# Swampland distance conjecture (SDC)

Look at GR in 5D with $y \equiv x_5$ on an $S^1$:

$$\mathcal{M}_{4,1} = \mathcal{M}_{3,1} + S^1_y$$

$\rightarrow$ KK-compactification:

$$ds^2 = R_0^2 e^{-\frac{2}{3}\sigma(x)} \cdot dy^2 + e^{\frac{1}{3}\sigma(x)} \cdot d\vec{x}_4^{\,2}$$

$$\equiv G_{MN}\, dx^M dx^N \Big|_{M,N=0\ldots3,5}$$

$R = R_0 \cdot e^{-\frac{\sigma}{3}}$ radius of $S^1_y$

$$d\vec{x}_4^{\,2} = g_{\mu\nu}\, dx^\mu dx^\nu$$



$$\det G_{MN} = \det g_{\mu\nu} \cdot R_0^2 \cdot e^{\frac{2\sigma}{3}}$$

$$G^{MN} R_{MN} = G^{55} R_{55} + G^{\mu\nu} R_{\mu\nu}$$

$$= \frac{e^{\frac{2\sigma}{3}}}{R_0^2} \cdot R_{yy} + e^{-\frac{\sigma}{3}} \underbrace{g^{\mu\nu} R_{\mu\nu}}_{= {}^{(4)}R}$$

5D gravity action on $S^1$:

$$S = \frac{M^3}{2} \cdot \int d^5x \sqrt{-G}\, R$$

$$= \int d^4x \sqrt{-g} \cdot \frac{M^3}{2} \int_0^{2\pi} dy\, e^{\frac{\sigma}{3}} R_0 \cdot \left( e^{-\frac{\sigma}{3}}\, {}^{(4)}R + \ldots \right)$$

$$\Rightarrow M_p^2 = M^3 \cdot 2\pi R_0$$

work out Christoffels (or vielbeins):

$$\Gamma^R_{MN} = \frac{1}{2} G^{RS} \left( \partial_M G_{SN} + \partial_N G_{SM} - \partial_S G_{MN} \right)$$

only $\Gamma^\rho_{\mu\nu}$, $\Gamma^\rho_{55}$, $\Gamma^5_{\mu 5} \neq 0$

Curvature:

$$R_{55} = \partial_\rho \Gamma^\rho_{55} - \Gamma^R_{5P} \Gamma^P_{R5} + \Gamma^R_{RP} \Gamma^P_{55}$$

$$\Gamma^5_{\mu 5} = \frac{1}{2} G^{55} \partial_\mu G_{55}$$

$$\Gamma^\rho_{55} = -\frac{1}{2} G^{\rho\sigma} \partial_\sigma G_{55}$$



$$\Rightarrow R_{55} = -\frac{1}{4} G^{55} G^{\mu\nu} \partial_\mu G_{55} \partial_\nu G_{55}$$

$$S = \frac{M^3}{2} \int_{M_4 \times S^1} d^5x \sqrt{-G}\, R$$

$$= \int d^4x \sqrt{-g} \cdot \left[ \frac{M^3}{2} \int_0^{2\pi} dy \sqrt{G_{55}} \cdot e^{-\frac{6}{3}\sigma}\, {}^{(4)}R + \frac{M^3}{2} \int_0^{2\pi} dy \sqrt{G_{55}} \cdot G^{55} R_{55} \right]$$

$$= \int d^4x \sqrt{-g} \cdot \left[ \underbrace{\frac{M^3 2\pi R_0}{2}}_{= M_p^2} \cdot {}^{(4)}R + \frac{M^3 2\pi R_0}{2} \cdot \frac{e^{\sigma}}{R_0^2} \cdot R_{55} \right]$$

$$= \int d^4x \sqrt{-g}\Big[\frac{M_P^2}{2}\cdot {}^{(4)}R$$

$$- \frac{M_P^2}{8}\cdot\frac{e^{\sigma}}{R_0^2}\cdot\frac{e^{\frac{2\sigma}{3}}}{R_0^2}\cdot e^{-\frac{\sigma}{3}} g^{\mu\nu}\cdot\left(-\frac{2}{3}R_0^2 e^{-\frac{2\sigma}{3}}\partial_\mu\sigma\right)$$

