The $O(\alpha^2)$ initial state QED corrections to $e^+e \rightarrow \gamma^* Z^*$ at very high luminosity colliders

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based on:

J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B791 (2019) 206-209
J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B801 (2020) 135196 and in preparation.

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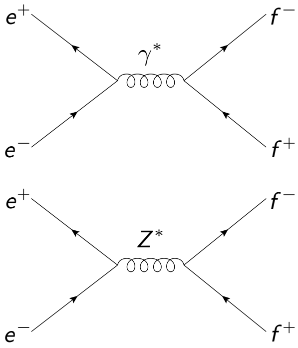
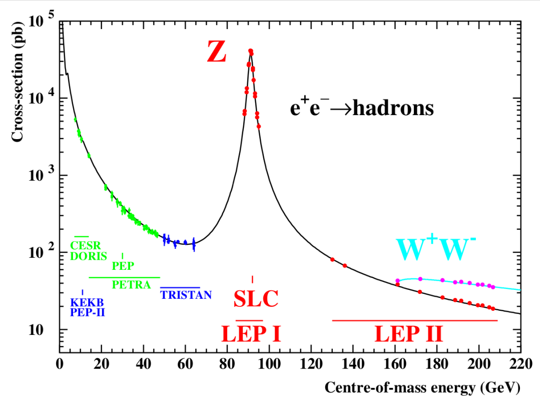
Introduction





- These corrections are important for the prediction of the Z-boson peak and for $t\,\bar{t}$ production at LEP, ILC and FCC-ee, and at Higgs factories through $e^+\,e^- \to Z^*\,H^0$.
- We revisit the initial state corrections to $e^+\,e^-$ annihilation to a (virtual) neutral vector boson, since only one comprehensive calculation existed: Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))

We look at the process:

$$e^- + e^+ \to \gamma^*/Z^* \to f^- + f^+$$

with the invariants

$$(p_- + p_+)^2 = s, \qquad p_-^2 = p_+^2 = m_e^2, \qquad (p_f + p_{\bar{f}})^2 = q^2 = s'$$

The initial state radiation (ISR) of n particles can be described by:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s'} = \frac{\sigma^0(s')}{4s}\mathcal{H}\left(\alpha, z \equiv \frac{s'}{s}, \rho \equiv \frac{m_e^2}{s}\right),$$

where $\sigma^0(s')$ describes the leading order process $e^+ e^- \to f\,\bar{f}$ and $\mathcal{H}(\alpha, z, \rho)$ radiator functions described by the Drell-Yan process with massive initial states, $\rho = m_e^2/s$,

$$\mathcal{H}(\alpha, z, \rho) = \delta(1-z) + \sum_{k=1}^{\infty}\left(\frac{\alpha}{4\pi}\right)^k \sum_{l=0}^{k} h_{kl} \ln^l(s/m_e^2)$$

The History

- Numerous calculations at $O(\alpha)$ are in agreement, since about the time of PETRA and PEP.
- First higher order universal [leading log] results are calculated using QED-factorization.
- The accuracy reached at LEP requires the $O(\alpha^2)$ corrections
- 1987: First $O(\alpha^2)$ calculation Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))
- 1990: $O(\alpha_s^2)$ QCD corrections to the massless Drell-Yan process Hamberg, van Neerven, Matsuura, (Nucl.Phys. B359 (1991))
  [Contains processes yet missing in Berends et al. (1987) and accounts for γ_5 correctly]
- $\sim$ 1996: A first attempt to perform the $O(\alpha^2)$ corrections using massive OMEs; Mertig, van Neerven, Scharf; led to `tarcer`. Calculation did not converge.

W.L. van Neerven: "I will get into all this, after they have gotten fermion-number conservation."

- 2002: A new start: JB, De Freitas, van Neerven.
- At this time modern integration technologies for massive 2- and higher loop calculations, also in the inclusive case, did not yet exist. We just had harmonic sums and harmonic polylogarithms Vermaseren 1998, JB and Kurth 1998, Vermaseren and Remiddi 1999.
  We integrated in a rather baroque and lengthy way, applying a lot of tricks. In a way it was painful, but we wanted to get the result.
- 2007: Willy worked on the project until one hour before he died and we still had exchanged e-mails. The work has been done shared at Leiden, Caracas and Zeuthen, e-mail and phone assisted.
- 2008: We finished the calculation, but disagreed with Berends et al. in all the non logarithmic terms at $O(\alpha^2)$; this also applied to Kniehl, Krawczyk, Kühn, Stuart, Phys. Lett. B209 (1988)
  We tried to clarify these differences for 3 years, without success.
- 2011: JB, De Freitas, van Neerven, Nucl.Phys. B855 (2012)
  Does the massive Drell-Yan process not factorize from 2–loop onward?
  (There has been a proof of factorization by J. Collins.)
- $>$ June 2018: After having a lot of experience in massive 2- and 3-Loop calculations in QCD, we decided to perform the QED $O(\alpha^2)$ corrections without any approximation to finally decide, which result is correct.

The Born Cross Section : $e^+e^- \to f, \overline{f} \quad f \neq e$

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f} \sqrt{1 - \frac{4m_f}{s}} \left[\left(1 + \cos^2\theta + \frac{4m_f^2}{s} \sin^2\theta \right) G_1(s) - \frac{8m_f^2}{s} G_2(s) + 2\sqrt{1 - \frac{4m_f^2}{s}} \cos\theta\, G_3(s) \right]$$

$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s} N_{C,f} \sqrt{1 - \frac{4m_f}{s}} \left[\left(1 + \frac{2m_f^2}{s} \right) G_1(s) - 6\frac{m_f^2}{s} G_2(s) \right]$$

$$\begin{aligned} G_1(s) &= Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}[\chi_Z(s)] + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2 \\ G_2(s) &= (v_e^2 + a_e^2) a_f^2 |\chi_Z(s)|^2 \\ G_3(s) &= 2Q_e Q_f a_e a_f \mathrm{Re}[\chi_Z(s)] + 4 v_e v_f a_e a_f |\chi_Z(s)|^2. \end{aligned}$$

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$

The $\mathcal{O}(\alpha)$ Corrections

- The first radiative corrections come from the process

$$e^+ + e^- \to \gamma^*/Z^* + \gamma.$$

- To stay in $d = 4$, we can split the contributions into hard, soft and virtual photons.
- The hard part is characterized by demanding $k^0 > \frac{\sqrt{s}\Delta}{2}$.
- The soft and virtual parts of the cross section have to be made infrared finite by introducing a small photon mass λ.
- The cross section is then given by

$$\frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s}\left(\frac{\alpha}{\pi}\right)\left[\delta\left(1-z\right)\left(\delta_1^{S_1}(\lambda, \Delta) + \delta_1^{V_1}(\lambda)\right) + \theta(1-z-\Delta)\delta_1^{H_1}(z)\right].$$

- The result is given by

$$\frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s}\frac{\alpha}{\pi}\left[\delta\left(1-z\right)\left(-2 + \frac{3}{2}L + 2\zeta_2 + 2(L-1)\ln(\Delta)\right)\right.$$
$$\left. + \theta(1-z-\Delta)\frac{1+z^2}{1-z}(L-1) + \mathcal{O}\left(\frac{m_e^2}{s}\right)\right]$$

with $L = \ln(s/m_e^2)$.

