**Integrate** the **holomorphic differential** along the two independent cycles.



# Periods of an elliptic curve

## Example

The Legendre form:

$$y^2 = x(x-1)(x-\lambda)$$

The periods are

$$\omega_1 = 2\int\limits_0^\lambda \frac{dx}{y} = 4K\left(\sqrt{\lambda}\right) \qquad \omega_2 = 2\int\limits_1^\lambda \frac{dx}{y} = 4iK\left(\sqrt{1-\lambda}\right)$$

# Picard-Fuchs operator

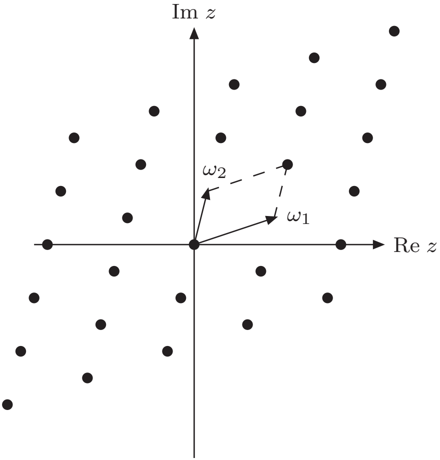
The elliptic curve $y^2 = x(x-1)(x-\lambda)$ depends on a parameter $\lambda$, and so do the periods $\omega_1(\lambda)$ and $\omega_2(\lambda)$.

How do the periods change, if we change $\lambda$?

The variation is governed by a second-order differential equation:
With $t = \sqrt{\lambda}$ we have

$$\underbrace{\left[ t\left(1-t^2\right)\frac{d^2}{dt^2} + \left(1-3t^2\right)\frac{d}{dt} - t \right]}_{\text{Picard-Fuchs operator}} \omega_j \quad = \quad 0$$

# Representing an elliptic curve as $\mathbb{C}/\Lambda$




Points inside fundamental parallelogram $\Leftrightarrow$ Points on elliptic curve

# Back and forth

- **Weierstrass normal form** $\rightarrow \mathbb{C}/\Lambda$:

  Given a point $(x, y)$ with $y^2 - 4x^3 + g_2 x + g_3 = 0$ the corresponding point $z \in \mathbb{C}/\Lambda$ is given by

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}$$

- $\mathbb{C}/\Lambda \rightarrow$ **Weierstrass normal form**:

  Given a point $z \in \mathbb{C}/\Lambda$ the corresponding point $(x, y)$ on $y^2 - 4x^3 + g_2 x + g_3 = 0$ is given by

$$(x, y) = \left(\wp(z), \wp'(z)\right)$$

# Notation

Convention: Normalise $(\omega_2, \omega_1) \to (\tau, 1)$, where

$$\tau = \frac{\omega_2}{\omega_1}$$

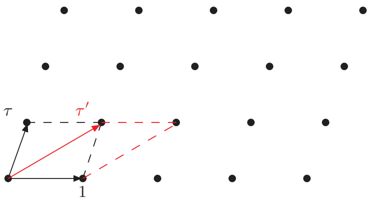
and require $\text{Im}(\tau) > 0$.

**Definition (The complex upper half-plane)**

$$\mathbb{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$$

# Modular transformations

The periods $\omega_1$ and $\omega_2$ generate a lattice. Any other basis as good as $(\omega_2, \omega_1)$.




Change of basis: $\left( \begin{array}{c} \omega_2' \\ \omega_1' \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} \omega_2 \\ \omega_1 \end{array} \right),$

Transformation should be invertible: $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}),$

In terms of $\tau$ and $\tau'$: $\tau' = \frac{a\tau + b}{c\tau + d}$

# Modular forms

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of modular weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$ if

1. $f$ transforms under modular transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

2. $f$ is holomorphic on $\mathbb{H}$,
3. $f$ is holomorphic at $i\infty$.