$$\cdot\left(-\frac{2}{3}R_0^2 e^{-\frac{2\sigma}{3}}\partial_\nu\sigma\right)\Big]$$

$$= \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}{}^{(4)}R - \frac{M_P^2}{18}\partial_\mu\sigma\partial^\mu\sigma\right]$$

with : $R(x^\mu) = R_0\cdot e^{-\frac{\sigma(x^\mu)}{3}}$

$$= \frac{M_P^2}{2}\cdot\int d^4x\sqrt{-g}\left({}^{(4)}R - \frac{1}{2}\frac{\partial_\mu R\partial^\mu R}{R^2}\right)$$

radius $R(x^\mu)$ of $S^1$ is massless 4D scalarfield - modulus



$S^1$ extra-dim. allows a bit more :

$$\sigma(x^\mu) = 3\ln R \to \sigma(x^\mu, y)$$

$$\sigma(x^\mu, y) = \sigma(x^\mu) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\sigma^{(n)}(x^\mu)\cdot e^{i\frac{n\cdot y}{R}}$$

e.o.m. for $\sigma^{(n)}(x^\mu)$:

$${}^{(4)}\Box\,\sigma^{(n)} + \partial_y^2\sigma^{(n)} =$$

$$= {}^{(4)}\Box\,\sigma^{(n)} + \frac{n^2}{R^2}\sigma^{(n)} = \left({}^{(4)}\Box + m_n^2\right)\sigma^{(n)}$$

KK-tower of states $\sigma^{(n)}(x^\mu)$ with mass $m_n^2 = \frac{n^2}{R^2}$

$\phi = \frac{M_P}{3}\sigma$ : canonically normalized radial modulus scalar

$$\frac{M_P^2}{18}\partial_\mu\sigma\partial^\mu\sigma = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

$$[\sigma] = 1\,,\ [\phi] = \text{mass}$$

$\Rightarrow$ KK-tower of zero mode $\sigma(x^\mu)$ has masses scaling with $\phi$ as:

$$m_n = \frac{n}{R} = \frac{n}{R_0}\cdot e^{-\frac{\sigma}{3}} = \frac{n}{R_0}\cdot e^{-\frac{\phi}{M_P}}$$



so, if at $\phi = 0$ your EFT has a cutoff $\Lambda$ and there KK-tower has:

$$m_n(\phi=0) = n\cdot\Lambda\cdot e$$

$\Rightarrow$ for every $\Delta\phi = M_P$ one KK-state decreases mass exponentially below $\Lambda$ $\rightarrow$ EFT without KK-states invalid

<u>correct statement:</u> states come down one by one – e.g. for $\Delta\phi = 5M_P$ get 5 new states below $\Lambda$ ...

need to "integrate in" finite # of states at finite $\Delta\phi$ into EFT

$\rightarrow$ maybe EFT modified but working, maybe fails ...

drastic statement:

"an infinite tower comes down exp. fast at $\Delta\phi = M_P$

$\Rightarrow$ EFT fails at $\Delta\phi = M_P$"

... well ... (see above) ...

further observation:



QFT knows finite decoupling — can arrange

$$\frac{m_{KK}(\phi=0)}{\Lambda} \equiv \frac{m_{KK}^{(0)}}{\Lambda} \gg 1$$

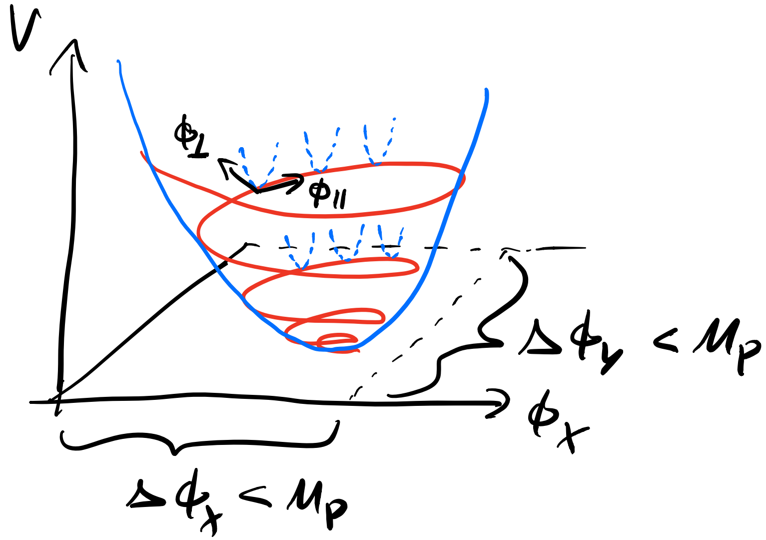
$\Rightarrow$ 1st KK-state decreases mass below $\Lambda$ only after $\Delta\phi = \ln\left(\frac{m_{KK}^{(0)}}{\Lambda}\right) \cdot M_P$