The Differences between Two Calculations

ISR corrections have been finally calculated up to $O(\alpha^2)$ in the asymptotic limit $m_e^2/s \ll 1$ with two different techniques:

1. Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988)):
   - Full calculation with massive electrons in the limit $m_e^2 \ll s$ calculation in $d = 4$ with soft-hard separation, including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.
   - Calculational Technique:

     o Direct integration over the phase space in $d = 4$ with soft-hard photon separator and photon mass to regulate the infrared.

     o Expansion in $m_e^2 \ll s$ on integrand level (no details given).
2. JB, De Freitas, van Neerven (Nucl. Phys. B855 (2012))
   - Direct calculation of the asymptotic limit $m_e^2 \ll s$ using massive light-cone operator matrix elements.
   - The technique is based on asymptotic factorization.

     Buza, Matiounine, Smith, Migneron, van Neerven (Nucl.Phys. B472 (1996))
   - It was already used in Berends et al., but only for the logarithmically enhanced terms, claiming it works only at that level.

The Differences to Kniehl et al.: The NS Case

Kniehl, Krawczyk, Kühn, Stuart, Phys. Lett. B209 (1988):

Use as input: Baier et al. (1966):
(This is a massless amplitude, except the final state.)

$$\frac{d^2\sigma}{ds'ds''} = \sigma_0(s')\lambda^{1/2}(s,s',s'')P(s'',m_e^2)\left(\frac{\alpha}{\pi}\right)^2 \times \left[-2+\frac{(s'+s'')^2+s^2}{\lambda^{1/2}(s,s',s'')(s-s'-s'')}\ln\left[\frac{s-s'-s''+\lambda^{1/2}(s,s',s'')}{s-s'-s''-\lambda^{1/2}(s,s,'s'')}\right]\right]$$

- Berends et al. (1987) picked this expression up in their 1988 Erratum and agreed.
- **However, keeping m_e everywhere,** lead to our 2011 result.
- This Epiphany has finally brought us on the right track:

No neglection of m_e in all integrands!

- The result by Kniehl et al. (1988), however, fully applies for ISR radiated pairs like $\mu^+\mu^-, \tau^+\tau^-$ and heavy quarks.
- Keeping m_e finite everywhere will lead to monstrous expressions at intermediary steps.
  A calculation based on this has been impossible to anybody back in 1987, given the available computing resources and the lack of mathematical methods, known only since very recently, for a rigorous treatment.

Factorization in the Asymptotic Region: The Method of Massive Operator Matrix Elements

We first consider the Method of massive OMEs.

In the asymptotic region the cross section factorizes

$$\frac{\mathrm{d}\sigma_{ij}(s')}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{l,i}\left(z, \frac{\mu^2}{m_e^2}\right) \otimes \tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right) \otimes \Gamma_{k,j}\left(z, \frac{\mu^2}{m_e^2}\right)$$

into

- massless cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$

  Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))

  Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))
- massive operator matrix elements $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence

  JB, De Freitas, van Neerven (Nucl.Phys. B855 (2012))

$\sigma^{(0)}(s')$ is the Born cross section and the convolution $\otimes$ is given by

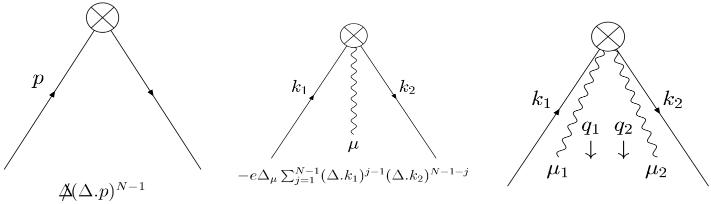
$$f(z) \otimes g(z) = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, f(z_1) g(z_2) \delta(z - z_1 z_2).$$

The comparison between both calculations shows:

- the one-loop, i.e. $O(\alpha)$, agrees between both calculations
- the logarithmically enhanced terms at two-loops ($O(\alpha^2)$) agree between both calculations
- the constant terms do not agree

$\Rightarrow$ breakdown of asymptotic factorization or errors?

Factorization in the Asymptotic Region



$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e|\, O_F^{\mathrm{NS,S}}\, |e\rangle\,,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma|\, O_F^{\mathrm{S}}\, |\gamma\rangle\,,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e|\, O_V^{\mathrm{S}}\, |e\rangle\,,$$

$$O^{\mathrm{NS,S}}_{F;\mu_1,\ldots,\mu_N} = i^{N-1}\mathrm{S}\left[\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{traces},$$
$$O^{\mathrm{S}}_{V;\mu_1,\ldots,\mu_N} = 2i^{N-2}\mathrm{S}\left[F_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha}_{\mu_N}\right] - \text{traces}$$

- The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$.
- In the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process
    Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))

We represent the observable in Mellin space transforming $z = s'/s \; \epsilon[0,1]$:

The differential scattering cross section $\Sigma(z) = d\sigma_{ij}(z)/ds'$ is considered. This quantity reads in Mellin space

$$\mathbf{M}[\Sigma(z)](N) = \int_0^1 dz z^{N-1} \Sigma(z) \; .$$

In this representation the different Mellin convolutions to be performed in z–space simplify to ordinary products. The following representation is obtained

$$\frac{d\sigma_{ij}}{ds'}(N) = \frac{1}{s}\sigma^{(0)}(N) \sum_{l,k} \Gamma_{l,i}\left(N, \frac{\mu^2}{m^2}\right) \tilde{\sigma}_{lk}\left(N, \frac{s'}{\mu^2}\right) \Gamma_{k,j}\left(N, \frac{\mu^2}{m^2}\right) \; .$$

- Here Γ_{li} denote massive operator matrix elements and $\tilde{\sigma}_{lk}$ the massless Wilson coefficients, both being calculated in the $\overline{\mathrm{MS}}$ scheme.
- μ is the factorization scale, which cancels in the physical cross section.
- The initial state fermion mass dependence is solely encoded in Γ_{li}.

The solutions of these equations are

$$\Gamma_{ee}\left(N, a, \frac{\mu^2}{m^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)} L + \Gamma_{ee}^{(0)}\right] + a^2\left[\left\{\frac{1}{8}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} - 2\beta_0\right) + \frac{1}{8}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\} L^2 \right.$$
$$\left. + \frac{1}{2}\left\{-\gamma_{ee}^{(1)} + 2\beta_0\Gamma_{ee}^{(0)} - \gamma_{ee}^{(0)}\Gamma_{ee}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\right\} L + \Gamma_{ee}^{(1)}\right] + O(a^3),$$

$$\tilde{\sigma}_{ee}\left(N, a, \frac{s'}{\mu^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)}\lambda + \tilde{\sigma}_{ee}^{(0)}\right] + a^2\left[\left\{\frac{1}{2}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} + \beta_0\right) + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\}\lambda^2 \right.$$
$$\left. + 1 + \left\{-\gamma_{ee}^{(1)} - \beta_0\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\right\}\lambda + \tilde{\sigma}_{ee}^{(1)}\right] + O(a^3),$$

$$\Gamma_{\gamma e}\left(N, a, \frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{\gamma e}^{(0)} L + \Gamma_{\gamma e}^{(0)}\right] + O(a^2)$$

$$\tilde{\sigma}_{e\gamma}\left(N, a, \frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{e\gamma}^{(0)}\lambda + \tilde{\sigma}_{e\gamma}^{(0)}\right] + O(a^2)\ ,$$

with the logarithms $L = \ln\left(\frac{\mu^2}{m^2}\right)$ and $\lambda = \ln\left(\frac{s'}{\mu^2}\right)$

