Notes on 4-forms

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This note is about the physics of 4-forms on a 4-dimensional manifold, and their use to address the cosmological constant problem and the Higgs Hierarchy problem. This note do not pretend to be complete nor correct, and represent just the partial understanding of the author at this moment.

1 The "dynamics" of 4-forms

1.1 Basics and conventions

Consider a differential manifold M, with $\dim M = 4$, which will be our spacetime. We define on M a 3-form gauge potential A_3, which in component reads

$$A_3 = \frac{1}{3!} A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \tag{1}$$

and which has a gauge symmetry

$$A_3 \to A_3 + d\Lambda_2. \tag{2}$$

We then define a 4-form field strength as

$$F_4 = dA_3 \tag{3}$$

which in components reads

$$F_4 = \frac{1}{4!} F_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{4a}$$

$$F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} - \partial_\sigma A_{\mu\nu\rho} + \partial_\rho A_{\sigma\mu\nu} - \partial_\nu A_{\rho\sigma\mu} \tag{4b}$$

and notice immediately that F_4 is invariant under (2), as $d^2 = 0$

We would like to make this field dynamical, so we declare an action for this, which is just an analog of Maxwell action but for a higher form-field.

$$S_{kin} = \frac{1}{2} \int d^4x \; F \wedge \star F \tag{5}$$

which in component reads

$$S_{kin} = \frac{1}{2 \cdot 4!} \int d^4x \sqrt{-g} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \tag{6}$$

1.2 Equation of motion

We can set to zero the variation of the action (5) with respect to A_3 and compute the equations of motion. They read

$$D_\mu \left(\sqrt{-g} F^{\mu\nu\rho\sigma}\right) = 0 \tag{7}$$

This can be shown to be equivalent to

$$\partial_\mu \left(\sqrt{-g} F^{\mu\nu\rho\sigma}\right) = 0 \tag{8}$$

Now we solve this equation. We notice that since F is completely antisymmetric, it has off-shell a single degree of freedom. We can therefore write

$$F^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}} f(x) \epsilon^{\mu\nu\rho\sigma} \tag{9}$$

and the equation of motion will be a differential equation for $f(x)$:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu f(x) = 0 \ . \tag{10}$$

Equation (10) can be contracted with an epsilon symbol on both sides, giving

$$\partial_\mu f(x) = 0 \ , \tag{11}$$

which implies that $f(x)$ is constant over spacetime. It can be shown that this is the only solution. This in turn implies that

$$F^{\mu\nu\rho\sigma}(x) = \frac{c}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} \tag{12}$$

and lowering the indices with the metric we get that

$$F_{\mu\nu\rho\sigma}(x) = c\sqrt{-g}\epsilon_{\mu\nu\rho\sigma} \tag{13}$$

Introducing the Levi-Civita tensors

$$\mathcal{E}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} \tag{14a}$$

$$\mathcal{E}_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon^{\mu\nu\rho\sigma} \tag{14b}$$

we see that we can write in a more compact way

$$F^{\mu\nu\rho\sigma}(x) = c\mathcal{E}^{\mu\nu\rho\sigma} \tag{15a}$$

$$F_{\mu\nu\rho\sigma}(x) = c\mathcal{E}_{\mu\nu\rho\sigma} \tag{15b}$$

In this section we have learned a first important fact: *The equation of motion of F_4 imply that F_4 is simply proportional to the determinant of the metric. It has zero propagating degrees of freedom on-shell.*

This fact is commonly referred to as F_4 *is constant*, in the literature.

Up to what said so far, the constant c can be any real number. We will see later that only some specific values of c are allowed.

1.3 Contribution of F to the cosmological constant

So far we considered the dynamics of the 4-form F_4 in a curved but static gravitational background. Now we will couple this field to gravity, namely we consider an action given by the Einstein-Hilbert action plus the action S_{kin} of above. This is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa_4} R - \Lambda_0 - \frac{1}{2 \cdot 4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right) + S_{GHY} + S_b \ . \tag{16}$$

The terms S_{GHY} and S_b are respectively the Gibbons-Hawking-York term, and a boundary term for F_4. They must be added for consistency, but they do not affect the equations of motion and play no role in this discussion. (However they are crucial for the action to be well-defined and *they do* affect the physics in other circumstances).