– what's wrong with that?

moreover – 2-field systems:

- 2 fields $\phi_x, \phi_y$
- canonical kinetic terms

• imagine potential:




geodesic distance $\Delta\phi$ in 2D field space always $< M_p$

but: terraces with deep trough in quadratic potential 'bowel' walls $\rightarrow m^2_{\phi_\perp} \gg m^2_{\phi_\parallel}$

('Dante's Inferno' by



Berg, Pajer & Sjörs 2009)

$\Rightarrow \Delta\phi_\parallel \gg M_p$ along trajectory

Hence, suddenly demanding that now $\Delta\phi_\parallel > M_p$ denotes a bad EFT, despite coming from $m^2_{\phi_\perp} \gg m^2_{\phi_\parallel}$, amounts to declaring that decoupling in QFT:

$$\frac{m^2_{\phi_\perp}}{m^2_{\phi_\parallel}} \rightarrow \text{large}$$

is forbidden! That's a

tall order:

Decoupling in QFT follows from the same axioms that both QFT and ST satisfy (Weinberg books) – so how can one expect ST to <u>generically</u> violate decoupling?



## Swampland de Sitter conjecture(s) (SdSC)

observation in type II string theories:

- compactify on CY ($R_{mn}=0$) or twisted tori ($R_{mn}<0$)

  $\boxed{\text{worldsheet CFT description}}$

- allow only classical stress energy sources:
  - fluxes
  - space-time filling D-branes
  - O-planes
  - negative curvature of compactification

$\Rightarrow$ only AdS vacua

or

dS critically points:

$$\partial_{\phi_i} V = 0, \; V > 0$$

$\llcorner$ moduli

with:

$$\text{min eig. val.} (\partial_{\phi_i} \partial_{\phi_j} V) \lesssim -V$$

e.g.: Hertzberg et al. 0711.2512

Flauger et al. 0812.3886

any many later ones...



counter example:

type IIB on Riemann surfaces with only classical sources gives stable dS vacua

(Saltman, Silverstein hep-th/0411271)

$\rightarrow$ if you discard this & any stabilizing quantum effects a la KKLT

$\rightsquigarrow$ conjecture:

$$V > 0 \Rightarrow |\vec{\nabla}_\phi V| > c \cdot V, \; c = \mathcal{O}(1)$$

(Obied, Ooguri, Spodyneiko & Vafa 1806.08362) if true: <u>no</u> single-field inflation without higher-derivative kinetic terms ('DBI/k-inflation') & <u>no</u> dS critical points ...

<u>however</u>: conjecture is <u>false</u> in our universe

... there is a Higgs!

$$V_H = \lambda(|H|^2 - v^2)^2, \quad V_H(|H| = v) = 0$$

$$\Rightarrow \Delta V_H = V_H(H=0) = \lambda v^4 \sim (100\,\mathrm{GeV})^4 \sim 10^{-64} M_P^4$$



at $H = 0$: $\frac{\partial V_H}{\partial H} = 0$

SdSC forbids that, so add quintessence scalar field:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - V_Q$$

$$V_\varphi = V_0 \cdot e^{-\lambda \cdot \phi}$$

$$V_0 \sim 10^{-120} M_P^4, \quad \lambda \lesssim 0.6$$

$$\Rightarrow \frac{|\partial_\phi V_Q|}{V_Q} = \lambda \sim O(1)$$

ok with (SdSC)

