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bctp Bethe Center for Theoretical Physics

Analytic Structure of Banana Amplitudes

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Joint work with Kilian Bönisch, Fabian Fischbach, Albrecht Klemm & Reza Safari

"The l-loop Banana Amplitude from GKZ Systems and relative Calabi-Yau Periods" [1912.06201]
"Analytic Structure of all Loop Banana Amplitudes" [2008.10574]

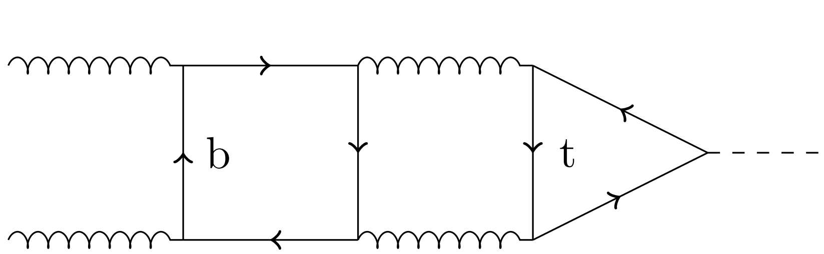
DESY Virtual Theory Forum
September 24, 2020

Motivation

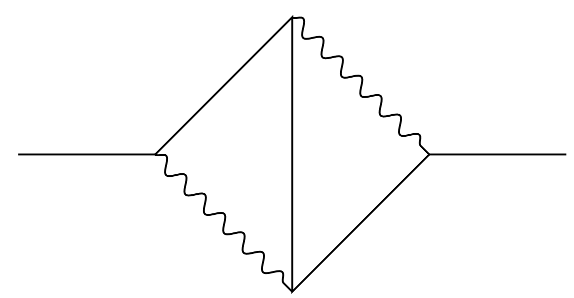
Physical interest:

Scattering processes are calculated perturbatively through Feynman diagrams

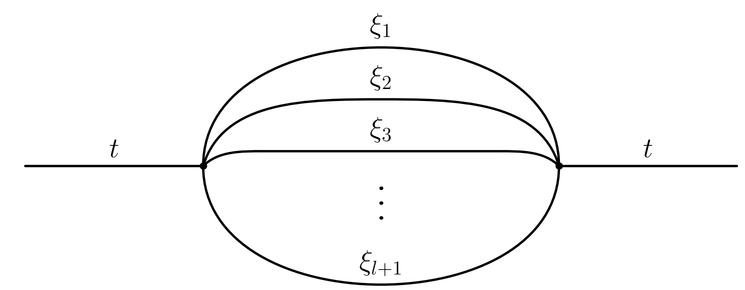
Precision measurements require high loop calculations



three loop Higgs correction



kite diagram



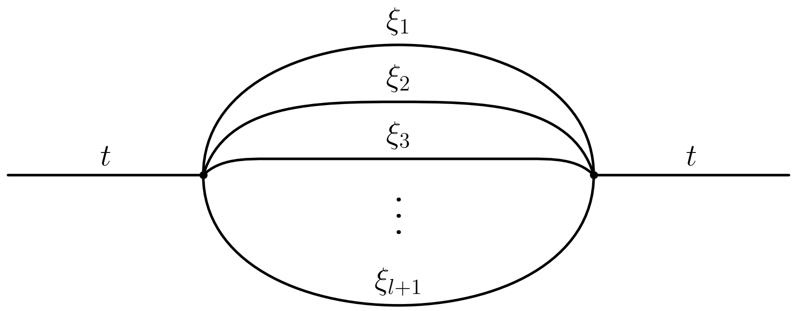
banana diagram

Mathematical interest:

Appearance of special functions (elliptic Polylogarithms, iterated integrals, Bessel functions, …) ➡ Function space of Feynman integrals?

