Direct Integration for Multi-leg Amplitudes: Tips, Tricks, and When They Fail

[arXiv:1712.02785] with J. Bourjaily, A. Mcleod, M. Spradlin, and M. Wilhelm

[arXiv:1805.09326] with J. Bourjaily, Y.-H. He, A. Mcleod, and M. Wilhelm

[arXiv:1805.10281] with J. Bourjaily, A. Mcleod, and M. Wilhelm

[arXiv:1810.07689] with J. Bourjaily, A. Mcleod, and M. Wilhelm

[arXiv:1910.01534] with J. Bourjaily, A. Mcleod, C. Vergu, M. Volk, and M. Wilhelm

[arXiv:1910.14224] with J. Bourjaily, A. Mcleod, C. Vergu, M. Volk, and M. Wilhelm

[arXiv:1912.05690] with J. Bourjaily, M. Volk

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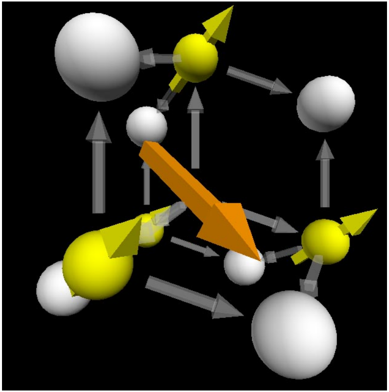
Disclaimer: Speaker is Supersymmetric



- I work with $\mathcal{N} = 4$ SYM
  - Thus multi-leg: beta function vanishes, amplitudes through five points known in the planar limit [see Papathanasiou's and Bartels's talks]
  - Integrals chosen UV finite, can IR regulate with masses to stay in four dimensions
  - Uniform transcendentality [see Henn's talk]
  - Nice integrals, *even as a basis for other theories*!

What I mean by Direct Integration

Direct hyperlogarithmic integration:

- Rewrite hyperlogarithms in the integrand so that the integration variable appears only in the argument, via a fibration basis
- Partial-fraction rational functions in the integrand in terms of the integration variable
- Up to integration by parts (and regularizing singularities), can then just apply the definition of the hyperlogarithm

$$G(w_1, w_2, \ldots; z) = \int_0^z \frac{1}{x - w_1} G(w_2, \ldots; x) dx$$

- Implemented in Erik Panzer's `HyperInt`, [Also see his talk]

What can go wrong: Algebraic Roots

- Both the fibration basis and partial-fractioning can introduce algebraic roots in the remaining integration variables
- This stops the algorithm: can't fiber or partial-fraction if integration variable is in an algebraic root - not linearly reducible
- In some cases, can find systematic change of variables to fix this [see Besier, Van Straten, Weinzierl, also Raab's talk]

$$\begin{aligned}
&\int_0^\infty \frac{d\alpha}{\alpha^2+2f\alpha+g} \\
&\quad = \int_0^\infty \frac{d\alpha}{2\sqrt{f^2-g}}\left(\frac{1}{\alpha+f-\sqrt{f^2-g}}-\frac{1}{\alpha+f+\sqrt{f^2-g}}\right) \\
&\quad = \frac{1}{2\sqrt{f^2-g}}\log\left(\frac{f-\sqrt{f^2-g}}{f+\sqrt{f^2-g}}\right)
\end{aligned}$$

Tips, Tricks, and When They Fail

From my “supersymmetric perspective”, looking at a variety of integrals, attempting direct integration

- Sometimes, tricks to avoid algebraic roots (simple, but unexpectedly powerful!)
- Sometimes, roots cancel - seeing this can be tricky!
- Sometimes, tricks fail - in interesting ways!