BUT: full potential to leading order

$$V = V_H(h) + V_\varphi(\phi)$$

$$\Rightarrow \left.\frac{|\vec{\nabla} V|}{V}\right|_{h=0,\,\phi} = \frac{\partial_\phi V_\varphi}{\Delta V_h} \sim \frac{10^{-120} M_P^4}{10^{-64} M_P^4} \sim 10^{-56} \ll 1$$

(Denef, Hebecker & Wrase 1807.05681)

so ... (SdSC) is wrong.



weakened conjecture:

$$\begin{cases} |\vec{\nabla} V| > c \cdot V, \quad c = \mathcal{O}(1) \\ \qquad \text{or} \\ \text{min eig.val}(\partial_{\phi_i}\partial_{\phi_j} V) < -\tilde{c} \cdot V \\ \text{with } \tilde{c} = \mathcal{O}(1) \end{cases}$$

$(\widetilde{\text{SdSC}})$

(Garg & Krishnan 1807.05193)

(Ooguri, Palti, Shiu & Vafa 1810.05506)

can only be argued in the limit: $V_{\text{extra-dim.}} \to \infty$

$g_s \to 0$

however, all controlled ST dS vacua like e.g. KKLT or LVS work out:

$\mathcal{V}, g_s$ finite

so $(\widetilde{SdSC})$ does not apply, but may have something about ST vacua close to the boundary of moduli space ($\mathcal{V}\to\infty, g_s\to 0$).



## Addendum

Further support for SDC in $\mathcal{N}=2$ CY compactifications of type II ST

$\rightsquigarrow$ question:

For constant volume moduli, all KK-masses are constant, but can still try to take complex structure moduli to large VEVs ... where is the SDC-tower?

Look at c.s. moduli space $\mathcal{M}^{2,1}$

of CYs, example quintic:

$$P: x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$$

has: $h^{2,1} = 101$ c.s. moduli

$\hat{=}$ 101 3-cycles and their Poincaré duals

integral basis of 3-cycles:

$$A_i \cap B^j = -B^j \cap A_i = \delta_i{}^j$$

$$A_i \cap A_j = B^i \cap B^j = 0$$

CY has holomorphic 3-form

$$\Omega_{3,0} \equiv \Omega$$



dual cohomology basis $(\alpha_i, \beta^j)$ of 3-forms on a CY:

$$\int_{A^j} \alpha_i = \int \alpha_i \wedge \beta^j = \delta_i{}^j, \quad \int_{B_j} \beta^i = \int \beta^i \wedge \alpha_j = -\delta^i_j$$

expand $\Omega$:

$$\Omega = Z^i \alpha_i - F_j \beta^j$$

where we define projective 'special' holomorphic coord.s on $\mathcal{M}^{2,1}$:

$$Z^i = \int_{A_i} \Omega, \quad F_i(Z^j) = \int_{B^i} \Omega$$

metric $G_{i\bar{j}}$ on $\mathcal{M}^{2,1}$ given by:

$$g^{2,1}_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\bar{z}^{\bar{j}}} K^{2,1}$$

$$(*)\quad K^{2,1} = -\ln\left[-i\left(z^i \bar{F}_i - \bar{z}^i F_i\right)\right]$$

for certain choice of $z^i$ the limit $z^i \to 0$ implies:

$$P = 0, \quad \frac{\partial P}{\partial z^i} = 0 \quad \forall i$$

This is a singularity of the CY of conifold type.

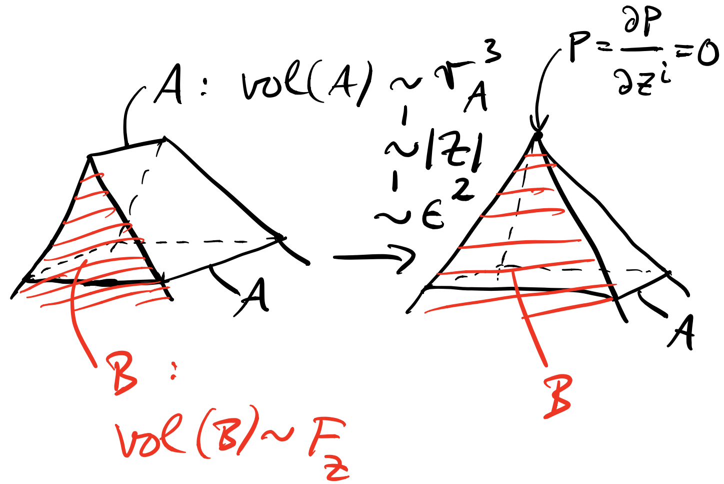
Call the single $z^i$ for which this happens at $z^i \to 0$ : $Z$.