The Renormalization Group Technique

Introducing the splitting functions in N-space

$$P_{ij}^{(I)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(I)}(z) = -\gamma_{ij}^{(I)}(N)$$

one obtains

$$\begin{aligned}
\frac{d\sigma_{e^+e^-}}{ds'} \; &= \; \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P_{ee}^{(0)}\mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right] \\
&+ a_0^2\Bigg\{\left[\frac{1}{2}P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{\beta_0}{2}P_{ee}^{(0)} + \frac{1}{4}P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}\right]\mathbf{L}^2 \\
&+ \left[P_{ee}^{(1)} + P_{ee}^{(0)} \otimes \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right) - \beta_0\tilde{\sigma}_{ee}^{(0)} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right]\mathbf{L} \\
&+ \left(2\Gamma_{ee}^{(1)} + \tilde{\sigma}_{ee}^{(1)}\right) + 2\Gamma_{ee}^{0)} \otimes \tilde{\sigma}_{ee}^{(0)} + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}\Bigg\}\Bigg\}
\end{aligned}$$

with

$$\mathbf{L} = \ln\left(\frac{s'}{m^2}\right) = \ln\left(\frac{s}{m^2}\right) + \ln(z); \quad \hat{\mathbf{L}} \equiv \ln(s/m^2)\;.$$

It is convenient to represent the differential scattering cross section in terms of three contributions, the flavor non-singlet terms with a single fermion line (I), those with a closed fermion line (II), and the pure-singlet terms (III). These contributions are :

$$\begin{aligned}
\frac{d\sigma^{\mathrm{I}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P^{(0)}_{ee}\mathbf{L} + \left(\tilde{\sigma}^{(0)}_{ee} + 2\Gamma^{(0)}_{ee}\right)\right] \\
&\quad + a_0^2\Bigg\{\frac{1}{2}P^{(0)}_{ee}\otimes P^{(0)}_{ee}\mathbf{L}^2 + \left[P^{(1),\mathrm{I}}_{ee} + P^{(0)}_{ee}\otimes\left(\tilde{\sigma}^{(0)}_{ee} + 2\Gamma^{(0)}_{ee}\right)\right]\mathbf{L} \\
&\quad + \left(2\Gamma^{(1),\mathrm{I}}_{ee} + \tilde{\sigma}^{(1),\mathrm{I}}_{ee}\right) + 2\Gamma^{0)}_{ee}\otimes\tilde{\sigma}^{(0)}_{ee} + \Gamma^{(0)}_{ee}\otimes\Gamma^{(0)}_{ee}\Bigg\}\Bigg\} \\
\frac{d\sigma^{\mathrm{II}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\left\{-\frac{\beta_0}{2}P^{(0)}_{ee}\mathbf{L}^2 + \left[P^{(1),\mathrm{II}}_{ee} - \beta_0\tilde{\sigma}^{(0)}_{ee}\right]\mathbf{L} + \left(2\Gamma^{(1),\mathrm{II}}_{ee} + \tilde{\sigma}^{(1),\mathrm{II}}_{ee}\right)\right\} \\
\frac{d\sigma^{\mathrm{III}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Bigg\{\frac{1}{4}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\mathbf{L}^2 + \left[P^{(1),\mathrm{III}}_{ee} + P^{(0)}_{\gamma e}\otimes\tilde{\sigma}^{(0)}_{e\gamma} + \Gamma^{(0)}_{\gamma e}\otimes P^{(0)}_{e\gamma}\right]\mathbf{L} \\
&\quad + \left(2\Gamma^{(1),\mathrm{III}}_{ee} + \tilde{\sigma}^{(1),\mathrm{III}}_{ee}\right) + 2\tilde{\sigma}^{(0)}_{e\gamma}\otimes\Gamma^{(0)}_{\gamma e}\Bigg\}
\end{aligned}$$

- $\tilde{\sigma}^{(k)}_{ij}$ denotes the respective contribution of the massless Drell-Yan (DY) cross section.

Different ingredients to the calculation :

- Splitting functions P_{ij} to $O(\alpha^2)$

E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545];
Nucl. Phys. B **152** (1979) 493;
A. Gonzalez-Arroyo, C. Lopez and F.J. Yndurain, Nucl. Phys. B **153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429;
E.G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417;
G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;
W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437;
R. Hamberg and W.L. van Neerven, Nucl. Phys. B **379** (1992) 143;
R.K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356;
S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **573** (2000) 853;
J. Ablinger *et al.*, Nucl. Phys. B **882** (2014) 263; Nucl. Phys. B **886** (2014) 733; Nucl. Phys. B **890** (2014) 48;
Nucl. Phys. B **922** (2017) 1.

- massless Drell-Yan Cross Section $\tilde{\sigma}_{ij}$ to $O(\alpha^2)$

R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403];
R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.

- massive OMEs Γ_{ij} to $O(\alpha^2)$ $\Longrightarrow$ JB et al. 2011

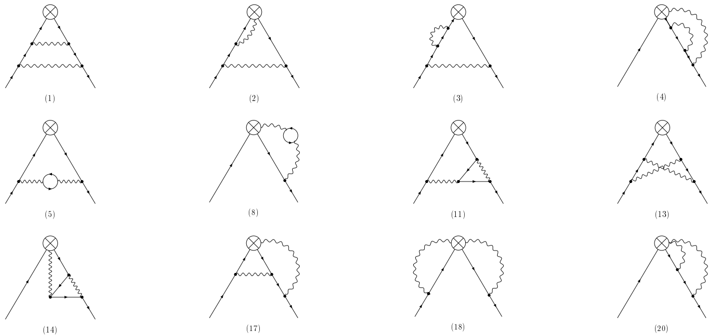
The 1-Loop OMEs

The $O(\varepsilon^0)$ terms are

$$
\begin{aligned}
\Gamma_{ee}^{(0)}(x) &= -8\mathcal{D}_1(x) - 4\mathcal{D}_0(x) + 4\delta(1-x) + 2(1+x)\left[2\ln(1-x)+1\right] \\
&= -4\left[\frac{1+x^2}{1-x}\left\{\ln(1-x)+\frac{1}{2}\right\}\right]_+ \\
\Gamma_{e\gamma}^{(0)}(x) &= 0 \\
\Gamma_{\gamma e}^{(0)}(x) &= -2\frac{1+(1-x)^2}{x}\left[2\ln(x)+1\right],
\end{aligned}
$$