Algebraic and geometric structure behind Feynman integrals ➡ Projective varieties, motives, period integrals, $\widehat{\Gamma}$-class conjecture, …

Representations of Feynman Integrals



E: # propagators	➡	$l+1$
v: # vertices	➡	2
l: # loops	➡	l
ν_k: # powers propagators	➡	1 or higher $\nu = \sum \nu_k$

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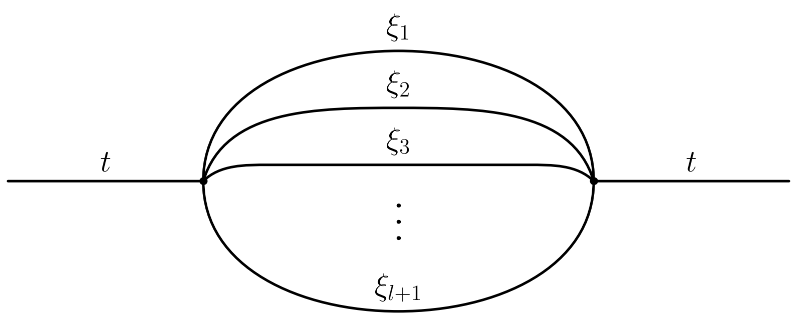

Feynman rules

graph theory

$\ldots$

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Feynman Representation

$$\mathcal{I} \sim \int_{(\mathbb{R}^{1,D-1})^l} \frac{\prod_{k=1}^{l} \mathrm{d}^D l_k}{\prod_{k=1}^{l} (q_k^2 - \xi_k^2 + i\epsilon)^{\nu_k}}$$

Symanzik Representation

$$\mathcal{I} = \int_{x_k \geq 0} \prod_{k=1}^{E} x_k^{\nu_k - 1} \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}} \mu_l$$

$$\mu_l = \sum_{k=1}^{l+1} (-1)^{k+1} x_k \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_{l+1}$$

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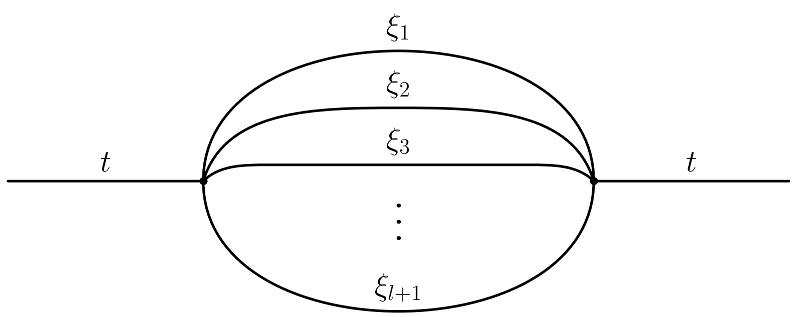

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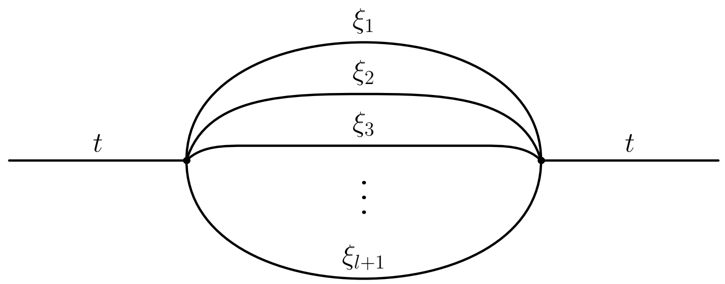
Graph polynomials

- First Symanzik polynomial $\mathcal{U}$ — homogeneous polynomial of degree l in x_i
  No kinematic dependence
- Second Symanzik polynomial $\mathcal{F}$ — homogeneous polynomial of degree $l+1$ in x_i
  dependence on masses and momenta

Geometric Realization

In $D = 2$ the banana diagram is given by:



$$\mathcal{I}_l(t, \xi_i) = \int_{\sigma_l} \frac{\mathcal{U}^0}{\mathcal{F}^1} \mu_l = \int_{\sigma_l} \frac{\mu_l}{P_{\Delta_l}(t, \xi_i; x_i) \prod_{k=1}^{l+1} x_k}$$

$$P_{\Delta_l}(t, \xi_i; x_i) := \left(t - \left(\sum_{k=1}^{l+1} \xi_k^2 x_k \right) \left(\sum_{k=1}^{l+1} \frac{1}{x_k} \right) \right)$$

$$\sigma_l := \left\{ [x_1 : \ldots : x_{l+1}] \in \mathbb{RP}^l | x_i \geq 0, \forall i \right\}$$