$\Rightarrow$ eq. of deformed conifold:

$$\sum_{i=1}^{4} z_i^2 = Z \equiv \epsilon^2 > 0$$

Can make a sketch:




$$Z > 0 \quad \longrightarrow \quad Z = 0$$

in local 1D moduli space of $Z$, $Z = 0$ is co-dim. = 1 and thus a point.

Go around $z=0$ in moduli space: $A \cap B = 1$ must stay unchanged, hence can have:

$$\underbrace{A \to A}_{\text{go once}}, \quad \underbrace{B \to B - A}_{\text{around } z=0}$$

"monodromy"

$\Rightarrow z \to z, \; F_z \to F_z - z$

can write monodromy transf. as:

$$\begin{pmatrix} F_z \\ z \end{pmatrix} \to T \cdot \begin{pmatrix} F_z \\ z \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$



since $F_z = F_z(z)$, monodromy $T$ must arise as:

$$T F_z - F_z = \oint_{z=0} F_z \overset{!}{=} -z$$

$$\Rightarrow F_z = \frac{1}{2\pi i} z \cdot \ln z + f(z)$$

$f(z)$: holomorphic, $f(z) = 0$ for $z = 0$

from CY holomorphic 3-form $\Omega_{3,0}$ also $0$-period pair:

$$z_0 = \int \Omega_{3,0}, \quad F_{z_0} = g(z_0)$$

$g(z_0)$: holomorphic for $z_0 > 0$

$$\Rightarrow K^{2,1} = -\ln\Big[\underbrace{-2i\cdot \mathrm{Im}\big(z_0 \bar{F}_{\bar{z}_0}\big)}_{= \mathrm{Im}\, z_0 \bar{g}(\bar{z}_0)} - z\bar{z}\ln z\bar{z}\Big]$$

one can show (see Appendix) that:

$$-2i\cdot \mathrm{Im}\big(z_0 \bar{F}_{\bar{z}_0}\big) \equiv c > 0$$

$$\Rightarrow K^{2,1} \xrightarrow[\epsilon \sim \sqrt{|z|} \to 0]{} \frac{z\bar{z}}{c}\cdot \ln z\bar{z}$$

$$\Rightarrow G_{z\bar{z}} = \frac{1}{c}\big(2 + \ln z\bar{z}\big)$$

$$\to -\infty \text{ for } z \to 0$$

and: $\mathcal{R}(G_{z\bar{z}}) = G^{z\bar{z}}\partial_z \ln G_{z\bar{z}} \sim \dfrac{1}{|z|\cdot(\ln|z|+1)^2}$

$$\to \infty \text{ for } z \to 0$$



$z=0$ is a curvature singularity of $\mathcal{M}^{2,1}$

however, note 2 things:

- geodesic distance to singularity

$$d_z = \int_0^{|z|^{1/3}} dr \sqrt{-G_{z\bar{z}}}\ , \quad r = \sqrt{z\bar{z}}$$

$$\sim \sqrt{\frac{2}{c}} \int_0^{|z|^{1/3}} dr \sqrt{1 + \ln r} < \infty \text{ finite}$$