Minimal Representations: Loop-by-Loop

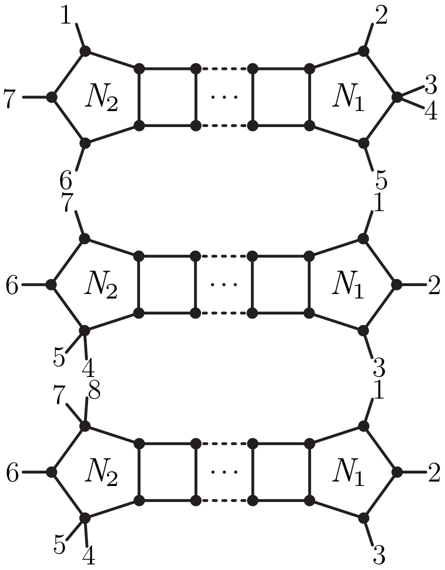
- Conjecturally, hyperlogarithms in Feynman integrals in 4D have bounded transcendental weight $\leq 2L$
- When possible, want to represent as $2L$-fold integrals
- This is not generally true for Symanzik form, one variable per propagator
- Can get closer by parametrizing "loop by loop" [Analogous to loop-by-loop Baikov, see Frellesvig's talk]
- Heuristically, fewer "extra integrals" $\rightarrow$ fewer chances to introduce spurious algebraic roots

Good Variables: Momentum Twistor Parametrizations

[1805.10281, Bourjaily, McLeod, MvH, Wilhelm]

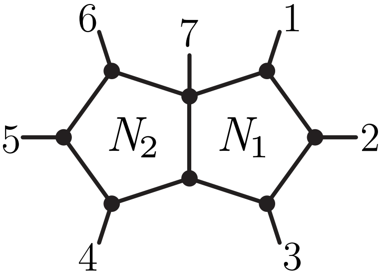
- Planar $\mathcal{N} = 4$ SYM has dual conformal symmetry: conformal in dual space $x_i - x_{i+1} = p_i$
- Momentum twistor variables are linear in this symmetry, trivialize momentum conservation, masslessness, by assigning each dual point x_i to a line $\{z_{i-1}, z_i\}$ in $\mathbb{P}^3$ [Hodges]
- Space of lines is six-dimensional, so these variables make Gramian matrices $G = \{G^a_b = (x_a - x_b)^2\}$ generically rank six
- Minors larger than 6×6 should vanish, implying relations between kinematic variables that involve square roots of 6×6 determinants. Momentum twistors rationalize these square roots.

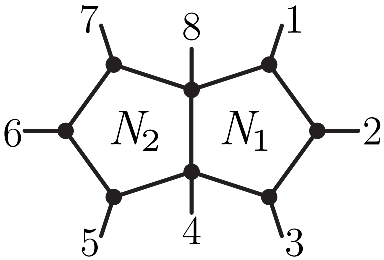
Sometimes, this is enough...



These seven- and eight-point classes of integrals run “out of the box” through four loops!

Sometimes it isn't





Splitting the Integration Path

$$
\mathcal{I}(\boldsymbol{\alpha},\boldsymbol{\beta}) \xrightarrow{\int d^2\boldsymbol{\beta}} \mathcal{I}(\boldsymbol{\alpha}) =: \begin{cases}
\mathcal{I}_A[\not\ni Q_1(\alpha_1)] \xrightarrow{\int d\alpha_1,\int d\alpha_2} \begin{cases} \mathcal{I}_{A_0}[\not\ni \sqrt{q_1},\sqrt{q_2}] \\ \mathcal{I}_{A_1}[\ni \sqrt{q_1(\alpha_3,\alpha_4)}] \\ \mathcal{I}_{A_2}[\ni \sqrt{q_2(\alpha_3,\alpha_4)}] \end{cases} \xrightarrow[\alpha_4\to 1]{\text{“}\int d\alpha_3\text{”}} \begin{cases} I_{A_0} \\ I_{A_1} \\ I_{A_2} \end{cases} \\
\mathcal{I}_B[\ni Q_1(\alpha_1)] \xrightarrow{\int d\alpha_4,\int d\alpha_3} \begin{cases} \mathcal{I}_{B_0}[\not\ni \sqrt{\widetilde{q}_1},\sqrt{\widetilde{q}_2}] \\ \mathcal{I}_{B_1}[\ni \sqrt{\widetilde{q}_1(\alpha_1,\alpha_2)}] \\ \mathcal{I}_{B_2}[\ni \sqrt{\widetilde{q}_2(\alpha_1,\alpha_2)}] \end{cases} \xrightarrow[\alpha_2\to 1]{\text{“}\int d\alpha_1\text{”}} \begin{cases} I_{B_0} \\ I_{B_1} \\ I_{B_2} \end{cases}
\end{cases}
$$