The linear term in $\varepsilon\,\overline{\Gamma}_{ee}^{(0)}(x)$ reads

$$
\begin{aligned}
\overline{\Gamma}_{ee}^{(0)}(x) &= -4\mathcal{D}_2(x) - 4\mathcal{D}_1(x) - \zeta_2\mathcal{D}_0(x) - \left(4+\frac{3}{4}\zeta_2\right)\delta(1-x) \\
&\quad +2(1+x)\left[\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right] \\
&= -2\left[\frac{1+x^2}{1-x}\left\{\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right\}\right]_+ . \\
\mathcal{D}_m(x) &= \left(\frac{\ln^m(1-x)}{1-x}\right)_+
\end{aligned}
$$

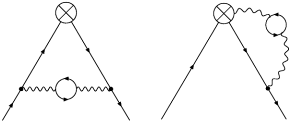


Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

The result for the the matrix element $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$ is

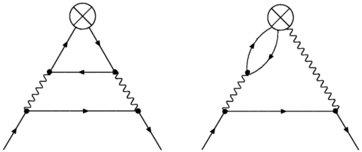
$$
\begin{aligned}
&\Big\{ \frac{1+3x^2}{1-x}\left[6\zeta_2 \ln(x) - 8\ln(x)\mathrm{Li}_2(1-x) - 4\ln^2(x)\ln(1-x)\right] + \left(\frac{122}{3}x + 22 + \frac{32}{1-x}\right)\zeta_2 + (8-112\zeta_2)\,\mathcal{D}_1(x) \\
&+16\frac{1+x^2}{1-x}\left[2\mathrm{Li}_3(-x) - \ln(x)\mathrm{Li}_2(-x)\right] + \frac{80}{3(1-x)} + 56(1+x)\zeta_2\ln(1-x) + (16 - 52\zeta_2 + 128\zeta_3)\,\mathcal{D}_0(x) \\
&+\left(\frac{22}{3}x + 32 + \frac{64}{3(1-x)^2} - \frac{51}{1-x} - \frac{16}{3(1-x)^3}\right)\ln^2(x) - (92+20x)\ln^2(1-x) + 14(x-2)\ln(1-x) + 120\mathcal{D}_2(x) \\
&+\left(\frac{178}{3} - 36x + \frac{64}{3(1-x)^2} - \frac{140}{3(1-x)} - \frac{48}{1+x}\right)\ln(x) - \frac{1}{3}(1+x)\ln^3(x) + 4\frac{x^2-8x-6}{1-x}\ln(x)\ln(1-x) \\
&-2\frac{1+17x^2}{1-x}\ln(x)\ln^2(1-x) - \frac{112}{3}(1+x)\ln^3(1-x) + 32\frac{1+x}{1-x}\left[\ln(x)\ln(1+x) + \mathrm{Li}_2(-x)\right] - 22x - \frac{62}{3} \\
&-4\frac{13x^2+9}{1-x}\mathrm{S}_{1,2}(1-x) + 4\frac{5-11x^2}{1-x}\left[\ln(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x) - 2\zeta_3\right] + \frac{4(16x^2-10x-27)}{3(1-x)}\mathrm{Li}_2(1-x) \\
&+\frac{224}{3}\mathcal{D}_3(x) + \left[\frac{433}{8} - \frac{67}{45}\pi^4 + \left(\frac{37}{2} - 48\ln(2)\right)\zeta_2 + 58\zeta_3\right]\delta(1-x)\Big\} + (-1)^n\Big\{\frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2} \\
&+\frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3}\ln(x) + \frac{4(x^2+10x-3)}{3(1+x)}\left(\zeta_2 + 2\mathrm{Li}_2(-x) + 2\ln(x)\ln(1+x)\right) \\
&+\frac{1+x^2}{1+x}\Big[36\zeta_3 - 24\zeta_2\ln(1+x) + 8\zeta_2\ln(x) - \frac{2}{3}\ln^3(x) + 40\mathrm{Li}_3(-x) - 4\ln^2(x)\ln(1+x) - 24\ln(x)\ln^2(1+x) \\
&-24\ln(x)\mathrm{Li}_2(-x) - 48\ln(1+x)\mathrm{Li}_2(-x) - 8\ln(x)\mathrm{Li}_2(1-x) - 16\mathrm{S}_{1,2}(1-x) - 48\mathrm{S}_{1,2}(-x)\Big] \\
&-\frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3}\mathrm{Li}_2(1-x) + 4x\frac{1-x-5x^2+x^3}{(1+x)^3}\ln^2(x)\Big\}
\end{aligned}
$$

Result: Process II



The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$ is

$$\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\mathrm{II}} \quad = \quad & \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x}\right)\ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\
& - \frac{32}{9}\left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4}\right)\ln(x) + \frac{16}{3}(1+x)\left(\ln(1-x) + \ln^2(1-x)\right) \\
& - \frac{2(1+x^2)}{3(1-x)}\ln^2(x) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2\right)\mathcal{D}_0(x) + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3}\left(\mathcal{D}_1(x) + \mathcal{D}_2(x)\right) \\
& + \left(\frac{8}{3}\zeta_3 + 10\zeta_2 - \frac{1411}{162}\right)\delta(1-x)
\end{aligned}$$



The result for $\hat{\Gamma}^{(1),\mathrm{III}}_{ee}$ is

$$
\begin{aligned}
\hat{\Gamma}^{(1),\mathrm{III}}_{ee} \quad = \quad & \frac{2}{x}(1-x)(4x^2+13x+4)\zeta_2 + \frac{1}{3x}(8x^3+135x^2+75x+32)\ln^2(x) \\
& + \left[\frac{304}{9x} - \frac{80}{9}x^2 - \frac{32}{3}x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3}\right]\ln(x) - \frac{224}{27}x^2 \\
& + 16\frac{1-x}{3x}(x^2+4x+1)\left[2\ln(x)\ln(1+x) - \mathrm{Li}_2(1-x) + 2\mathrm{Li}_2(-x)\right] \\
& + (1+x)\Bigg[4\zeta_2\ln(x) + \frac{14}{3}\ln^3(x) - 32\ln(x)\mathrm{Li}_2(-x) - 16\ln(x)\mathrm{Li}_2(x) + 64\mathrm{Li}_3(-x) \\
& + 32\mathrm{Li}_3(x) + 16\zeta_3\Bigg] - \frac{182}{3}x + 50 - \frac{32}{1+x} + \frac{800}{27x} + \frac{64}{3(1+x)^2}
\end{aligned}
$$

The first moment vanishes for all three contributions $\hat{\Gamma}^{(1),\mathrm{I}}_{ee}$, $\hat{\Gamma}^{(1),\mathrm{II}}_{ee}$ and $\hat{\Gamma}^{(1),\mathrm{II}}_{ee}$.

$\rightarrow$ Fermion number conservation is satisfied.