- form of $F_z$ reminiscent of integrating a <u>massless</u> state

consider a D3-brane wrapping $A^i$ $n^i$ times and $B_j$ $p_j$ times if $\mathcal{C}$ is a Lagrangian cycle:

$$\mathcal{C} = n^i A^i - p_j B_j$$

with Poincare dual form:

$$\Gamma = n^i \alpha_i - p_j \beta^j$$

$\Rightarrow$ D3 is a BPS state:

$$M = e^{\frac{1}{2}K^{2,1}} \left| \int_{\mathcal{C}} \Omega \right|$$

$$= e^{\frac{K^{2,1}}{2}} \cdot \left| \int_{CY} \Omega \wedge \Gamma \right|$$

$$= e^{\frac{1}{2}K^{2,1}} \left| n^i F_i - p_j z^j \right|$$

set $n^i = 0$, $p_j = 0$ $\forall z_j \neq z$



$$\Rightarrow M_{D3 \text{ on } A} = e^{\frac{1}{2}K^{2,1}} |p| \cdot z$$

$\Rightarrow$ for $z \to 0$ $\quad M_{D3 \text{ on } A} \to 0$

1 D3 BPS state wrapping A gets massless when approaching singularity $\to$ integrating in this light state resolves it.

$\Rightarrow$ guess: put $n$ conifolds on top of each other ...

$\rightsquigarrow$ monodromy $T = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

monodromy order $\to \infty$ for $n \to \infty$

$\rightsquigarrow$ distance to joint singularity $\sim n \to \infty$ for $n \to \infty$

$\rightsquigarrow$ $n$ D3-branes wrapping the $n$ copies of the A-cycle get light for $z \to 0$

$\Rightarrow$ for $n \to \infty$ singularity is at infinite distance; approaching it, an infinite tower of wrapped D3 BPS states gets light, which arises from an infinite-order monodromy around the singular point in $\mathcal{M}^{2,1}$.

$\rightsquigarrow$ work by Grimm, Palti, Valenzuela since 2018 spells this out



and constructs and attempts to classify all infinite distance singularities of c.s. moduli space using their infinite-order monodromies to reconstruct the metric at the singularity as well as the BPS tower & show that the tower masses respect the SDC when moving large distances in c.s. moduli space upon approaching the singularity.

## Final comment:

Cecotti showed in his book that for $\mathcal{N} \geq 4$ SUSY all swampland conjectures can be proven.

$\rightsquigarrow$ Sooo... maybe they are an artifact of too much SUSY?



## Appendix

The finiteness of the distance in c.s. moduli space to the conifold singularity rests crucially on the non-zero constant c in $e^{-K^{2,1}}$. We sketch here one of the arguments due to Candelas, Green & Hübsch (1990) proving $c \neq 0$:

First we note that expression

(*) for $K^{2,1}$ can be written as:

$$K^{2,1} = -\ln\left(-i\int_{\mathcal{M}_z} \Omega\wedge\bar{\Omega}\right)$$

where $\mathcal{M}_z$ denotes the deformed conifold with deformation $z = \epsilon^2$ as above. Hence, we have

$$c>0 \iff -i\int_{\mathcal{M}_z}\Omega\wedge\bar{\Omega} > 0 \quad \forall z \geq 0$$



We now use the presence of conifolds in the moduli space of the quintic $CY_3$:

$[4||5]$

Consider the ineffective splitting into:

$$\left[\begin{array}{c||cc} 4 & 4 & 1 \\ 1 & 1 & 1 \end{array}\right] \begin{array}{l} (z_1,\dots,z_5) \\ (y_1,y_2) \end{array}$$

with the defining polynomials as:

$$\begin{aligned} x(z_i)\,y_1 + u(z_i)\,y_2 &= 0 \\ y(z_i)\,y_1 + v(z_i)\,y_2 &= 0 \end{aligned} \qquad (P)$$

with: $\begin{cases} \deg X = \deg U = 4 \\ \deg \underline{Y} = \deg V = 1 \end{cases}$

Since $y_1 \neq 0 \vee y_2 \neq 0$ on $\mathbb{P}^1$, enforcing (P) requires:

$$\ell \equiv \det \begin{pmatrix} X & U \\ V & \underline{Y} \end{pmatrix} = X\underline{Y} - UV = 0$$

we have a nodal singularity $\ell = d\ell = 0,\ d^2\ell \neq 0$ for:

$$X = \underline{Y} = U = V = 0$$



Convenient choice of $X, \underline{Y}, U, V$:

$$\overbrace{\left(z_2^4 + z_4^4 - z_5^4\right)}^{X} y_1 + \overbrace{\left(z_1^4 + z_3^4 + z_5^4\right)}^{U} y_2 = 0$$

$$\underset{\substack{\uparrow \\ V}}{z_1} y_1 + \underset{\substack{\uparrow \\ \underline{Y}}}{z_2} y_2 = 0$$

$$\Rightarrow \begin{cases} z_1 = z_2 = 0 \\ z_3^4 + z_5^4 = 0 \\ z_4^4 - z_5^5 = 0 \end{cases} \quad \left(\begin{array}{c} 4 \times 4 \text{ zeros} \\ \text{in } z_3, z_4 \end{array}\right)$$