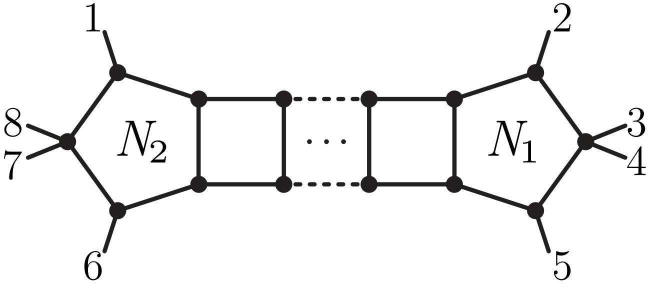
Two types of tricks:

- Divide integrand into pieces which are linearly reducible via one path vs. another, by which contain specific polynomials
- Divide integrand into pieces depending on different square roots, use different change of variables in each

Some integrals require both!

Kinematic Roots Remain

- Not all kinematic roots are rational in momentum twistors.
- Some are spurious, introduced by the splitting procedure, while some are physical.
- Example of the latter: four-mass box cuts



Identifying Spurious Roots

- Sometimes can rationalize with a change of variables (e.g. Euler substitution), fibrate, then transform back and see the roots cancel
- Sometimes can match series expansion to expansion of an ansatz without square roots
- If these don't work, can at least check the symbol

$$df^{(n)} = \sum f_i^{(n-1)} d \ln \phi_i \quad \rightarrow \quad S(f^{(n)}) = \sum S(f_i^{(n-1)}) \otimes \phi_i$$

(Lack Of) Unique Factorization

- "The symbol trivializes all identities" if all letters are *rational* - factor all letters, then expand

$$\cdots \otimes a \times b \otimes \cdots \quad = \quad \cdots \otimes a \otimes \cdots \quad + \quad \cdots \otimes b \otimes \cdots$$

- This fails for algebraic letters: no unique factorization

$$9 = 3 \times 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

- For small number of letters, can find relations by brute force
- Works for example for seven-point two-loop MHV in planar $\mathcal{N} = 4$: 22 letters can be written in terms of 5, which all drop out [1912.05690 Bourjaily, Volk, MvH]

Factorization in Prime Ideals

- For more complicated cases, want to do something systematic
- Let's focus on a specific kinematic point: all our symbol entries are (algebraic) numbers
- Instead of working with numbers, work with *ideals*:

$$(a) = \{ma | m \in \mathbb{Z}\}\,, \quad (a, b) = \{ma + nb | m, n \in \mathbb{Z}\}$$

- Then for the previous example $(3), (2+\sqrt{-5})$, and $(2-\sqrt{-5})$ all factor further, giving a unique factorization in prime ideals:

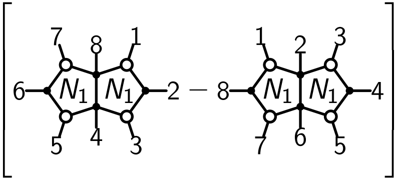
$$(9) = (3)(3) = (2+\sqrt{-5})(2-\sqrt{-5}) = (3, 1+\sqrt{-5})^2(3, 1-\sqrt{-5})^2$$

Application: an NMHV Octagon

[1910.14224 Bourjaily, Mcleod, Vergu, Volk, MvH, Wilhelm]

- Particular supercomponent of eight-point two-loop NMHV amplitude:

$$\int d\eta_1^1 d\eta_3^2 d\eta_5^3 d\eta_7^4 \; \mathcal{A}_8^{L=2} = \frac{1}{\langle 1357 \rangle}$$



- After integration: symbols in 2000 letters with 10 million terms, many distinct algebraic roots
- After factorizing letters: symbols in 35 letters with 5000 terms, only two distinct “physical” square roots
- Cancel in the difference!

