Bilinear quark operators in RI/SMOM scheme at three loops

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outlook

- introduction
- scheme dependence in QCD
- RI/MOM, RI/SMOM schemes and the lattice matrix elements
- conversion factors from $\overline{\mathrm{MS}}$ to RI/SMOM for scalar, vector and trensor currents
- numerical results
- conclusions

operator product expansion

- operator product expansion on the light cone

$$J^A(0)J^B(x) \overset{x^2\to 0}{\sim} \sum_n C_n^{AB}(x^2)\, O_{\mu_1\ldots\mu_n}\, x^{\mu_1}\ldots x^{\mu_n} + \text{higher twists}$$

- short distance coefficients $C_n^{AB}(x^2)$ are computed in perturbation theory
- long distance matrix elements $O_{\mu_1\ldots\mu_n}$ are subject to non-perturbative considerations:
  - fits from experimental data
  - sum rules
  - lattice Monte Carlo simulations
  - smth else?
- evolution of the operators $O_{\mu_1\ldots\mu_n}$ with the scale μ
  $\Longrightarrow$ anomalous dimensions
  $\Longrightarrow$ renormalization group equation
- analysis is usually done in $\overline{\mathrm{MS}}$ scheme

operators

- the following operators are of the most interest for the moments of the structure functions and the moments of light-cone distribution amplitudes
- twist 2 operators

$$\mathcal{S}\,\bar{\psi}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi$$
$$\mathcal{S}\,\bar{\psi}\gamma_{\mu_1}\gamma_5\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi$$

- and twist 3 operators (e.g. spin structure function $g_2(x, Q^2)$)

$$\bar{\psi}\gamma_{\mu_1}\,\mathcal{S}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi$$
$$\bar{\psi}\gamma_{\mu_1}\gamma_5\,\mathcal{S}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi$$

scheme dependence in QCD

- we distiguish between bare and renormalized quantities, e.g.

$$\alpha_0 = \bar{\mu}^{\varepsilon} Z_\alpha \alpha_R \quad \psi_0 = Z_2^{1/2}\psi_R, \quad X_0 = Z_X X_R, \quad X = m, \xi, O_\Gamma, \dots$$

- in $\overline{\mathrm{MS}}$ scheme all Z's have the form
$Z_X = 1 + \sum_{i>0} \frac{z_X^{(i)}}{\varepsilon^i}, \qquad z_X^{(i)} = \sum_{j \geq i} a^j c_X^{(i,j)}, \qquad a = \frac{\alpha_s}{4\pi}$
- in momentum subtraction (MOM) scheme we require the properly chosen Green functions to be equal to their *tree values* at some fixed μ-dependent configurations of external momenta
- the finiteness of all renormalized parameters, fields, operators, etc. means within perturbation theory that

$$X_{\overline{\mathrm{MS}}}(\bar{\mu}) = C_X(\bar{\mu}, \mu, \dots) X_{\mathrm{MOM}}(\mu), \qquad X = \psi, A_\mu, m, O_\Gamma, \dots$$

where conversion functions C_X are independent on cut-offs and

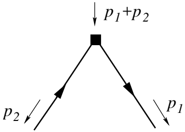
$$C_X = 1 + \sum_{i>0} a^i C_X^{(i)}(\bar{\mu}, \mu, \xi, \dots)$$

lattice QCD and RI-schemes

- lattice QCD provides a direct way to estimate matrix elements of local operators from first principles
- the renormalization of the operators is required in order to get finite results as the lattice cut-off $a \to 0$
- usually renormalized quantities are obtained in some appropriate renormalization scheme such as RI(RI')/(S)MOM
- on the other hand perturbative calculations in continuum QCD are conventionally done in dimensional regularization and $\overline{\mathrm{MS}}$ scheme
- matching of lattice matrix elements to those defined in a continuum perturbative scheme requires the calculation of the corresponding multiplicative conversion factors
- 3-loop SMOM $\to \overline{\mathrm{MS}}$ conversion needed to reduce the systematic uncertainty

RI/MOM

- consider Green's function of bilinear quark operator O



- by renormalization in some scheme we express bare renormalized parameters in terms of bare ones

$$\psi_R = Z_q^{1/2}\psi_0, \qquad m_R = Z_m m_0, \qquad O_R = Z_O O_0$$

- RI/MOM ($\mathrm{RI}'/\mathrm{MOM}$) schemes Martinelli et al '95
  RI $\longrightarrow$ "regularizarion independent"