Define the $|_k\gamma$ operator by

$$(f|_k\gamma)(\tau) \quad = \quad (c\tau+d)^{-k} \cdot f(\gamma(\tau))$$

# Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence **subgroups**, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "**nice**" transformation properties only for subgroup $\Gamma$ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

# Modular forms

For a congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ denote by $\mathcal{M}_k(\Gamma)$ the **space of modular forms of weight $k$**.

We have the inclusions

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathcal{M}_k(\Gamma_0(N)) \subseteq \mathcal{M}_k(\Gamma_1(N)) \subseteq \mathcal{M}_k(\Gamma(N))$$

For $f \in \mathcal{M}_k(\Gamma(N))$:

$$\begin{aligned} f|_k\gamma &= f, & \gamma &\in \Gamma(N) \\ f|_k\gamma &\in \mathcal{M}_k(\Gamma(N)), & \gamma &\in \mathrm{SL}_2(\mathbb{Z})\backslash\Gamma(N) \end{aligned}$$

# Notation

For $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ set

$$\bar{q} = \exp\left(2\pi i \tau\right), \qquad \bar{w} = \exp\left(2\pi i z\right)$$

**Maps** the complex **upper half-plane** $\tau \in \mathbb{H}$ **to** the **unit disk** $|\bar{q}| < 1$.

Trivialises periodicity with period 1:

$$\bar{q}\left(\tau+1\right) = \bar{q}\left(\tau\right), \qquad \bar{w}\left(z+1\right) = \bar{w}\left(z\right)$$

Shifts with $\tau$ correspond to multiplication with $\bar{q}$:

$$\bar{q}\left(\tau+\tau\right) = \bar{q}\left(\tau\right) \cdot \bar{q}\left(\tau\right), \qquad \bar{w}\left(z+\tau\right) = \bar{w}\left(z\right) \cdot \bar{q}\left(\tau\right)$$

# Iterated integrals of modular forms

Let $f_1, \dots, f_n$ be modular forms.

$$I(f_1, f_2, ..., f_n; q) = (2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

As basepoint we usually take $\tau_0 = i\infty$.

An integral over a modular form is in general **not** a modular form.

Analogy: An integral over a rational function is in general not a rational function.

# Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a $\bar{q}$-expansion

$$f_k(\tau) = a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + \dots, \qquad \bar{q} = \exp(2\pi i \tau)$$

The transformation $\bar{q} = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $\bar{q} = 0$ and we have

$$2\pi i \, f_k(\tau) d\tau = \frac{d\bar{q}}{\bar{q}} \left( a_0 + a_1 \bar{q} + a_2 \bar{q}^2 + \dots \right).$$

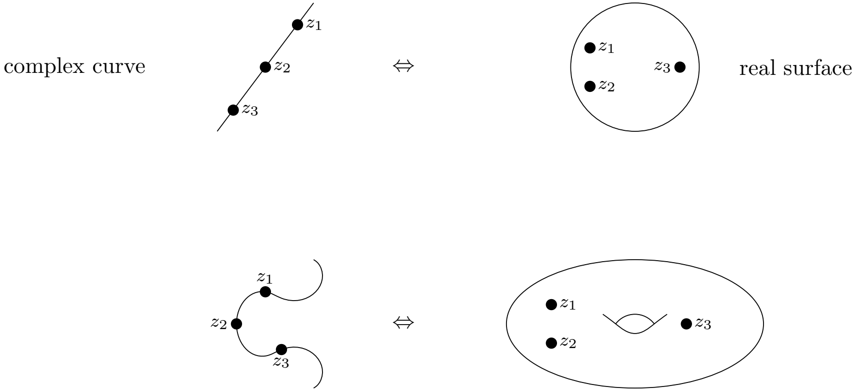
Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a **simple pole** at $\bar{q} = 0$.

Section 2

# Moduli spaces

# Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus $g$ with $n$ marked points**.



# Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \ldots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \ldots, \mathbf{z_{n-1}})$

# Iterated integrals

For $\omega_1$, ..., $\omega_k$ differential 1-forms on a manifold $M$ and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of $\omega_j$ to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \;=\; \gamma^* \omega_j.$$

The iterated integral is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) \;=\; \int\limits_0^\lambda d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Chen '77

# Iterated integrals on $\mathcal{M}_{0,n}$

We are interested in differential one-forms, which have only simple poles:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{dy}{y - z_j}.$$

**Multiple polylogarithms**:

$$G(z_1, \dots, z_k; y) = \int_0^y \frac{dy_1}{y_1 - z_1} \int_0^{y_1} \frac{dy_2}{y_2 - z_2} \dots \int_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}, \qquad z_k \neq 0$$

# Iterated integrals on $\mathcal{M}_{1,n}$

- Coordinates are $(\tau, z_1, ..., z_{n-1})$
- Decompose an arbitrary path along $d\tau$ and $dz_j$
- Two classes of iterated integrals:
  1. Integration along $z$
  2. Integration along $\tau$
- What are the differential one-forms we want to integrate?

# The Kronecker function

The **first Jacobi theta function** $\theta_1(z, q)$:

$$\theta_1(z, q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)z}, \qquad q = e^{i\pi\tau}$$

The **Kronecker function** $F(z, \alpha, \tau)$:

$$F(z, \alpha, \tau) = \pi\theta_1'(0, q) \frac{\theta_1(\pi(z+\alpha), q)}{\theta_1(\pi z, q)\, \theta_1(\pi\alpha, q)} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z}, \tau) \alpha^k$$

We are mainly interested in the coefficients $g^{(k)}(z, \tau)$ of the Kronecker function.

# The coefficients $g^{(k)}(z,\tau)$ of the Kronecker function

Properties of $g^{(k)}(z,\tau)$:

1. **only simple poles** as a function of $z$
2. **quasi-periodic** as a function of $z$: Periodic by 1, quasi-periodic by $\tau$.

$$
\begin{aligned}
g^{(k)}(z+1,\tau) &= g^{(k)}(z,\tau), \\
g^{(k)}(z+\tau,\tau) &= \sum_{j=0}^{k} \frac{(-2\pi i)^j}{j!} g^{(k-j)}(z,\tau)
\end{aligned}
$$

3. **almost modular**:

$$
g^{(k)}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \sum_{j=0}^{k} \frac{(2\pi i)^j}{j!} \left(\frac{cz}{c\tau+d}\right)^j g^{(k-j)}(z,\tau)
$$

# Iterated integrals on $\mathcal{M}_{1,n}$: Integration along $z$

Differential one-forms:

$$\omega^{\mathrm{Kronecker},z}_{k,K}(z_j,\tau) = (2\pi i)^{2-k} g^{(k-1)}(z-z_j, K\tau)\,dz$$

**Elliptic multiple polylogarithms**:

$$\tilde{\Gamma}\left( \begin{smallmatrix} n_1 & \dots & n_r \\ z_1 & \dots & z_r \end{smallmatrix} ; z; \tau \right) = (2\pi i)^{n_1+\dots+n_r-r} I\left( \omega^{\mathrm{Kronecker},z}_{n_1+1,1}(z_1,\tau), \dots, \omega^{\mathrm{Kronecker},z}_{n_r+1,1}(z_r,\tau); z \right)$$

Broedel, Duhr, Dulat, Tancredi, '17

- $\tau = \mathrm{const}$
- meromorphic version, only simple poles
- not double periodic!

