Plabic Graphs and Symbol Alphabets in $\mathcal{N} = 4$ super-Yang-Mills Theory

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w/ Mago, Spradlin, and Volovich

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We will study scattering amplitudes in maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM).

Amplitudes are a collection of functions with interesting properties.

They are functions on $Gr(4, n)$ (momentum twistors: n-points on $\mathbb{P}^3$ [Hodges, 0905.1473]).

$Gr(4, n)$ has a cluster algebraic structure [Fomin, Zelevinsky, math/0104151], [Gekhtman, Shapiro, Vainshtein, math/0208033], [Scott, math/0311148].

The analytic structure of amplitudes in $\mathcal{N} = 4$ SYM is (conjecturally) strongly controlled by cluster structure of $Gr(4, n)$ [Golden, Goncharov, Spradlin, Vergu, Volovich, 1305.1617].

An example of a cluster algebra

[Fomin, Zelevinski, math/0104151], Cluster Algebra Portal
What is a cluster algebra? A_2 cluster algebra

Seed quiver:

$$\mathrm{a}_1 \longrightarrow \mathrm{a}_2$$

$$b_{ij} = (\#\text{ arrows } i \to j) - (\#\text{ arrows } j \to i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Mutation rule $a_k a_k' = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}}$ and flip all arrows connected to the node you're mutating, e.g.

$$a_3 \equiv a_1' = \frac{1}{a_1}\left[a_1^0 a_2^0 + a_1^0 a_2^1\right] = \frac{1+a_2}{a_1} .$$

All cluster coordinates

$$a_1\,,\; a_2\,,\; a_3 = \frac{1+a_2}{a_1}\,,\; a_4 = \frac{1+a_1+a_2}{a_1 a_2}\,,\; a_5 = \frac{1+a_1}{a_2} .$$

[Fomin, Zelevinski, math/0104151], [Gekhtman, Shapiro, Vainshtein, math/0208033], [Scott, math/0311148]

$Gr(4, n)$ seed quiver:

$$
\begin{tikzcd}
\boxed{\langle 1\,2\,3\,4\rangle} \arrow[dr] & & & & & \\
& \langle 1\,2\,3\,5\rangle \arrow[r] \arrow[d] & \langle 1\,2\,3\,6\rangle \arrow[d] & \cdots & \langle 1\,2\,3\,n\text{-}1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1\,2\,3\,n\rangle} \\
& \langle 1\,2\,4\,5\rangle \arrow[r] \arrow[d] & \langle 1\,2\,5\,6\rangle \arrow[ul] \arrow[d] & \cdots & \langle 1\,2\,n\text{-}2\,n\text{-}1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1\,2\,n\text{-}1\,n\rangle} \arrow[ul] \\
& \langle 1\,3\,4\,5\rangle \arrow[r] \arrow[d] & \langle 1\,4\,5\,6\rangle \arrow[ul] \arrow[d] & \cdots & \langle 1\,n\text{-}3\,n\text{-}2\,n\text{-}1\rangle \arrow[r] \arrow[d] & \boxed{\langle 1\,n\text{-}2\,n\text{-}1\,n\rangle} \arrow[ul] \\
& \boxed{\langle 2\,3\,4\,5\rangle} & \boxed{\langle 3\,4\,5\,6\rangle} \arrow[ul] & \cdots & \boxed{\langle n\text{-}4\,n\text{-}3\,n\text{-}2\,n\text{-}1\rangle} & \boxed{\langle n\text{-}3\,n\text{-}2\,n\text{-}1\,n\rangle} \arrow[ul]
\end{tikzcd}
$$

Cases: $n = 6, 7$ are finite type cluster algebras. $n \geq 8$ are of infinite type!

Amplitudes in $\mathcal{N} = 4$ SYM: Current Status

Amplitudes are conjectured to be functions of cluster coordinates of $Gr(4, n)$ [Golden, Goncharov, Spradlin, Vergu, Volovich, 1305.1617].