The equation of motion of F_4 stay the same, while the equation of motion of the metric will be of course given by Einstein's field equation. Now we will integrate out F_4 at tree level. Namely we take its equation of motion, we solve it, and we plug it back in the action, finding a new effective action for the metric. Notice that since

$$F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} = c^2 \mathcal{E}^{\mu\nu\rho\sigma} \mathcal{E}_{\mu\nu\rho\sigma} = -24 c^2 \ . \tag{17}$$

the new effective action will be

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa_4} R - \Lambda_{eff} \right) \tag{18}$$

with

$$\Lambda_{eff} = \Lambda_0 - \frac{c^2}{2} \tag{19}$$

Here the crucial point is the following. *The value of the cosmological constant that we measure is Λ_{eff}. There is no way to measure independently Λ_0 and c. Therefore, it is possible that Λ_0 is large and c is also large and close to Λ, such that a big cancellation happens. If there were a nice mechanism to explain why c and Λ_0 take a similar value, this would resolve the cosmological constant problem.*

This fact was first pointed out by Hawking in [1]. Note also that at this point we said nothing about the possible values that c can take, so in principle one could think that we can set to zero Λ_{eff} by sufficiently tuning c somehow.

2 Coupling to membranes

In the discussion so far we neglected a crucial ingredient. A 3-form gauge potential couples naturally to a membrane!

A membrane by itself will have an action given by

$$S_{brane} = -T \int d^3\xi \sqrt{-\gamma} \tag{20}$$

$$S_{int} = e \int_W A_{\mu\nu\rho} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \frac{\partial x^\rho}{\partial \xi^c} \epsilon^{abc} \tag{21}$$

Let us compute now the equations of motion of F_4 in the presence of the membrane. To do that we need to compute the variation with respect of A of the action

$$S = \int F \wedge \star F + S_{int} \tag{22}$$

We see that S_{kin} is an integral over 4d space, while S_{int} is an integral over the brane worldvolume. By a standard trick we can rewrite

$$S_{int} = e \int d^4x \int d^3\xi \delta^{(4)}(x - x(\xi)) \; A_{\mu\nu\rho} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \frac{\partial x^\rho}{\partial \xi^c} \epsilon^{abc} \tag{23}$$

so now the equation of motion will read

$$\partial_\mu \left(\sqrt{g} F^{\mu\nu\rho\sigma}\right) = -e \int d^3\xi \delta^{(4)}(x - x(\xi)) \frac{\partial x^\nu}{\partial \xi^a} \frac{\partial x^\rho}{\partial \xi^b} \frac{\partial x^\sigma}{\partial \xi^c} \epsilon^{abc} \; . \tag{24}$$

From this we learn a crucial fact: c is constant on both sides of the membrane, but the values will be different. Let us compute the amount of the change of c. Suppose that we are in Minkowski spacetime now, and we have a brane at $x^3 = 0$, meaning the brane spans completely x^0, x^1, x^2. Let us evaluate this expression. We have 3 open indices in the left and in the right, so let us evaluate it for $\mu = 3, \nu = 0, \rho = 1, \sigma = 2$. We get to

$$\partial_3 c = -e \int d^3\xi \delta^4(x - x(\xi)) \frac{\partial x^0}{\partial \xi^a} \frac{\partial x^1}{\partial \xi^b} \frac{\partial x^2}{\partial \xi^c} \epsilon^{abc} \tag{25}$$

Now in order to do the derivatives in the RHS we need to use 3 out of the 4 deltas to set x as a function of ξ.

$$\begin{aligned} \int d^3x \partial_3 c &= -e \int d^3\xi \delta(x^3 - x^3(\xi)) \frac{\partial x^0(\xi)}{\partial \xi^a} \frac{\partial x^1(\xi)}{\partial \xi^b} \frac{\partial x^2(\xi)}{\partial \xi^c} \epsilon_{abc} \\ &= -e \int d^3\xi \delta(x^3 - x^3(\xi)) \delta^0_a \delta^1_b \delta^2_c \epsilon^{abc} = \\ &= -e \int d^3\xi \delta(x^3 - x^3(\xi)) \end{aligned} \tag{26}$$

From this we find

$$\begin{aligned} \text{vol} \cdot \int_{-\epsilon}^{\epsilon} dx^3 \partial_3 c = -e \int d^3\xi \int_{-\epsilon}^{\epsilon} dx^3 \delta(x^3 - x^3(\xi)) = \\ = -e \int d^3\xi = -e \cdot \text{vol} \end{aligned} \tag{27}$$

So we have

$$c(\epsilon) - c(-\epsilon) = -e \tag{28}$$

Taking now the limit for $\epsilon \to 0$ we find that

$$\Delta c = e \tag{29}$$

This computation was done under two assumptions

1. Minkowski background
2. Brane at $z = 0$

It can be shown that the result is the same regardless of these hypothesis, namely $\Delta c = ne$ also in curved backgrounds.