The Scattering Cross Section

The 2–loop corrections to the process $e^+e^- \to Z^0$ can be organized in the following form :

$$\boxed{\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0\left[T_{11}\hat{\mathbf{L}} + T_{10}\right] + a_0^2\left[T_{22}\hat{\mathbf{L}}^2 + T_{21}\hat{\mathbf{L}} + T_{20}\right]\right\}}$$

• Universal Corrections : $T_{ii}(z)$ $\Longrightarrow$ depend on LO splitting functions and β_0

$$\begin{aligned}
T_{11} &= 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4\left[\frac{1+z^2}{1-z}\right]_+ , \\
T_{22} &= \left\{64\mathcal{D}_1(z) + 48D_0(z) + (18 - 32\zeta_2)\delta(1-z)\right. \\
&\quad \left. -32\frac{\ln(z)}{1-z} - 32(1+z)\ln(1-z) + 24(1+z)\ln(z) - 8(5+z)\right\}_{\mathrm{I}} \\
&\quad + \frac{2}{3}\left\{8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z)\right\}_{\mathrm{II}} \\
&\quad + 16\left\{\frac{1}{2}(1-z)\ln(z) + \frac{1}{4}(1-z) + \frac{1}{3}\frac{1}{3z}(1-z^3)\right\}_{\mathrm{III}} .
\end{aligned}$$

The Cross Section at $O(\alpha)$ and the Logarithmic 2-Loop Contributions

- $O(\alpha)$ Term : $T_{10}(z)$ $\Longrightarrow$ depend on LO OME + LO DY

$$
\begin{aligned}
T_{10} &= -4\left[\frac{1+z^2}{1-z}\right]_+ + 2(4\zeta_2-1)\delta(1-z) \\
T_{11}\hat{\mathbf{L}} + T_{10} &= P_{ee}^{(0)}(z)\left[\hat{\mathbf{L}}-1\right] + 2(4\zeta_2-1)\delta(1-z)\ .
\end{aligned}
$$

Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$ Terms : $T_{21}(z)$ $\Longrightarrow$ depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

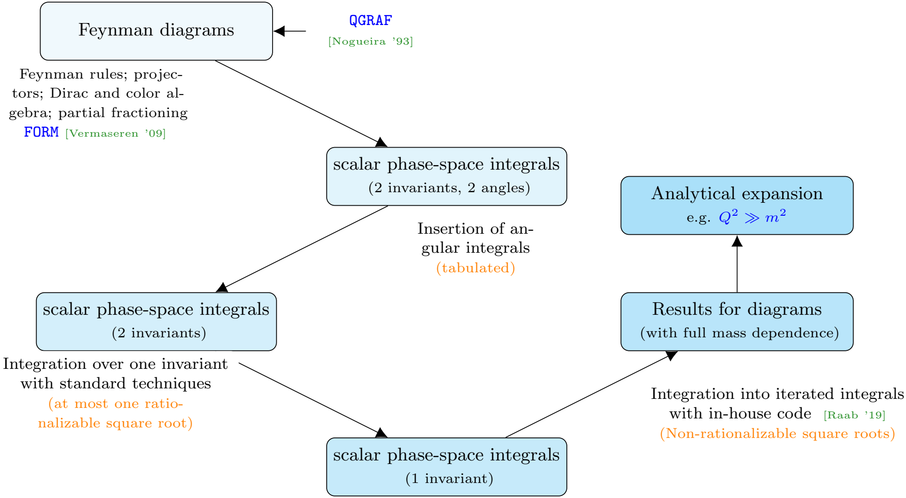
$$
\begin{aligned}
T_{21}^{\mathrm{I}} = 16\Biggl\{ & -8\mathcal{D}_1(z) - (7-4\zeta_2)\mathcal{D}_0(z) + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3\right)\delta(1-z) \\
& + \left(\frac{1+z^2}{1-z}\right)\left[\ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z)\right] \\
& + (1+z)\left[4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2\right] - \ln(z) + 3 + 4z\Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
T_{21}^{\mathrm{II}} &= 16\left\{\frac{4}{3}\mathcal{D}_1(z) - \frac{10}{9}\mathcal{D}_0(z) - \frac{17}{12}\delta(1-z)\right.\\
&\left. -\frac{2}{3}\frac{\ln(z)}{1-z} - \frac{1}{3}(1+z)\left[2\ln(1-z) - \ln(z)\right] - \frac{1}{9} + \frac{11}{9}z\right\}\\
T_{21}^{\mathrm{III}} &= 16\left\{(1+z)\left[2\mathrm{Li}_2(1-z) - \ln^2(z) + 2\ln(z)\ln(1-z)\right]\right.\\
&+\left(\frac{4}{3}\frac{1}{z} + 1 - z - \frac{4}{3}z^2\right)\ln(1-z) - \left(\frac{2}{3}\frac{1}{z} + 1 - \frac{1}{2}z - \frac{4}{3}z^2\right)\ln(z)\\
&\left. -\frac{8}{9}\frac{1}{z} - \frac{8}{3} + \frac{8}{3}z + \frac{8}{9}z^2\right\}
\end{aligned}
$$

Up to this point, we find agreement with Berends et al. (1988), but disagree for T_{20}.

The $\mathcal{O}(\alpha^2)$ Corrections by Direct Calculation

- In Berends et al. the $\mathcal{O}(\alpha^2)$ corrections have been split up into four distinct processes:
  - Process I, photon radiation
  - Process II, non-singlet fermion pair production
  - Process III, pure-singlet fermion pair production
  - Process IV, interference between non-singlet and pure-singlet fermion pair production
- In the calculation of Blümlein et al. (Nucl. Phys. B855 (2012)) process I and IV had to be treated combined due to the nature of the OMEs, which avoids cutting techniques.
- We have to distinguish between vector and axial/couplings of the Z-boson. $\Longrightarrow$ We throroughly work in $\mathrm{d} = 4$ dimensions.
- We have to recalculate and to add contributions due to diagrams not having been considered before.



Feynman diagrams
QGRAF
[Nogueira '93]
Feynman rules; projectors; Dirac and color algebra; partial fractioning
FORM [Vermaseren '09]
scalar phase-space integrals
(2 invariants, 2 angles)
Insertion of angular integrals
(tabulated)
scalar phase-space integrals
(2 invariants)
Integration over one invariant with standard techniques
(at most one rationalizable square root)
scalar phase-space integrals
(1 invariant)
Integration into iterated integrals with in-house code [Raab '19]
(Non-rationalizable square roots)
Results for diagrams
(with full mass dependence)
Analytical expansion
e.g. $Q^2 \gg m^2$

Recalculation

- We want to use iterated integrals so we can work in a differential field.

$$\mathrm{H}_{w_1,\dots,w_n}(x) = \int_0^x dt\, f_{w_n}(t)\, \mathrm{H}_{w_1,\dots,w_{n-1}}(x), \quad \tilde{\mathrm{H}}_{w_1,\dots,w_n}(x) = \int_x^1 dt\, f_{w_n}(t)\, \mathrm{H}_{w_1,\dots,w_{n-1}}(x).$$

- The steps to transform the last integrand to iterated integrals include:
  - Express all logarithms and polylogarithms in terms of iterated integrals evaluated at the last integration variable through linear differential equations.
  - Find relations between the occurring letters and square roots to get rid of redundancies.
  - Compactify the integrand expressed in terms of iterated integrals as far as possible.
    $\rightarrow$ Since we express everything in linearly independent quantities, the complexity of the last integral can be drastically reduced in this step.
- The same technique has been successfully applied to calculate the full mass dependence of the pure-singlet structure functions in deep-inelastic-scattering.
- In total we need 37 letters to express the contributions due to fermion pair production.