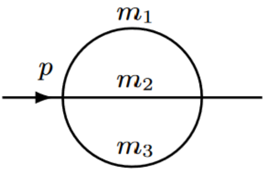
When the Tricks Fail

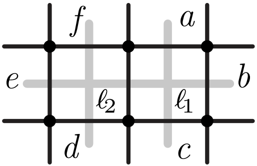
- In all of the above cases, we could at least complete the integration, getting hyperlogarithms (that depend on some kinematic root)
- This is not always the case: sometimes, still have non-rationalizable square roots in integration parameters:

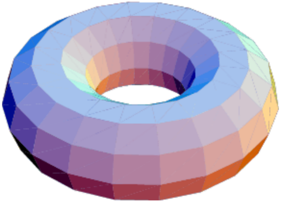
$$\sqrt{P(\overrightarrow{\alpha})}$$

Simplest Cases are Elliptic

$$\sqrt{(\alpha - e_1)(\alpha - e_2)(\alpha - e_3)(\alpha - e_4)}$$

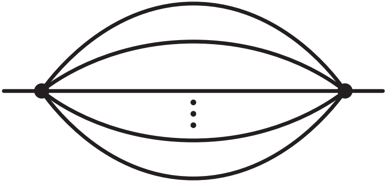


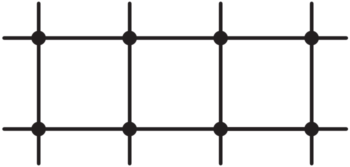




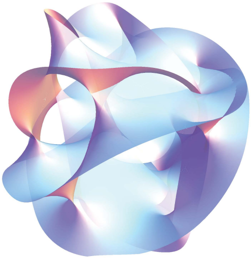
[see Weinzierl's talk]

(Some) More Complicated Cases are Calabi-Yau





[Bloch, Kerr, Vanhove; Broadhurst] [Bourjaily, He, Mcleod, MvH, Wilhelm]



$$\sqrt{P(\alpha_1, \alpha_2, \cdots)}$$

What is a Calabi-Yau?

- Compact Kähler manifold with vanishing first Chern class
- Ricci-flat
- Preserves N=1 supersymmetry of compactifications

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- Ricci-flat
- Preserves N=1 supersymmetry of compactifications
- **. . . not helpful!**

How do you **diagnose** a Calabi-Yau?

Embed the patient in a weighted projective space!

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$$(x_1, x_2, \ldots) \sim (\lambda x_1, \lambda x_2, \ldots)$$

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$$(x_1, x_2, \ldots) \sim (\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots)$$

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- If the sum of the coordinate weights equals the overall scaling (degree), it's Calabi-Yau!

Cleanest Example: Scalar Marginal Integrals

- Start with Symanzik form:

$$\Gamma(E - LD/2) \int_{x_i \geq 0} [d^{E-1} x_i] \frac{\mathfrak{U}^{E-(L+1)D/2}}{\mathfrak{F}^{E-LD/2}}$$

- Graph polynomials $\mathfrak{U}$ and $\mathfrak{F}$ defined by:

$$\mathfrak{U} \equiv \sum_{\{T\} \in \mathfrak{T}_1} \prod_{e_i \notin T} x_i, \quad \mathfrak{F} \equiv \left[\sum_{\{T_1, T_2\} \in \mathfrak{T}_2} s_{T_1} \Big(\prod_{e_i \notin T_1 \cup T_2} x_i \Big) \right] + \mathfrak{U} \sum_{e_i} x_i m_i^2$$