$\Rightarrow \quad z_3, z_4, z_5 \neq 0$

$\rightsquigarrow$ choose $z_5 = 1 \Rightarrow$ 16 nodes

On the $\mathbb{P}^1$ patch $y_1 \neq 0$ we can now eliminate the $\mathbb{P}^1$ coordinates to get:

$$\ell = z_2 \cdot (z_2^4 + z_4^4 - z_5^4) - z_1 (z_1^4 + z_3^4 + z_5^4) = 0$$

The deformation is now described by:

$$\ell = 0 \to M_z : q = \ell + \epsilon(z_i) = 0$$

$$\epsilon(z_i) = |z| \cdot p \,,\; p \equiv \sum_{i=1}^{5} z_i^5$$



$\Rightarrow$ $q = dq = 0$ has no solution, in particular $U = V = X = Y = 0$ where $\ell = d\ell = 0$ has $dp \neq 0$

$\Rightarrow$ $q = 0$ is non-singular

Now, cut out small 3-balls $B_k$ around (deformed) node loci, big enough to contain the deformation $S^3$.

Hence we have:

$$-i\int_{\mathcal{M}_z} \Omega \wedge \bar{\Omega} > -i\int_{\{q=0\}\setminus\{B_k\}} \Omega \wedge \bar{\Omega} \quad (i)$$

since: $-i\int_{\mathcal{M}_z} \Omega \wedge \bar{\Omega} \sim \|\Omega\|^2 \cdot \mathcal{V}$

and: $\mathcal{V} = \text{volume}(\mathcal{M}_z) > 0$

while the norm:

$$\|\Omega\|^2 \geq 0$$

cutting out 3-balls from



$\mathcal{M}_z$ reduces $\mathcal{V}$, and thus $\|\Omega\|^2 \mathcal{V} \Rightarrow (i)$.

Now, on $\{q=0\}\setminus\{B_k\}$ we can safely take $|z| = 0$ and hence in the $\mathbb{P}^4$-patch $z_1 \neq 0$ $\Omega_{3,0}$ takes the form:

$$\Omega_{3,0}^{\{q=0\}\setminus\{B_k\}} = \frac{z_1 \cdot dz_2 \wedge dz_3 \wedge dz_4}{\partial q / \partial z_5} = \frac{dz_2 \wedge dz_3 \wedge dz_4}{4 \cdot (z_5)^3}$$

Since $\{q=0\}\setminus\{B_k\}$ is smooth, we have:

$$\frac{\partial q}{\partial z_5} \neq 0 \text{ on } \{q=0\}\setminus\{B_k\}$$

$\Rightarrow$ $\Omega_{3,0}^{\{q=0\}\setminus\{B_k\}}$ is non-vanishing on $\{q=0\}\setminus\{B_k\}$.



Hence:

$$\Rightarrow -i\int_{\{q=0\}\setminus\{B_k\}} \Omega \wedge \overline{\Omega} \neq 0 \quad \forall |z| \geq 0$$

$$\wedge \quad -i\int_{\{q=0\}\setminus\{B_k\}} \Omega \wedge \overline{\Omega} \sim \|\Omega\|^2 \cdot V \geq 0$$

$$\Rightarrow -i\int_{\{q=0\}\setminus\{B_k\}} \Omega \wedge \overline{\Omega} > 0 \quad \forall |z| \geq 0$$

by (i) this implies:

$$-i \int_{\mathcal{M}_z} \Omega \wedge \overline{\Omega} > 0 \quad \forall\, |z| \geq 0$$

$\Rightarrow \quad C > 0 \qquad$ q.e.d.