$\mathrm{RI}^{(\prime)}/\mathrm{MOM}$ renormalization conditions

- in the RI/MOM scheme the renormalization of the external fermion legs is fixed by the conditions (constant Z_q)

$$\lim_{m_R \to 0} \frac{1}{12 m_R} \mathrm{Tr}\, S_R^{-1}(p) \Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\, \gamma_\nu \frac{\partial S_R^{-1}(p)}{\partial p_\nu} \Big|_{p^2=-\mu^2} = -1$$

where $S(p)$ is the fermion propagator

- "light" version: the $\mathrm{RI}'/\mathrm{MOM}$ scheme; Z_q is fixed from

$$\lim_{m_R \to 0} \frac{1}{12 p^2} \mathrm{Tr}\, \not{p} S_R^{-1}(p) \Big|_{p^2=-\mu^2} = -1$$

- additionaly we define renormalized mass as

$$m_0 = (Z_{\bar\psi\psi})^{-1} m_R$$

where

$$(\bar\psi\psi)_0 = Z_{\bar\psi\psi} (\bar\psi\psi)_R$$

$\mathrm{RI}^{(\prime)}/\mathrm{MOM}$ renormalization conditions

- not true renormalization schemes!
  $\longrightarrow$ need renormalization of the overall divergences
- nontrivial statment to prove is that within this framework all unrenormalized Green functions expressed in terms of α_R, m_R are cut-off independent up to some multiplicative factor, i.e.

$$\Gamma_0(\{p\},\ \alpha_0 Z_\alpha,\ m_0/Z_{\bar{\psi}\psi},\ \mu, \Lambda)\, Z_\Gamma(\Lambda, \mu)$$

is independent of Λ as $\Lambda \to \infty$ Weinberg '73

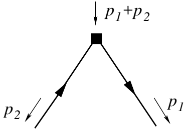
RI/MOM vs RI/SMOM

fixing the renormalization conditions for the overall divergences

Sturm et al '09

RI/MOM:

$$p_1^2 = p_2^2 = -\mu^2$$
$$(p_1 + p_2)^2 = 0$$



RI/SMOM:

$$p_1^2 = p_2^2 = -\mu^2$$
$$(p_1 + p_2)^2 = -\mu^2$$

- $\mathrm{RI}^{(\prime)}/\mathrm{MOM}$ prescription suffers from strong sensitivity to IR effects
- dominant source of uncertainty on the lattice
- large corrections in conversion factors: e.g. for the quark the mass (Landau gauge, $\mu = 2\text{GeV}$, $n_f = 4$, $\alpha_s/\pi = 0.1$) Chetyrkin, Retey '00

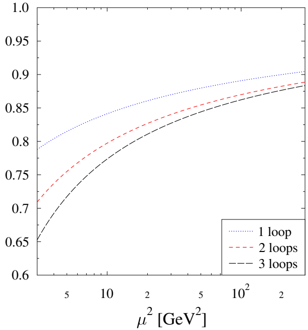
$$m_{\overline{\mathrm{MS}}} = \Big(1 - 0.13333 - 0.0754 - 0.0495\Big) m_{\mathrm{RI/MOM}}$$

Example: NS quark-antiquark operator with two derivatives

Second Gegenbauer moment of the K-meson LCDA [‡]

[‡] G. S. Bali *et al.*, [RQCD collaboration], arXiv:1903.08038

RI'-MOM→ $\overline{\mathrm{MS}}$ matching



RI'-SMOM→ $\overline{\mathrm{MS}}$ matching

scheme	$a_2^K(\mu = 2\ \mathrm{GeV})$
SMOM, 1-loop	$0.066^{+11}_{-13}(17)_r(13)_a(5)_m$
SMOM, 2-loop	$0.089^{+11}_{-12}(11)_r(11)_a(4)_m$
SMOM, 3-loop	???

setup

in this work we consider operators without derivatives:

- scalar $J^S = \bar{\psi}\psi$
- vector $J^V_\mu = \bar{\psi}\gamma_\mu\psi$
- tensor $J^T_{\mu\nu} = \bar{\psi}\sigma_{\mu\nu}\psi$

$$\langle\bar{\psi}\psi\rangle = \delta_{ij}\,\Lambda^S\,, \qquad \langle\bar{\psi}\gamma_\mu\psi\rangle = \delta_{ij}\,\Lambda^V_\mu\,, \qquad \langle\bar{\psi}\sigma_{\mu\nu}\psi\rangle = \delta_{ij}\,\Lambda^T_{\mu\nu}$$

