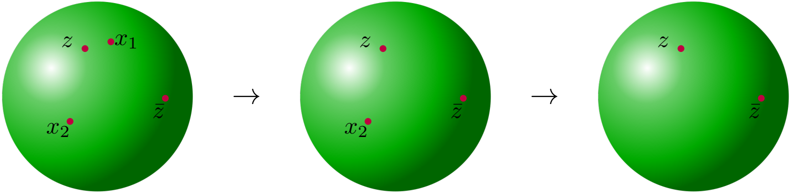
Hyperlogarithms

Differential and difference equations from linear fibrations



Erik Panzer 5 October 2020 Royal Society UK

Euler-Mellin integral

$$I(z,s) = \int_{\mathbb{R}_+^n} x^s \ f_1(x,z)^{s_{n+1}} \cdots f_m(x,z)^{s_{n+m}} \ \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n}$$

polynomials

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polynomials

- **Beta function:**

$$B(s,t) = \int_0^\infty \frac{x^s}{(1+x)^{s+t}} \frac{\mathrm{d}x}{x}$$

- **Hypergeometrics:**

$${}_2F_1\left(\begin{matrix} s, t \\ u \end{matrix} \middle| z\right) = \frac{\Gamma(u)}{\Gamma(s)\Gamma(t)} \int_0^\infty \frac{x^s(1+x)^{t-u}}{(1+x(1-z))^t} \frac{\mathrm{d}x}{x}$$

- **Feynman integrals:**

$$\left(\prod_{k=1}^{L} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \ell_k}{\pi^{D/2}}\right) \prod_{i=0}^{n} \frac{1}{P_i^{s_i}} = \frac{\Gamma(t)}{\Gamma(s_0)\cdots\Gamma(s_n)} \int_{\mathbb{R}_+^n} \frac{x_1^{s_1} \cdots x_n^{s_n}}{\mathcal{U}^{D/2-t}\mathcal{F}^t} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n}$$

$P_i = k_i^2 + m_i^2$ quadrics

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Properties:

- converges in a non-empty cone of $\Re(s)$ (for suitable z)

$$B(s,t) = \int_0^\infty \frac{x^s}{(1+x)^{s+t}} \frac{\mathrm{d}x}{x} \qquad \text{converges for} \qquad \Re(s), \Re(t) > 0$$

- extends to a meromorphic function of $s \in \mathbb{C}^m$ with linear poles

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \qquad \text{has poles on} \qquad s,t \in \mathbb{Z}_{\leq 0}$$

- vector space generated by shifts in s and derivatives in z,

$$\mathfrak{M} := \sum_{\delta \in \mathbb{Z}^{n+m}} \sum_k \mathbb{Q}(z,s) \cdot \frac{\partial^k}{\partial z^k} I(z, s+\delta) \quad \text{has finite dimension}$$

($= 1$ for B, $= 2$ for ${}_2F_1$)

Euler-Mellin integral

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Computational problems:

- expansion around integers $\bar{s} \in \mathbb{Z}^{n+m}$: (wlog $\bar{s}$ convergent)

$$I(z,s) = \sum_{k \in \mathbb{Z}_{\geq 0}^{n+m}} (s - \bar{s})^k I_k(z, \bar{s})$$

- difference equations:

basis of $\mathfrak{M}$

$$I(z, s+\delta) = c_1(z,s)\,\mathfrak{m}_1 + \cdots + c_N(z,s)\,\mathfrak{m}_N$$

polynomials $\in \mathbb{Q}(z,s)$

- differential equations:

$$\frac{\partial^k}{\partial z^k} I(z,s) = c_1(z,s)\,\mathfrak{m}_1 + \cdots + c_N(z,s)\,\mathfrak{m}_N$$

Euler-Mellin integral

$$I(z, s) = \int_0^\infty x^{s_1} \; f_1(x, z)^{s_2} \cdots f_m(x, z)^{s_{m+1}} \; \frac{\mathrm{d}x}{x}$$

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Expansion at leading order: $s = \bar{s} \in \mathbb{Z}^{m+1}$ is integral

1. Factorize: $f_i(x,z) = a_i \prod_\sigma (x - \sigma(z))^{d_{i\sigma}}$
2. Partial fractions:

$$x^{s_1} f_1(x,z)^{s_2} \cdots f_m(x,z)^{s_{m+1}} = \sum_{k \geq 0} c_k x^k + \sum_\sigma \sum_{k>0} \frac{c_{\sigma,k}}{(x-\sigma)^k}$$

3. Integrate by parts:

$$I(z,s) = r(z) + \int_0^\infty \sum_\sigma c'_\sigma \frac{\mathrm{d}x}{x-\sigma} = r(z) - \sum_\sigma c'_\sigma(z) \ln(-\sigma(z))$$

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Higher orders:

$$\int \ln(x-\sigma_1)^{k_1} \cdots \ln(x-\sigma_k)^{k_r} \frac{\mathrm{d}x}{x-\sigma_0} = ?$$

Hyperlogarithm (Poincaré, Lappo-Danilevskyy, Goncharov, . . .)