The Letters:

$$v_1 = \frac{1}{\sqrt{1-4t}\sqrt{16t^2 - 8(1+z)t + (1-z)^2}}$$

$$v_2 = \frac{1}{t\sqrt{1-4t}\sqrt{16t^2 - 8(1+z)t + (1-z)^2}}$$

$$v_3 = \frac{1}{\sqrt{1-4t}\big(4t - (1+x)\big)\sqrt{16t^2 - 8(1+z)t + (1-z)^2}},$$

$$v_4 = \frac{1}{t\sqrt{1-t}},$$

$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_4 = \frac{1}{\big(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\big)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_5 = \frac{t}{\big(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\big)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_6 = \frac{1}{\big(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\big)\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_7 = \frac{t}{\big(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\big)\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_8 = \frac{1-z}{\big(4\rho - (1-z)t\big)\sqrt{1-t}},$$

$$d_9 = \frac{1}{\big(16\rho^2 + 4\big(z - 2\rho(1+z)\big)t + (1-z)^2t^2\big)\sqrt{1-t}},$$

$$d_{10} = \frac{t}{\big(16\rho^2 + 4\big(z - 2\rho(1+z)\big)t + (1-z)^2t^2\big)\sqrt{1-t}},$$

$$d_{11} = \frac{1}{t\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$

$$d_{12} = \frac{1}{16\rho^2 + 4\big(z - 2\rho(1+z)\big)t + (1-z)^2t^2},$$

$$d_{13} = \frac{t}{16\rho^2 + 4\big(z - 2\rho(1+z)\big)t + (1-z)^2t^2},$$

$$d_{14} = \frac{1}{t(1-z) - 4\rho},$$

$$d_{15} = \frac{1}{\sqrt{1-t}(t(1-z) - 4\rho)},$$

$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2 - 16\rho^2}},$$

$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z) - 4\rho)\sqrt{t(1-z)^2 - 16\rho^2}},$$

$$d_{18} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}},$$

$$d_{19} = \frac{1}{\sqrt{t}(t(1-z) - 4\rho)\sqrt{t(1-z)^2 - 16\rho^2}},$$

$$d_{20} = \frac{1}{\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{23} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)},$$

$$d_{24} = \frac{1}{\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{25} = \frac{t}{\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{26} = \frac{1}{\sqrt{1-t}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{27} = \frac{t}{\sqrt{1-t}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{29} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)},$$

$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho(1+z)t + 16\rho^2}},$$

$$d_{31} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\big(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\big)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{33} = \frac{t}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}.$$

$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_4 = \frac{1}{\left(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\right)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_5 = \frac{t}{\left(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2\right)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}},$$
$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2 - 16\rho^2}},$$
$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z) - 4\rho)\sqrt{t(1-z)^2 - 16\rho^2}},$$
$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{26} = \frac{1}{\sqrt{1-t}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{27} = \frac{t}{\sqrt{1-t}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho(1+z)t + 16\rho^2}},$$
$$d_{31} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\left(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2\right)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$

$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}},$$
$$d_{33} = \frac{t}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}.$$

- 16 of these letters introduce elliptic structures, since multiple square roots cannot be rationalized at once.

$\rho = m_e^2/s, \quad z = s'/s, \qquad t$ - integration variable.

The Size of the Calculation

Size of amplitudes:

process I	$10Gb$
process II	$25kb$
process III	$56kb$
process IV	$124kb$

Computation time:

	Reduction to Basis	Integration	
process I		$30h$	several months of code design
process II	1 day	min's	
process III	1 month	$2h$	
process IV	2 months	$5h$	

The Non-Singlet Case

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} = {} & \frac{\sigma^{(0)}(s')}{s}\, a^2 \Biggl\{ \frac{64}{3} z(1-z)(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_7} + \frac{256}{3} z\rho(1+z-4\rho)\tilde{\mathrm{H}}_{v_4,d_6} \\
& + \frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2}\tilde{\mathrm{H}}_{d_8,d_7} \\
& + \frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3}\tilde{\mathrm{H}}_{d_8,d_6} \\
& + \frac{16}{9(1-z)^2}\Bigl[(1+z)^2\bigl(4-9z+4z^2\bigr) + 2\bigl(9-16z+13z^2-2z^3\bigr)\rho + 32\rho^2\Bigr]\tilde{\mathrm{H}}_{d_2} \\
& + \frac{512z\rho}{9(1-z)^4}\Bigl[3(1-z)^4 z - (1-z)^3\bigl(4+z^2\bigr)\rho - 2\bigl(9-29z+38z^2-17z^3+3z^4\bigr)\rho^2 \\
& - 4(2-z)\bigl(3+6z-5z^2\bigr)\rho^3 + 16\bigl(7-8z+9z^2\bigr)\rho^4 + 128(3-z)\rho^5\Bigr]\tilde{\mathrm{H}}_{d_4} \\
& - \frac{16}{9(1-z)^4}\Bigl[3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8\bigl(2-16z+9z^2 \\
& + 4z^3-5z^4+2z^5\bigr)\rho + 16z\bigl(12-13z+18z^2-z^3\bigr)\rho^2 + 32\bigl(1+22z-7z^2\bigr)\rho^3\Bigr]\tilde{\mathrm{H}}_{d_1} \\
& - \frac{128z}{9(1-z)^4}\Bigl[1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2\bigl(7-55z+54z^2 \\
& + 16z^3-17z^4+3z^5\bigr)\rho - 4\bigl(39-16z+16z^2+4z^3+5z^4\bigr)\rho^2 \\
& + 16\bigl(8-23z+22z^2+9z^3\bigr)\rho^3 + 128\bigl(7+2z-z^2\bigr)\rho^4\Bigr]\tilde{\mathrm{H}}_{d_5} - \frac{64}{3}(2z+(1-z)\rho)\tilde{\mathrm{H}}_{d_3} \\
& + \biggl[\frac{16}{3\sqrt{1-4\rho}}(1+z-4\rho)\tilde{\mathrm{H}}_{v_4} + \frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}}\tilde{\mathrm{H}}_{d_8}\biggr] \\
& \times \ln\left(\frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}\right)\Biggr\}
\end{aligned}
$$

Complete cross section without any approximation. Note the new iterated integrals, which are found using Risch-algorithm techniques.

The Non-Singlet Case

- The explicit expansion of the analytical result in the limit $m_e^2 \ll s$ gives

$$\frac{d\sigma^{(2),\mathrm{II}}(z)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ \frac{8}{3}\frac{1+z^2}{1-z}L^2 - \Bigg[\frac{16}{9}\frac{11-12z+11z^2}{1-z} - \frac{16}{3}\frac{1+z^2}{1-z}\ln(z)$$
$$+ \frac{32}{3}\frac{1+z^2}{1-z}\ln(1-z)\Bigg]L + \frac{32}{9(1-z)^3}\left(7-13z+8z^2-13z^3+7z^4\right)$$
$$- \frac{16z}{9(1-z)^4}\left(3-36z+94z^2-72z^3+19z^4\right)\ln(z) - \frac{8z^2}{3(1-z)}\ln^2(z)$$
$$- \left(\frac{32}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{2+z^2}{1-z}\ln(z)\right)\ln(1-z) + \frac{32}{3}\frac{1+z^2}{1-z}\ln^2(1-z)$$
$$+ \frac{16z^2}{3(1-z)}\mathrm{Li}_2(z) - \frac{16\left(2+3z^2\right)}{3(1-z)}\zeta_2\Bigg\} + \mathcal{O}\left(\frac{m_e^2}{s}L^2\right),$$

with $L = \ln(m_e^2/s)$.