Formfactor decompositions at the symmetric point:

$$\Lambda^S = F_1^S\, I - F_2^S\, \frac{[\not{p}_1, \not{p}_2]}{\mu^2}$$

$$\Lambda^V_\mu = F_1^V\,\gamma_\mu + F_2^V\,\frac{\not{p}_1\gamma_\mu\not{p}_1 + \not{p}_2\gamma_\mu\not{p}_2}{\mu^2} - F_3^V\,\frac{\not{p}_1\gamma_\mu\not{p}_2}{\mu^2} - F_4^V\,\frac{\not{p}_2\gamma_\mu\not{p}_1}{\mu^2}$$

$$\Lambda^T_{\mu\nu} = F_1^T\,\sigma_{\mu\nu} + F_2^T\,\frac{\not{p}_1\not{p}_2\sigma_{\mu\nu} - \sigma_{\mu\nu}\not{p}_2\not{p}_1}{\mu^2} + F_3^T\,\frac{\not{p}_1\not{p}_2\sigma_{\mu\nu}\not{p}_1\not{p}_2}{\mu^4}$$

RI/MOM results at three loops

- RI/MOM, RI'/MOM 3-loop results are known for S,P,V,A,T currents
  Chetyrkin, Retey '00
- requires evaluation of massless 3-loop diagrams of the propagator type (p-integrals)
- reduction $\longrightarrow$ MINCER
- master integrals analytically in terms of ζ(n)
- e.g. for the quark field and the mass tranformations we have (Landau gauge, $\mu = 2\text{GeV}$, $n_f = 4$, $\alpha_s/\pi = 0.1$)

$$C_m^{\mathrm{RI/MOM}} = 1 - 0.13333 - 0.0754 - 0.0495$$
$$C_q^{\mathrm{RI/MOM}} = 1 + 0.0 - 0.00476 - 0.005108$$

$$m_{\overline{\mathrm{MS}}} = C_m^{\mathrm{RI/MOM}} m_{\mathrm{RI/MOM}}\,, \qquad \psi_{\overline{\mathrm{MS}}} = C_q^{\mathrm{RI/MOM}} \psi_{\mathrm{RI/MOM}}$$

RI/SMOM results at the 2-loop order

- 1-loop results

Sturm et al. '09
Gracey '11

- master integrals were evaluated analytically

Usyukina, Davydychev '94
Birthwright, Glover, Marquard '04

- 2-loop results

Almeida, Sturm '10
Gracey '11

$$C_m^{\mathrm{RI/SMOM}} = 1 - 0.6455188a - a^2(22.607688 - 4.01353947n_f) + O(a^3)$$
$$C_q^{\mathrm{RI/SMOM}} = C_q^{\mathrm{RI'/MOM}}$$

IBP reduction of Feynman integrals

Parametrization of L-loop Feynman integrals in d dimensions

$$I_{i_1,i_2,\ldots i_n} = \int \frac{dk_1 \ldots dk_L}{D_1^{i_1} D_2^{i_2} \ldots D_n{}^{i_n}}$$

where indices $i_1, \ldots, i_n$ can also be negative. The # of invariants D and indices i is $n = L(L+5)/2$ where L is the number of loops.

- algebraical reduction of the large number of integrals ($\sim 10^6$) with different indices to the small set of master-integrals
- it is possible to form $L(L+2)$ integration by part (IBP) relations that give linear algebraic relations between $I_{i_1,\ldots,i_n}$ with different sets of $i_1, \ldots, i_n$
- solution of the large system ($\sim 10^6$) of linear equations over the field $\mathbb{R}[d]$
- available software packages: Reduze ([arXiv:1201.4330]) , FIRE ([arXiv:1408.2372]) $\longrightarrow$ highly optimized, parallelization

evaluation

Some usefull statistics

	# of tops.	# of dias.	# of ind.	# of ints.	# of MIs
1-loop	1	2	3	~ 10	2
2-loop	4	33	7	$\sim 10^3$	8
3-loop	13	688	12	$\sim 10^6$	60

Evaluation of master-integrals

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- analytical evaluation is possible via differential equations
  $\longrightarrow$ needs IBP reduction in non-symmetric kinematics

not in this work!

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Evaluation of master-integrals

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  $\longrightarrow$ needs IBP reduction in non-symmetric kinematics

  not in this work!
- numerical solution using the methods of sector decomposition ([hep-ph/0004013]):
  analytical resolution of singularities and numerical integration of the residual parts
- available software: SecDec ([arXiv:1601.03982]) , FIESTA ([arXiv:1511.03614]) $\longrightarrow$ parallelization!