Current status:

- Steinmann Cluster Bootstrap [Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdogan, von Hippel, McLeod, Papathanasiou, 2005.06735]:
    - MHV: 6-point 7-loop, 7-point 4-loop.
    - NMHV: 6-point 6-loop, 7-point 4-loop.
- $\bar{Q}$-equation:
    - MHV: n-point two-loop [Caron-Huot, 1105.5606].
    - NMHV: 8-point two-loop [Zhang, Li, He, 1911.01290].

Problem

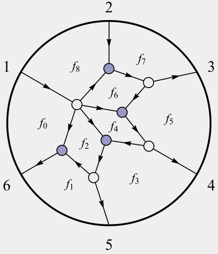
How do we extract a finite set of cluster coordinates from infinite type cluster algebras?

18 algebraic symbol letters (not cluster coordinates) appearing in $n = 8$ two-loop NMHV amplitude [Zhang, Li, He, 1911.01290].

Different approaches to algebraic letters: [Arkani-Hamed, Lam, Spradlin, 1912:08222], [Drummond, Foster, Gürdogan, Kalousios, 1912.08217], [Henke, Papathanasiou, 1912.08254]
See also Niklas Henke's talk!

We can predict letters that appear in symbols from plabic graphs - even algebraic letters! See also [He, Li, 2007.01574].

Yangian Invariants and leading singularities in $\mathcal{N} = 4$ SYM are represented by plabic graphs [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka, 1212.5605].

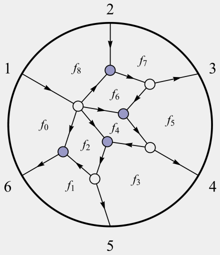


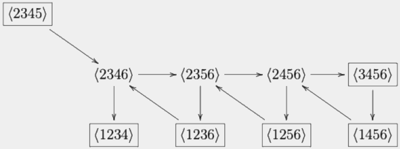
$$\Rightarrow Y(\{Z|\eta\}) = \int \prod_{i=1}^{4k} d\log f_i \; \delta^{4k|4k}(C(\{f\}) \cdot (Z|\eta))\,.$$

We will study solutions to $C \cdot Z = 0$.

Extract symbol alphabet from solutions!

Example: $n = 6$





Graphical moves $\Rightarrow$

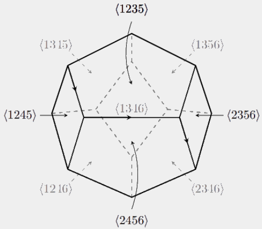
$$C \cdot Z = 0 \Rightarrow$$

$$f_0 = -\frac{\langle 1234\rangle}{\langle 2346\rangle}, \qquad f_1 = -\frac{\langle 2346\rangle}{\langle 2345\rangle}, \qquad f_2 = \frac{\langle 2345\rangle\langle 1236\rangle}{\langle 1234\rangle\langle 2356\rangle},$$

$$f_3 = -\frac{\langle 2356\rangle}{\langle 2346\rangle}, \qquad f_4 = \frac{\langle 2346\rangle\langle 1256\rangle}{\langle 2456\rangle\langle 1236\rangle}, \qquad f_5 = -\frac{\langle 2456\rangle}{\langle 2356\rangle},$$

$$f_6 = \frac{\langle 2356\rangle\langle 1456\rangle}{\langle 3456\rangle\langle 1256\rangle}, \qquad f_7 = -\frac{\langle 3456\rangle}{\langle 2456\rangle}, \qquad f_8 = -\frac{\langle 2456\rangle}{\langle 1456\rangle}.$$

This solution contains three non-frozen coordinates $\{\langle 2346\rangle, \langle 2356\rangle, \langle 2456\rangle\}$ that make up a cluster of $Gr(4,6)$!



[Drummond, Foster, Gürdogan, 1810.08149]

Summary of results

From plabic graphs, graphical moves, and solving $C \cdot Z = 0$, we are able to obtain:

- Full $n = 6, 7$ symbol alphabet.
- 18 algebraic letters appearing in the 8-point two-loop NMHV amplitude.
- Parts of $n = 8$ rational alphabet.

Thanks for listening!