$$L_{\sigma_1,\ldots,\sigma_w}(x) = \int_{x>t_1>\cdots>t_w>0} \frac{\mathrm{d}t_1}{t_1-\sigma_1}\cdots\frac{\mathrm{d}t_w}{t_w-\sigma_w}$$

$$L_\sigma(x) = \ln(1-x/\sigma) \qquad L_{0,1}(x) = \int_0^x \ln(1-t)\frac{\mathrm{d}t}{t} = -\operatorname{Li}_2(x)$$

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Properties of the $\mathbb{C}(x)$-vector space spanned by $L_{\vec{\sigma}}(x)$, $\sigma_i \in \mathbb{C}$:

- $L_{\vec{\sigma}}(x)$ are linearly independent
- an algebra $\quad L_\sigma(x)\cdot L_\tau(x) = L_{\sigma,\tau}(x) + L_{\tau,\sigma}(x)$
- graded by weight w
- closed under ∂ and $\int \mathrm{d}x \quad \partial_x L_{\sigma,\ldots}(x) = \frac{1}{x-\sigma}L_{\ldots}(x)$

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Corollary ($n = 1$)

All coefficients of the $s \to \bar{s}$ expansion of $I(z,s)$ are $\overline{\mathbb{Q}(z)}$-linear combinations of hyperlogarithms with algebraic arguments $\in \overline{\mathbb{Q}(z)}$.

$$I(z,s) = \int_0^\infty y^{s_2} \left(\int_0^\infty x^{s_1} f_1(x,y,z)^{s_3} \cdots f_m(x,y,z)^{s_{m+2}} \, \frac{\mathrm{d}x}{x} \right) \frac{\mathrm{d}y}{y}$$

Inner integral $= \sum c_\sigma L_{\sigma_1,\ldots,\sigma_w}(\sigma_0)$ with $\sigma_i \in \overline{\mathbb{Q}(z,y)}$

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Lemma

If $\{\sigma_0,\ldots\} \subset \overline{Q(z)}(y)$, then every $L_{\sigma_1,\ldots,\sigma_w}(\sigma_0)$ is a hyperlogarithm in y:

$$L_{\sigma_1(z,y),\ldots,\sigma_w(z,y)}(\sigma_0(z,y)) = \sum_\tau c_\tau(z) L_{\tau_1(z),\ldots,\tau_w(z)}(y)$$

Proof:

$$\mathrm{d} L_{\sigma_1,\ldots,\sigma_w}(\sigma_0) = \sum_{k=1}^{w} L_{\ldots\not{\sigma}_k\ldots}(\sigma_0)\, \mathrm{dlog} \frac{\sigma_k - \sigma_{k-1}}{\sigma_k - \sigma_{k+1}}$$

Corollary

If this is the case, all coefficients of the $s \to \bar{s}$ expansion of $I(z,s)$ are $\overline{\mathbb{Q}(z)}$-linear combinations of hyperlogarithms with algebraic arguments.

Example:

$$I(z,\bar{z}) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x}{(1+x+y)(xy+z\bar{z}x+(1-z)(1-\bar{z})y)} \mathrm{d}y$$

Singularities:

1. integrand: $S = \{1+x+y, xy+z\bar{z}x+(1-z)(1-\bar{z})y\}$, e.g.

$$x = \sigma_1 = -1-y \qquad \text{and} \qquad x = \sigma_2 = -\frac{y(1-z)(1-\bar{z})}{y+z\bar{z}}$$

2. after $\int_0^\infty \mathrm{d}x$:

$$S_x = \{\underbrace{1+y}_{\sigma_1=0}, \underbrace{y, 1-z, 1-\bar{z}}_{\sigma_2=0}, \underbrace{y+z\bar{z}}_{\sigma_2=\infty}, \underbrace{z+y, \bar{z}+y}_{\sigma_1=\sigma_2}\}$$

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$$= \int_0^\infty \frac{1}{(z+y)(\bar z+y)} \log\frac{(1+y)(z\bar z+y)}{(1-z)(1-\bar z)y}\mathrm{d}y$$

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$$= \frac{1}{z-\bar{z}}\int_0^\infty \left(\frac{1}{y+\bar{z}} - \frac{1}{y+z}\right)\log\frac{(y+1)(y+z\bar{z})}{(1-z)(1-\bar{z})y}\mathrm{d}y$$

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$$= \frac{2\,\mathrm{Li}_2(z) - 2\,\mathrm{Li}_2(\bar{z}) + [\log(1-z) - \log(1-\bar{z})]\log(z\bar{z})}{z-\bar{z}}$$

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Polynomial reduction (Brown)

- Suppose S is a set of polynomials $S \ni f = f^x x + f_x$ linear in x.
- Let S_x denote the union of irreducible factors of

$$\{f^x, f_x : \; f \in S\} \quad \text{and} \quad \{f^x g_x - f_x g^x : \; f, g \in S\}.$$

Lemma

$$\text{singularities}\Big(I\Big) \subset S \qquad \Rightarrow \qquad \text{singularities}\Big(\int_0^\infty I \, \mathrm{d}x\Big) \subset S_x$$