Feynman parametrization

$$I_{a_1,a_2,\ldots a_n} = \int \frac{dk_1 \ldots dk_L}{D_1^{a_1} D_2^{a_2} \ldots D_n^{a_n}} \, ,$$

Feynman parameters representation

$$I_{a_1,\ldots,a_n} = \frac{\Gamma(a - Ld/2)}{\prod_{j=1}^{n} \Gamma(a_j)} \int\limits_{[1,0]^n} \frac{D^{n-(L+1)d/2}}{F^{n-Ld/2}} \prod x_j^{a_j - 1} \, \delta\left(1 - \sum x_j\right)$$

In our case (euclidean kinematics) F and D are polynoms of $x_1, \ldots, x_n$

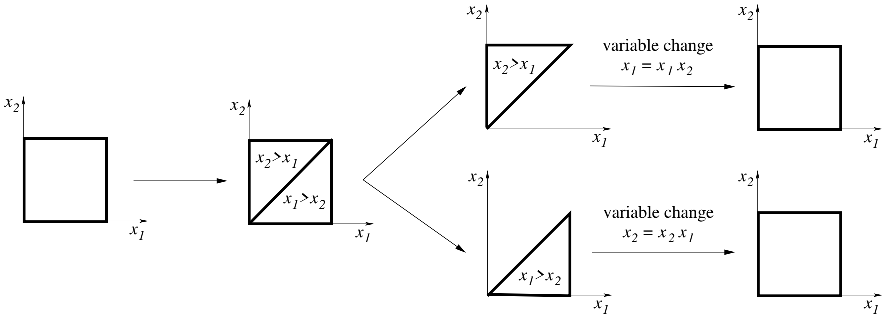
- positive definite
- never become zero inside the hypercube $[0,1]^n$
- the only singularities can occur when some of x's go to 0

resolution of singularities (sector decomposition I)

2-dimensional example:

$$\int_{[0,1]^2} \frac{\phi(x_1,x_2)}{(x_1+x_2)^\gamma}$$

$$\int_{[0,1]^2} \frac{\phi(x_1x_2,x_2)}{x_2^{\gamma-1}(1+x_1)^\gamma}$$



$$\int_{[0,1]^2} \frac{\phi(x_1,x_1x_2)}{x_1^{\gamma-1}(1+x_2)^\gamma}$$

resolution of singularities (sector decomposition II)

- in practice we have n-fold (up to $n = 9$) integrals
- e.g. in case of non-planar topology we have:

$$\int_{[0,1]^9} \frac{\phi(x_1, \ldots, x_9)}{(x_1 x_2 x_4 x_6 + x_1 x_3 x_4 x_6 + x_2 x_3 x_4 x_6 + \text{383 other terms})^{\gamma}}$$

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- recurrsively applying SD we get a lot of integrals of the form

$$\int\limits_{[0,1]^n} \frac{\phi(\ldots)}{x_1^{a_1}x_2^{a_2}\ldots x_p^{a_p}(1 + \text{very long polynom})^\gamma} \qquad (*)$$

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- in the worst case one can get $n!$ sector integrals (at 3-loops this would lead to $9! = 362880$)
- however in practice we have typically few hunderts of sectors (recurrsive algorithm stops earlier)
- numerical evaluation with Monte Carlo methods

(quasi) ε-finite basis

- the choice of MI's is not unique: any M linear independent integrals can be chosen as a basis $\tilde{\mathcal{M}}_i$

$$\text{amplitude} = \sum_{i=1}^{M} \tilde{c}_i(d)\tilde{\mathcal{M}}_i, \qquad d = 4 - 2\varepsilon$$

- if all $\tilde{c}_i(d)$ are finite as $d \to 4$ then we have ε-finite basis
- if no restrictions are imposed on MI's ε-finite basis can be constructed

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Algorithm to construct ε-finite basis

(1) Initialize basis $\{\mathcal{B}\} = \{\mathcal{M}\}$
(2) Find some int. $\mathcal{I} = \sum_{i=1}^{M} r_i(d)\mathcal{B}_i$, where $r_k(d)$ has max. power of $1/\varepsilon$
(3) Then set new basis as $\{\mathcal{B}\} \longrightarrow \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{k-1}, \mathcal{I}, \mathcal{B}_{k+1}, \ldots, \mathcal{B}_M\}$
(4) Repeat (2) (3) until you can find $\mathcal{I}$ with singular coeffient