Cleanest Example: Scalar Marginal Integrals

Two cases where things simplify, both for even dimensions:

- $E = LD/2$: Explored by mathematicians. Superficial divergence from gamma function, if there are no subdivergences can strip this off, no further need for dim reg. Only $\mathfrak{U}$ contributes.

$$\int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{U}^{D/2}}$$

- $E = (L+1)D/2$: **Marginal**. If finite, can again avoid dim reg. Only $\mathfrak{F}$ contributes.

$$\int_{x_i \geq 0} [d^{E-1} x_i] \frac{1}{\mathfrak{F}^{D/2}}$$

  - In $D = 2$, these are the sunrise/banana graphs!
  - Many more cases in $D = 4$

Marginal Integrals are Calabi-Yau

Let's look at our "special cases".

[Brown 0910.0114] explored the $E = LD/2$ case, argument for marginal integrals ($E = (L+1)D/2$) similar:

- $\mathfrak{F}$ is homogenous, degree $L+1$, so $\mathfrak{F}^{D/2}$ has degree $(L+1)D/2 = E$ in E variables
- Direct integration preserves this: each integration removes one variable, and decreases the degree of the denominator by one.
- Suppose we encounter a square root. Root $\sqrt{Q(x_i)}$ will contain a degree $2m$ polynomial in m variables.
- $y^2 = Q(x_i)$ defines a variety. Give the x_i weight 1, y weight m. Then sum of the weights is equal to degree $\to$ diagnosed Calabi-Yau!

Example: Massless $D = 4$

- Specialize to $D = 4$, massless propagators:

$$\int_{x_i \geq 0} [d^{2L+1} x_i] \frac{1}{\mathfrak{F}^2}$$

- $\mathfrak{F}$ is linear in every variable (x_i^2 only shows up in the mass term). We may integrate out any one parameter x_j. Writing $\mathfrak{F} \equiv \mathfrak{F}_0^{(j)} + x_j \mathfrak{F}_1^{(j)}$:

$$\int_{x_i \geq 0} [d^{2L} x_i] \frac{1}{\mathfrak{F}_0^{(j)} \mathfrak{F}_1^{(j)}}$$

- Each factor is still linear, so we can integrate in another variable x_k. Writing $\mathfrak{F}_i^{(j)} \equiv \mathfrak{F}_{i,0}^{(j,k)} + x_k \mathfrak{F}_{i,1}^{(j,k)}$:

$$\int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)}\right) - \log\left(\mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}\right)}{\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}}$$

Example: Massless $D = 4$

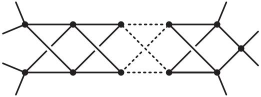
$$\int_{x_i \geq 0} [d^{2L-1} x_i] \frac{\log\left(\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)}\right) - \log\left(\mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}\right)}{\mathfrak{F}_{0,0}^{(j,k)} \mathfrak{F}_{1,1}^{(j,k)} - \mathfrak{F}_{0,1}^{(j,k)} \mathfrak{F}_{1,0}^{(j,k)}}$$

- Denominator is at most quadratic in each remaining variable.
- If irreducibly quadratic in all variables (and discriminants irreducibly cubic or quartic in all other variables), then Calabi-Yau with dimension $2L - 2$.
- Thus for massless marginal integrals in $4D$, Calabi-Yau dimension is **bounded**.

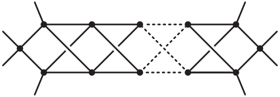
Is this bound saturated?

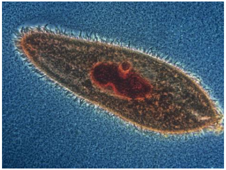
Yes!