- The result contains higher denominator powers which have not been obtained by Berends et al.

$$\delta_{II} = -\frac{128}{9}\left[3 + \frac{1}{(1-z)^3} - \frac{2}{(1-z)^2} - 2z\right] - 16\Bigg[1 + \frac{5z}{3} + \frac{8}{9}\frac{1}{(1-z)^4}$$
$$- \frac{20}{9}\frac{1}{(1-z)^3} + \frac{4}{9}\frac{1}{(1-z)^2}\Bigg]\ln(z) + \frac{8}{3}\frac{1+z^2}{1-z}\left[\frac{10}{9} - \frac{14}{3}\ln(z) - \ln^2(z)\right].$$

Process I

2 photon emission:

- $T_2^{\mathrm{S}_2}$: both emitted photons are soft $\surd$
- $T_2^{\mathrm{V}_2}$: both photons are virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{V}_1}$: one photon is soft, one is virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{H}_1}$: one photon is soft, one is hard $\surd$
- $T_2^{\mathrm{V}_1\mathrm{H}_1}$: one photon is virtual, one is hard: disagreement
- $T_2^{\mathrm{H}_2}$: both emitted photons are hard, $\surd$

Here and in the following we only report the vector-case.
There are differences in the axial-vector case (not clear from Berends et al.)
Since we can work in 4-dimensions (only Abelian couplings) we can treat γ_5 without a further finite renormalization.

- We find the difference:

$$\begin{aligned}\delta_I \quad &= \quad -8 + \frac{100}{33} z \ln^2(z) - 8\ln(1-z) + 4(1-z)\ln^2(1-z) - \frac{8(2-z)z}{1-z}\mathrm{Li}_2(z) \\ &\quad + \frac{8(2-2z+z^2)}{1-z}\zeta_2,\end{aligned}$$

Process III: Pure Singlet

$$\delta_{III} = \frac{160}{3} - \frac{32}{z} + \frac{128}{3(1+z)^2} - \frac{64}{1+z} + 96(1+z)\zeta_3 - \left[52(1-z) + \frac{64}{3z}(1-z^3)\right]\ln^2(z)$$
$$- \frac{56}{3}(1+z)\ln^3(z) + \left[24(1-z) + 16(1+z)\ln(z)\right]\zeta_2 + \ln(z)\left[\frac{104}{3} - \frac{32}{z} + \frac{128}{3(1+z)^3}\right.$$
$$\left. - \frac{256}{3(1+z)^2} - \frac{64}{1+z} + 64\left(1 - z + \frac{1-z^3}{3z}\right)\ln(1+z)\right] - \left[40(1-z) + \frac{64}{3z}(1-z^3)\right.$$
$$\left. + 48(1+z)\ln(z)\right]\mathrm{Li}_2(1-z) + 64\left[1 - z + \frac{1}{3z}(1-z^3) - (1+z)\ln(z)\right]\mathrm{Li}_2(-z)$$
$$+ 128(1+z)\mathrm{Li}_3(-z) - 96(1+z)S_{1,2}(1-z) + \mathbf{2}\delta^{\mathrm{PS}}_{\mathrm{interf}},$$

$$\frac{d\sigma^{(2),\mathrm{III}}_{\mathrm{interf}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2\left\{-160(1-z) - \left[16(5+4z) - 80(1+z)\mathrm{H}_{-1} + \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}^2\right]\mathrm{H}_0\right.$$
$$- \left[52z - \frac{40\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_0^2 - \frac{16}{3}z\mathrm{H}_0^3 + \left[8(5-4z)\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_0^2\right]\mathrm{H}_1$$
$$- \frac{4\left(4-6z+3z^2\right)}{z}\mathrm{H}_0\mathrm{H}_1^2 - \left[8(5-4z) - \frac{8\left(8-2z+5z^2\right)}{z}\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1\right]\mathrm{H}_{0,1}$$
$$- \left[80(1+z) + \frac{32\left(5+2z^2\right)}{z}\mathrm{H}_0 - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_{0,-1} - \frac{32\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,0,1}$$
$$+ \frac{16\left(10-10z+3z^2\right)}{z}\mathrm{H}_{0,0,-1} - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_{0,1,1} - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,-1,-1}$$
$$\left. + \left[8(10+z) + 160\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1 - \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\zeta_2 + 32(5+z)\zeta_3\right\}$$
$$+ \mathcal{O}\left(\frac{m^2}{s}\right)$$

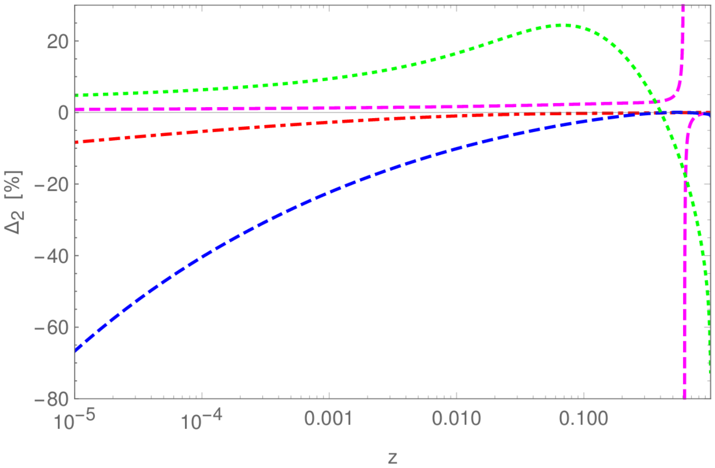
$\Longrightarrow$ First calculated by A.N. Schellekens (Thesis, Nijmegen, 1981)

Process IV: Non-Singlet Pure Singlet Interference

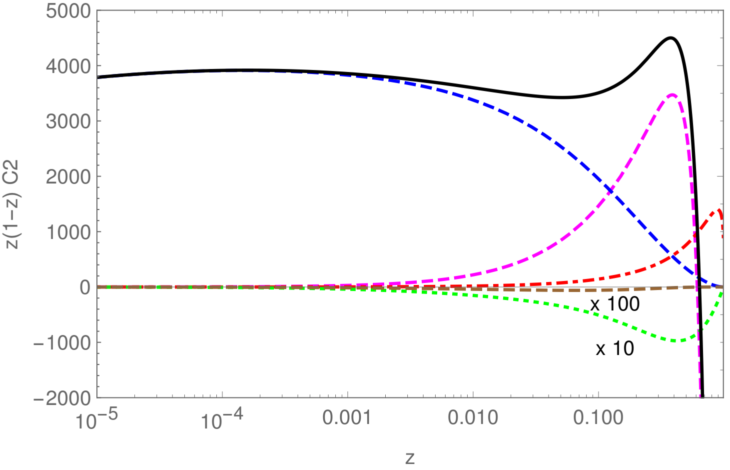
- We find the difference:

$$\delta_{IV} = \frac{2(53+994z+32z^2+742z^3-85z^4-8z^5)}{9(1-z)(1+z)^2} - 8\left[\frac{1-14z-56z^2+78z^3-25z^4}{(1-z^2)^2}\right.$$
$$\left. + \frac{1+z^2}{1-z}\ln(z)\right]\zeta_2 - \frac{8z(13+12z^2-20z^3+3z^4}{(1-z^2)^2}\ln^2(z)$$
$$+16\left[\frac{1-z+7z^2-3z^3}{(1+z)^2} + \frac{7+3z^2}{2(1-z)}\ln(z)\right]\mathrm{Li}_2(1-z) + \left[\frac{32(1+5z-4z^2)}{(1-z)^2}\ln(1+z)\right.$$
$$\left. - \frac{16(4-7z-6z^2-128z^3+2z^4-9z^5)}{3(1-z)^2(1+z)^3}\right]\ln(z) + \frac{32(1+5z-4z^2)}{(1-z)^2}\mathrm{Li}_2(-z),$$

$\Rightarrow$ Our results agree with the ones obtained in JB, De Freitas, van Neerven, 2011.

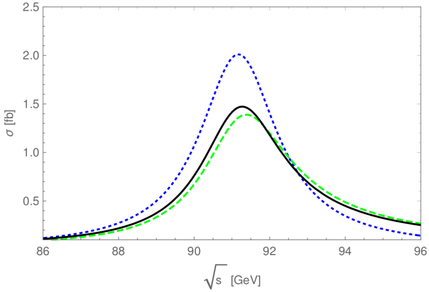


- Relative deviation of the photon (I), non-singlet (II), pure-singlet (III) and their interference (IV) contribution in %.



- Illustration of the Wilson coefficients for the photonic (I), non-singlet (II), pure-singlet (III), interference $\times 10$ (IV) and neglected contributions $\times 100$ multiplied with the factor $z(1-z)$. The black line represents the whole contribution to initial state radiation.

The Z-Peak



blue: Born

green: $\mathcal{O}(\alpha)$

red: $\mathcal{O}(\alpha^2)$

black: $\mathcal{O}(\alpha^2)$
+ soft exponentiation

- The mass and width of the Z-boson are measured very precisely:
  $\Delta M_Z = \pm 2.3\text{MeV}, \quad \Delta\Gamma_Z = \pm 2.1\text{MeV}$ (PDG)
- $O(\alpha^2)$ corrections and soft exponentiation have sizable impact on peak position and width.
- The differences we found can affect the width of the peak within the accuracy of the experiments.

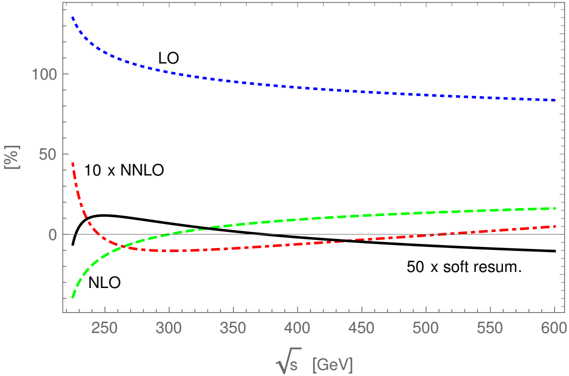
	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV	Width (MeV)
$O(\alpha)$ correction	210	603	210	602
$O(\alpha^2)$ correction	-109	-187	-109	-187
$O(\alpha^2)$: γ only	-110	-215	-110	-215
$O(\alpha^2)$ correction + soft exp.	17	23	17	23
Difference to $O(\alpha^2)$ [1]		4		4

$O(34\text{MeV})$ shift between fixed and s-dep. width in peak position

Berends et al., Bardin et al., Beenakker & Hollik.

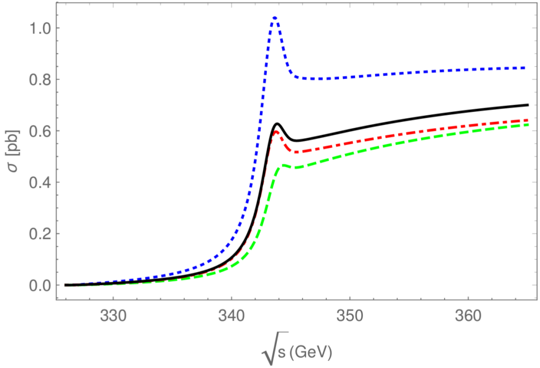
$\mathcal{O}(\alpha^2)$ QED Initial-State-Radiation – Phenomenology

$Z^0 H^0$ Production



- Initial-state-radiation has a big effect on the shape of the threshold.
- The $\mathcal{O}(\alpha^2)$ corrections are of the size of the anticipated accuracy; the sequence of corrections converges relatively quickly.
- Soft-photon resummation is less important than in other processes.

$t\,\bar{t}$ Production at Threshold



- Initial-state-radiation has a big effect on the shape of the threshold.
- Soft-photon resummation leads to a sizable effect.
- The $\mathcal{O}(\alpha^2)$ corrections are of the size of the anticipated accuracy.

Conclusions

- We calculated the $O(\alpha^2)$ massive operator matrix elements in QED, which contribute to the 2-loop initial state corrections for $e^+e^- \to Z^*/\gamma^*$ in the limit $m_f^2/s \to 0$ using the renormalization group method for the electron-contributions.
- We have obtained all logarithmic contributions $O((\alpha L)^2), O(\alpha^2 L), O(\alpha L)$ and the constant contributions $O(\alpha)$ correctly.
- The literal application of the $s \gg m_f^2$ expansion, as proposed by BBN seemed for nearly one decade not to deliver the result obtained by conventional integration. However, we found by a full calculation that the $O(\alpha^2)$ results of BBN are not correct.
- On the other hand, we obtained our results for the 2-loop matrix elements by two independent methods, which agree on the results. Furthermore, the complete OMEs obey Fermion number conservation, and renormalize as expected. The 2-loop anomalous dimensions are correctly obtained.
- In the case of massless external lines, massive OMEs can be calculated without any problem and the results agree in all cases investigated with that obtained in the limit $m^2/\mu^2 \to 0$. This now also applies for massive external states.

Conclusions

- There are contributions to the $O(\alpha^2)$ corrections with vanishing OME, which also appear in the massless Drell–Yan process. They have to be included (missing at BBN).
- Due to the axial-vector couplings of the Z-boson the corrections in the vector- and axial-vector case are not the same (as already known from the Drell-Yan process). We have accounted for these contributions as well.
- The differences at $O(\alpha^2)$ are large, reaching the order of the logarithmic terms in part of the kinematic region.
- We have performed phenomenological studies for key-processes at present and future e^+e^- colliders. Cutting at $s' > 4m_\tau^2$ leads to a difference of $4~\mathrm{MeV}$ for Γ_{Z^0} scanning the Z^0 peak, at a present accuracy of $2.3~\mathrm{MeV}$
- The newly obtained corrections are of relevance at high luminosity e^+e^- facilities planned for the future.