# Iterated integrals on $\mathcal{M}_{1,n}$: Integration along $\tau$

Differential one-forms:

$$\omega^{\mathrm{Kronecker},\tau}_{k,K}(z_j) = (2\pi i)^{2-k} K(k-1)\, g^{(k)}(z_j, K\tau)\, \frac{d\tau}{2\pi i}$$
$$= \frac{K(k-1)}{(2\pi i)^k} g^{(k)}(z_j, K\tau)\, \frac{d\bar{q}}{\bar{q}}$$

- Integrate in $\bar{q}$
- No poles in $0 < |\bar{q}| < 1$.
- Possibly a simple pole at $\bar{q} = 0$ (“trailing zero”)

Section 3

# Physics

# Standard tools

- **Integration-by-parts identities**

  Tkachov '81, Chetyrkin '81

- the **method of differential equations**

  Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99

- **Laporta algorithm** and computer implementations

  Laporta '01,

  `REDUZE` von Manteuffel, Studerus '12,

  `FIRE` Smirnov '15,

  `KIRA` Maierhöfer, Usovitsch, Uwer ' 17

# Notation

$\mathbf{N_F} = N_{\text{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

$\mathbf{N_B} = N_{\text{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_{N_B})$.

$\mathbf{N_L} = N_{\text{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, ..., \omega_{N_L})$.

# Differential equations

**System of differential equations**

$$dI + AI = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form $A$ satisfies the integrability condition

$$dA + A \wedge A = 0 \qquad \text{(flat Gauß-Manin connection).}$$

**Computation of Feynman integrals reduced to solving differential equations**!

# Simple differential equations

The system of differential equations is particular simple, if $A$ is of the form

$$A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j,$$

where

- $C_j$ is a $N_F \times N_F$-matrix, whose entries are (rational or integer) numbers,
- the **only dependence on $\varepsilon$** is **given by** the **explicit prefactor**,
- the differential one-forms $\omega_j$ have **only simple poles**.

Henn '13

# Transformations

- **Change the basis of the master integrals**

$$I' = UI,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x'_i = f_i(x), \qquad 1 \le i \le N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x'_j} dx'_j.$$

# The equal-mass sunrise

It is **not possible** to obtain an $\varepsilon$-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.
However by factoring off the (**non-algebraic**) expression $\omega_1/\pi$ from the master integrals in the sunrise sector one obtains an $\varepsilon$-form:

$$I_1 = 4\varepsilon^2 S_{110}\left(2-2\varepsilon, x\right) \quad I_2 = -\varepsilon^2 \frac{\pi}{\omega_1} S_{111}\left(2-2\varepsilon, x\right) \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\omega_1^2}{\pi^2} I_2$$

If in addition one makes a (**non-algebraic**) change of variables from $x$ to $\tau$, one obtains

$$\frac{d}{d\tau} I = \varepsilon\, A(\tau)\, I,$$

where $A(\tau)$ is an $\varepsilon$-independent $3\times3$-matrix whose **entries are modular forms**.

# The unequal-mass sunrise

After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to $(\tau, z_1, z_2)$ one finds

$$\mathbf{A} \;=\; \varepsilon \sum_{j=1}^{N_L} C_j\, \omega_j, \qquad \textbf{with } \omega_j \textbf{ only simple poles},$$

where $\omega_j$ is either

$$2\pi i \, f_k(\tau)\, d\tau,$$

where $f_k(\tau)$ is a modular form, or of the form

$$\omega_k(z_i, K\tau) \;=\; (2\pi i)^{2-k} \left[ g^{(k-1)}(z_i, K\tau)\, dz_i + K(k-1)\, g^{(k)}(z_i, K\tau) \frac{d\tau}{2\pi i} \right]$$

# Comments

- It is advantageous to integrate in $\tau$:
  - Analytic expressions shorter
  - Easier to evaluate numerically
- Boundary condition at $\tau = i\infty$:
  - Elliptic curve degenerates, geometric genus equals zero
  - Feynman integrals expressible in terms of multiple polylogarithms

# Conclusions

- Feynman integrals important in many areas of physics.
- Feynman integrals evaluating to multiple polylogarithms related to iterated integrals on $\mathcal{M}_{0,n}$.
- There is a class of Feynman integrals related to **elliptic curves** from two loops onwards, evaluating to **iterated integrals on $\mathcal{M}_{1,n}$**.
- Computation of Feynman integrals is trivial, as soon as the system of differential equations is transformed to

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for
- many Feynman integrals evaluating to multiple polylogarithms
- a few non-trivial elliptic examples