3 4-form quantization

Up to now we have said nothing at all about the value of the constant c. We have only said that it is constant in any region of space delimitated by a brane (domain wall) and jumps in a discrete way when such defect is crossed. However, it was found that the value of c is quantized [2].

$$c = en, \quad n \in \mathbb{Z} \tag{30}$$

This must be true if the 4d theory comes as an effective theory from Strings. Infact in String Theory both RR-forms and their magnetic duals obey all a Generalized Dirac quantization for forms.

In particolar, for every RR field-strength F_p it must happen that

$$\int_{\Sigma_p} F_p = \text{const} \cdot n, \quad n \in \mathbb{Z} \tag{31}$$

In general one can ask why we needs such quantization condition. For forms F_p with $p > 0$ this is equivalent to a consistency condition on the $(p-2)$brane amplitudes. However the case of F_0 is more tricky as it not clear what is a (-2) brane. Let us postpone this discussion, and assume for the moment that (31) holds true also for $p = 0$. We then would consider

$$F_0 = \star F_4 = c \tag{32}$$

and 0-dimensional cycles are just points. So we get

$$\int_{\Sigma_0} F_0 = F_0|_{pt} = \text{const} \cdot n \tag{33}$$

where with $F_0|_{pt}$ we denoted F_0 evaluated at a point. A short computation can be done to find the value of the constant, and we get to

$$c = en, \quad n \in \mathbb{Z} \tag{34}$$

4 Membrane nucleation

In the quantum theory the configurations of fixed n are unstable, and they can tunnel into each other. A phenomenon called *membrane nucleation* can occurr, in which a membrane bubble appears, and start immediately expanding at the speed of light. This is nothing exotic, it is simply the Schwinger effect. The membrane that appeared now separates the old region with $c = ne$ from a new region with $c = (n-1)e$ inside. The decrease of the energy of the field configuration in the new region with $c = (n-1)e$ is expected as the "missing energy" is actually just stored in the membrane boundary. The tunneling rates among various solutions have been studied.

In principle there are two possibilities for transitions

1. Transition that decrease n. They have a very large rate.

2. Transition that increase n. They still are possible due to a mechanism called *gravitational instanton*, but with an extremely smaller rate.

Consider a initial situation in which we have a patch of the Universe with a given $c = ne$. Then a chain of transition decreasing n in steps will happen, therefore reducing the effective cosmological constant step by step. This cascade will stop when we get to zero effective cosmological constant, or negative one.

Consider now a case in which we have *many* causally disconnected patches U_i in the Universe, in each one of them, the initial value of Λ_0 is different, say Λ_{0i}, and in each one of them there is this cascade process of lowering the cosmological constant. Then at the end of the process, in every patch there will be a different value of $\Lambda_{eff,i}$. One can then argue by using anthropic arguments that we live in the patch in which Λ_{eff} takes the measured value.

This process was initially introduced in [3] as a possible solution to the cosmological constant problem. We see immediately three crucial points

1. In order to solve the cc problem in this way, a necessary condition is that the "steps"-size should be comparable with Λ_{eff}. So they must be very small. Therefore we need a very small e.

2. In order for our Universe to be metastable, the transition from our vacuum into one with even lower cc (and possibly even AdS) must be longer then the life of the universe.

3. It relies on anthropics.

5 Coupling to the Higgs

Now we consider coupling the 4-form to the Standard Model Higgs

$$V_H = -\left(M_0^2 + \frac{y}{24}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\right)|H|^2 + \lambda|H|^4 \tag{35}$$

where M_0 is a mass $M_0 \sim M_{UV}$ and y is a dimensionless coupling constant. With this new term in the lagrangian, we now repeat the analysis of before. The equation of motion for F and H will read

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(c - \frac{y}{2}h^2\right) = -e\int d^3\xi\delta^4(x - x(\xi))\frac{\partial x^0}{\partial\xi^a}\frac{\partial x^1}{\partial\xi^b}\frac{\partial x^2}{\partial\xi^c}\epsilon^{abc} \tag{36a}$$

$$\Box h = (-M_0^2 + yc)h + \lambda h^3 \tag{36b}$$

let us consider the vacuum configuration

$$h = \langle h\rangle = v \ , \tag{37}$$

(where we took $H = (0, h/\sqrt{2})$).