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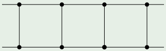
Iteration gives coarse upper bounds on the Landau varieties, depends on the order of the reduction of the variables:

$$S_{x,y} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-1\}\,,$$
$$S_{y,x} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-z-\bar{z}\}\,.$$

Improvements

- Fubini algorithm: $S_{x,y} \cap S_{y,x} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$
- **compatibility graphs** $\Rightarrow$ infinite families 

Definition

A set $\{f_1, \ldots, f_m\} \subset \mathbb{Q}[z, x]$ is **linearly reducible** if the variables x can be reordered such that all partial integrals $I^{(k)}$ are hyperlogarithms in x_{k+1},

$$I^{(k)}(z, x_{k+1}, \ldots, x_N) = \int_{\mathbb{R}_+^k} x^s f_1^{s_{n+1}} \cdots f_m^{s_{n+m}} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_k}{x_1 \cdots x_k}$$

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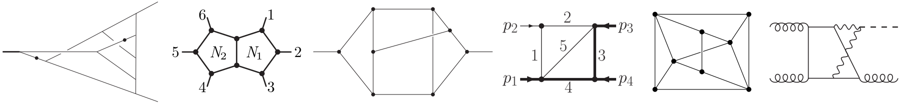
- dedicated programs like MPL, HyperInt, PolylogTools
- efficiency & output very sensitive to order & parametrisation
- find orders & predict letters σ_i with polynomial reduction (approximates Landau varieties also in non-reducible cases)



hyperlogarithmic constants:

- For $\sigma_i \in \overline{\mathbb{Q}}$, the numbers $L_{\vec{\sigma}}(\sigma_0)$ are periods of mixed Tate motives (Goncharov, Brown, Deligne, ...)
- reduction tables (multiple zeta value datamine & others)
- motivic coaction:

$$\Delta\zeta^{\mathfrak{m}}(2,3) = 1\otimes\zeta^{\mathfrak{m}}(2,3)+\zeta^{\mathfrak{dr}}(2,3)\otimes 1+2\zeta^{\mathfrak{dr}}(3)\otimes\zeta^{\mathfrak{m}}(2)+\zeta^{\mathfrak{dr}}(2)\otimes\zeta^{\mathfrak{m}}(3)$$

- motivic decomposition algorithm
- 'f-alphabet':

$$\sum_{\vec{n}} \mathbb{Q}\zeta^{\mathfrak{m}}(\vec{n}) \cong \mathbb{Q}[f_2] \otimes \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle$$

(and other roots of unity)

- HyperlogProcedures by Oliver Schnetz

Iterated integration by parts turns the $s \to \bar{s}$ expansion coefficients into

$$\int \frac{\mathrm{d}x_n}{x_n - \sigma_n(z)} \cdots \int \frac{\mathrm{d}x_1}{x_1 - \sigma_1(z, x_2, \ldots, x_n)} \prod_i \ln^{w_i} f_i(z, x)$$

- only **finitely many** possibilities $\sigma_i \in \Sigma_i$
- amounts to fibration by marked Riemann spheres, embedding:

$$H^\bullet_{dR}\Big(\mathbb{C}^{*n} \setminus \bigcup_i \{f_i = 0\}\Big) \hookrightarrow H^\bullet_{dR}(\mathbb{C} \setminus \Sigma_1) \otimes \cdots \otimes H^\bullet_{dR}(\mathbb{C} \setminus \Sigma_n)$$

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Twisted cohomology

Do not expand, treat $a = s - \bar{s}$ as formal variable. Consider

$$I_{\sigma_n, \ldots, \sigma_1}(z, a) := \int_0^\infty \frac{x_n^{a_n}\, \mathrm{d}x_n}{x_n - \sigma_n(z)} \cdots \int_0^\infty \frac{x_1^{a_1}\, \mathrm{d}x_1}{x_1 - \sigma_1(z, x_2, \ldots, x_n)} \prod_i f_i(z, x)^{a_{n+i}}$$

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Facts:

1. coefficients $\propto a^w$ have uniform weight $|w|$
2. The $\mathbb{C}(a)$-vector space $\mathfrak{M}'$ spanned by $I_{\vec{\sigma}}$s has

$$\dim \mathfrak{M}' = |\Sigma_1 - 1| \cdots |\Sigma_n - 1| < \infty$$

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$$\dim \mathfrak{M}' = |\Sigma_1 - 1| \cdots |\Sigma_n - 1| < \infty$$

3. $\mathfrak{M}'$ is closed under $\frac{\partial}{\partial z}$ and $a \mapsto a \pm \vec{e}_k$ (partial fractions & IBP)
4. $\mathfrak{M}' \supset \mathfrak{M} \ni I(z,s) = \sum_{\vec{\sigma}} c_{\vec{\sigma}} I_{\vec{\sigma}}$ ($c_{\vec{\sigma}}$ = coordinates on integrand space)

Corollary

linearly reducible $\Rightarrow$ differential equations and contiguous relations

Example: double box ($n = 6$ and $m = 2$ and $\dim \mathfrak{M} = 12$)