- IBP reduction with FIRE5
  $\Longrightarrow$ (1,4,13) reduction tables in (1,2,3)-loops ($\sim$ 700MB)
- master integrals evaluation, sector decomposition with FIESTA4
  MC integration with Vegas (dimension up to 9, # of calls 10^8)
  $\Longrightarrow$ aim at relative precision 10^{-8} in master integrals
  (but actually rel. prec. 10^{-6} is reached)
  TODO: try different integration algorithms
- construction of (quasi) ε-finite basis
  $\Longrightarrow$ ε-finite bases in (1,2)-loops
  $\Longrightarrow$ (quasi) ε-finite bases in 3-loops: maximal pole $1/\varepsilon^1$
  TODO: ε-finite basis in 3-loops
- renormalization

cancellation of ε-poles

Renormalized $O(\alpha_s^3)$ contribution to the vector current matrix element (coefficient in front of $(\alpha_s/(4\pi))^3$, $N_c = 3$)

expansion in ε	value (old basis)
$1/\varepsilon^6$	0.0002 ± 0.0012
$1/\varepsilon^5$	0.001 ± 0.014
$1/\varepsilon^4$	0.008 ± 0.080
$1/\varepsilon^3$	0.01 ± 0.36
$1/\varepsilon^2$	0.1 ± 2.1
$1/\varepsilon^1$	0.2 ± 11.3
$O(1)$	67 ± 59

- poles $1/\varepsilon^{6,5,4}$ are spurious
  $\longrightarrow$ should be exactly zero even before any renormalization
- poles $1/\varepsilon^{3,2,1}$ should be zero (renormalized!)

cancellation of ε-poles

Renormalized $O(\alpha_s^3)$ contribution to the vector current matrix element (coefficient in front of $(\alpha_s/(4\pi))^3$, $N_c = 3$)

expansion in ε	value (old basis)	value (new basis)
$1/\varepsilon^6$	0.0002 ± 0.0012	0
$1/\varepsilon^5$	0.001 ± 0.014	0
$1/\varepsilon^4$	0.008 ± 0.080	0
$1/\varepsilon^3$	0.01 ± 0.36	0.0001 ± 0.0001
$1/\varepsilon^2$	0.1 ± 2.1	0.0005 ± 0.0021
$1/\varepsilon^1$	0.2 ± 11.3	0.005 ± 0.014
$O(1)$	67 ± 59	70.55 ± 0.11

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results for the formfactors in $\overline{\mathrm{MS}}$ scheme

for $SU(3)$ we have in Landau gauge ($a = \alpha_s/(4\pi)$):

$$
\begin{aligned}
F_1^S &= 1 + 0.645519a + a^2(48.4885 - 6.34686n_f) \\
&+ a^3(2396.22(25) - 417.877(2)n_f + 8.6454(6)n_f^2) \\
&+ O(a^4) \\
F_1^V &= 1 - 1.04174a + a^2(-9.51764 - 0.134112n_f) \\
&+ a^3(70.55(11) - 29.638(7)n_f^2 + 1.4762(1)n_f^2) \\
&+ O(a^4) \\
F_1^T &= 1 - 0.138749a + a^2(-17.4648 + 1.68503n_f) \\
&+ a^3(-474.565(61) + 60.854(8)n_f - 0.4856(3)n_f^2) \\
&+ O(a^4)
\end{aligned}
$$

3-loop RI/SMOM conversion for the mass

Conversion factor for the mass in Landau gauge ($a = \alpha_s/(4\pi)$):

$$m_R^{\overline{\mathrm{MS}}} = C_m^{\mathrm{RI/SMOM}} \; m_R^{\mathrm{RI/SMOM}}$$

$$\begin{aligned} C_m^{\mathrm{RI/SMOM}} &= 1 \\ &- a\,(0.6455188) \\ &- a^2\,(22.607688 - 4.01353947 n_f) \\ &- a^3\left(860.34(25) - 164.747(25) n_f + 2.1845(6) n_f^2\right) \\ &+ O(a^4) \end{aligned}$$

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for $n_f = 3$ and $\alpha_s/\pi \sim 0.1$ (corresp. approx. to the value at scale 2GeV)

$$1. - 0.0161 - 0.0066 - 0.0060 + \ldots$$

conclusions

- we establish the framework for the evaluation of bilinear quark operators in RI/SMOM scheme up to three loops
- the 3-loop matrix elements and the anomalous dimensions are obtained for the scalar, vector and tensor currents
- conversion factors for the mass $C_m^{\mathrm{RI/SMOM}}$, quark field $C_q^{\mathrm{RI/SMOM}}$ and tensor structure $C_T^{\mathrm{RI/SMOM}}$ are obtained at the 3-loop order