From the analysis of above we now have if we cross a membrane

$$\left(\Delta c - \frac{y}{2}\Delta h^2\right) = e \tag{38}$$

We further know that $c = ne$ from the Dirac quantization condition. It follows that also $-\frac{y}{2}h^2$ must be quantized in the same way. We get to

$$\left(c - \frac{y}{2}h^2\right) = ne, \quad n \in \mathbb{Z} \tag{39}$$

5.1 Scanning at the same time Higgs and Λ_{eff}

From equations (38) and (39) together, we can see now that we can have the following solutions

1. Before EW breaking.

$$c = en \tag{40a}$$
$$\langle v^2 \rangle = 0 \tag{40b}$$

2. After EW breaking

$$c = \frac{yM_0^2 + 2\lambda en}{y^2 + 2\lambda} \tag{41a}$$
$$\langle v^2 \rangle = \frac{2(M_0^2 + \lambda en)}{y^2 + 2\lambda} \tag{41b}$$

We can compute in each of these vacua what is the value of the effective cosmological constant Λ_{eff}, and the Higgs mass parameter μ_H^2 (defined such that the Higgs potential is $V_H = \mu_H^2|H|^2 + \lambda|H|^4$)

1. Before EW breaking.

$$\Lambda_{eff} = \Lambda_0 - \frac{1}{2}e^2n^2 \tag{42a}$$
$$\mu_H^2 = -M_0^2 + yen \tag{42b}$$

2. After EW breaking

$$\Lambda_{eff} = \Lambda_0 - \frac{\lambda e^2n^2 + yM_0^2en - M_0^4/2}{y^2 + 2\lambda} \tag{43a}$$
$$\mu_H^2 = \frac{2\lambda(yen - M_0^2)}{y^2 + 2\lambda} \tag{43b}$$

We can now re-run the argument of membrane nucleation and the scan. We see that dynamically both the cosmological constant relaxes *and* the Higgs mass parameter relaxes, as the four-form field decharges itself via Schwinger effect. The papers [4] and [5] independently pointed out at this mechanism. For the solutions, they both rely on anthropics.

6 Coupling to an axion

Consider now the case in which our 4-form is defined in a non-gravitational background of Minkowski space, but it couples with an axion. We will see that a nice phenomenon will occur, known as *axion monodromy.*

The axion is a real scalar field with a discrete shift symmetry

$$\phi \to \phi + mf, \ m \in \mathbb{Z} \tag{44}$$

where f is the axion decay constant. We take the lagrangian to be

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2 \cdot 4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{\mu}{4!} \phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \tag{45}$$

Again we can compute the equation of motion for F and we can integrate F out. We obtain a potential for ϕ of the form

$$V(\phi) = \frac{1}{2}(c + \mu\phi)^2 \tag{46}$$

We notice that due to this potential the axion ϕ gets massive, with mass μ. However, even if the axion is massive, the shift symmetry is not broken. One can infact compensate the shift of the axion with a shift of the $4-form$. Shift at the same time

$$\begin{cases} \phi \to \phi + nf, & n \in \mathbb{Z} \\ c \to c - n\mu f, & n \in \mathbb{Z} \end{cases} \tag{47}$$

and the whole potential stay invariant.

However, from the previous discussion we know that in two zones separated by a n membranes we should have $c \to c+ne$. This poses a consistency condition

$$e = \mu f \tag{48}$$

We found the following: *If the coupling constant of the brane is equal to the product of the axion mass and its decay constant, then the axion will retain the shift symmetry even if it is massive. Therefore its mass is protected from radiative corrections.*

This is the starting point for models of axion monodromy inflation.

Some comments are due

1. The joint shift symmetry of the axion and the 4-form flux is a symmetry of the lagrangian. However, it is spontaneusly broken by the choice of a vacuum. (This is trivial, changing c will change the vacuum).

2. If the axion has a normal mass term (not coming from the 4-form) then the shift symmetry is completely broken.

References

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