Even $L \geq 2$
Tardigrades

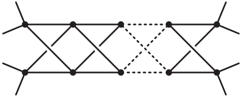


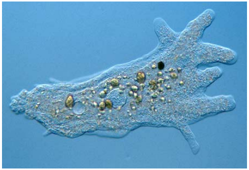
Odd $L \geq 1$
Paramecia





Odd $L \geq 5$
Amoebas

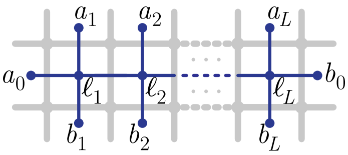




Observations:

- The $L = 2$ tardigrade is a two-loop, five-point (three external masses) K3!
- We've looked at other marginal integrals through seven loops, the majority have maximal dimension Calabi-Yaus.
- The $L = 3$ amoeba is oddly enough *not* maximal dimension.

What about the Traintracks?



- Not marginal: $E = 3L + 1 \neq (L+1)D/2$ for $L \neq 1$
- Not Symanzik, loop-by-loop:

$$\int_0^\infty [d^L\alpha] d^L\beta \frac{1}{(f_1 \cdots f_L) g_L}$$

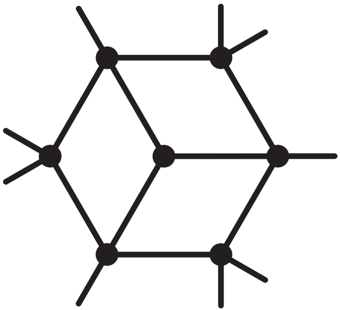
$$f_k \equiv (a_0 a_{k-1}; a_k b_{k-1})(a_{k-1} b_k; b_{k-1} a_0)(a_k b_k; a_{k-1} b_{k-1}) f_{k-1} + \alpha_0(\alpha_k + \beta_k) + \alpha_k \beta_k$$
$$+ \sum_{j=1}^{k-1} \left[\alpha_j \alpha_k (b_j a_0; a_j a_k) + \alpha_j \beta_k (b_j a_0; a_j b_k) + \alpha_k \beta_j (a_0 a_j; a_k b_j) + \beta_j \beta_k (a_0 a_j; b_k b_j) \right]$$

$$g_L \equiv \alpha_0 + \sum_{j=1}^{L} \left[\alpha_j (b_j a_0; a_j b_0) + \beta_j (a_0 a_j; b_0 b_j) \right]; \quad (ab; cd) \equiv \frac{x_{a,b}\, x_{c,d}}{x_{a,c}\, x_{b,d}}$$

Three-Loop K3

- Take codimension $L+1$ residue, obstructed by a square root
- Get $\sqrt{Q}$, where Q is degree 4 in α_2 and degree 6 in α_1 and α_0
- Weights $3+1+1+1=6$, like the marginal integrals

Wheel/Coccolithophore



- Once again, not marginal, not Symanzik
- Planar, relevant to $\mathcal{N} = 4$ sYM
- Can take a similar series of residues (and a handy re-projectivization) to find weights $4 + 1 + 1 + 1 + 1 = 8$, CY_3!

The Theme: Every Example We Understand is $\mathbb{WP}^{k,1,1,1,\ldots}$

- Marginal integrals, the three-loop traintrack, the wheel, all have weights $k+1+1+1... = 2k$
- Can think of our singular manifolds as special cases of smooth $\mathbb{WP}^{k,1,1,1,\cdots}$, reach through complex structure deformation
- Can calculate Hodge diamonds, etc.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{cccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 149 & & 149 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 976 & & 3952 & & 976 & & 1 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Further Questions

- How often can "planar $\mathcal{N} = 4$ -like" integrals be used?
- Can we do better than tips and tricks, more deterministic algorithm?
- Does direct integration uncover the same Calabi-Yau geometries as other methods? For example, recent leading singularity calculations for traintracks by Vergu & Volk.
- How special/rare is the Calabi-Yau property?

Thank You

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