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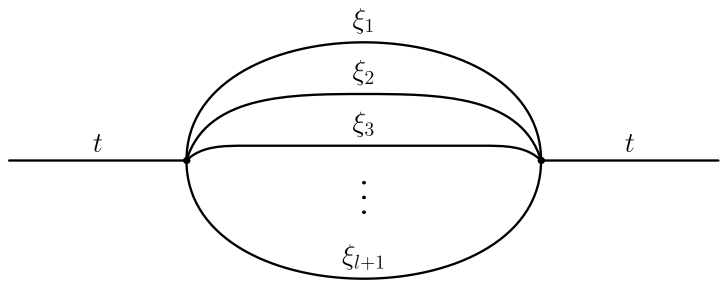

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1) Hypersurface in toric ambient space

$$M_{l-1} = \{P_{\Delta_l}(x) = 0 | x \in \mathbb{P}_{\hat{\Delta}_l}\}$$

- Clear connection from differential to geometry
- Generic hypersurface constraint hast far too many parameters

$\longrightarrow$ Subslice problem

- Differential equations for periods "for free"

2) Complete intersection model

$$W_{l-1} = \left(\begin{array}{c||cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array}\right) \subset \left(\begin{array}{c||c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 \end{array}\right) = F_l$$

- Hidden connection through maximal cut period
- Number of parameters fits to physical ones
- Differential equations for periods "for free"

Equal Mass Case: Operators

We set all masses to unity, i.e. $\xi_i = 1$ for $i = 1, \ldots, l+1$

Maximal cut integral

$$\mathcal{T}_{T^l}(s) = \int_{T^l} \frac{\mu_l}{\left(1/s - \left(\sum_{k=1}^{l+1} \xi_k^2 x_k\right)\left(\sum_{k=1}^{l+1} \frac{1}{x_k}\right)\right)}$$

$$\sim \sum_{n=0}^{\infty} s^{n+1} \sum_{|k|=n} \binom{n}{k_1, \ldots, k_{l+1}}^2 =: \varpi_0(s)$$

$s = 1/t$

Find differential operator annihilating maximal cut integral

$\longrightarrow$ Set of solutions (almost) describe Feynman integral as linear combination

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$\longrightarrow$ Set of solutions (almost) describe Feynman integral as linear combination

Observation: $\varpi_0(s)$ is double Borel sum of the $(l+1)$th symmetric power of $\sum_{k=0}^{\infty} \frac{1}{(k!)^2} z^k = I_0(2\sqrt{z})$

$\longrightarrow$ Get easily PF equations from this $\mathcal{L}_l \mathcal{T}_{T^l} = 0$

E.g. for $l = 4$ we find:

$$\mathcal{L}_4 = 1 - 5s + (-4 + 28s)\theta + \left(6 - 63s + 26s^2 - 225s^3\right)\theta^2 + \left(-4 + 70s - 450s^3\right)\theta^3 - (-1+s)(-1+9s)(-1+25s)\theta^4 \qquad \theta = s\partial_s$$

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For full Feynman integral we get inhomogeneity $\mathcal{L}_l \mathcal{F}_{\sigma_l} = S_l = -(l+1)!\ s$

Equal Mass Case: Local Solutions

Around the MUM point $s = 0$ we have a local Frobenius basis of the form:

$$\varpi_k = \sum_{j=0}^{k} \binom{k}{j} \log(s)^j \, \Sigma_{k-j} \qquad \text{for } k = 1, \ldots, l-1$$

$$\varpi_l = (-1)^{l+1}(l+1) \sum_{j=0}^{l} \binom{l}{j} \log(s)^j \, \Sigma_{l-j}$$

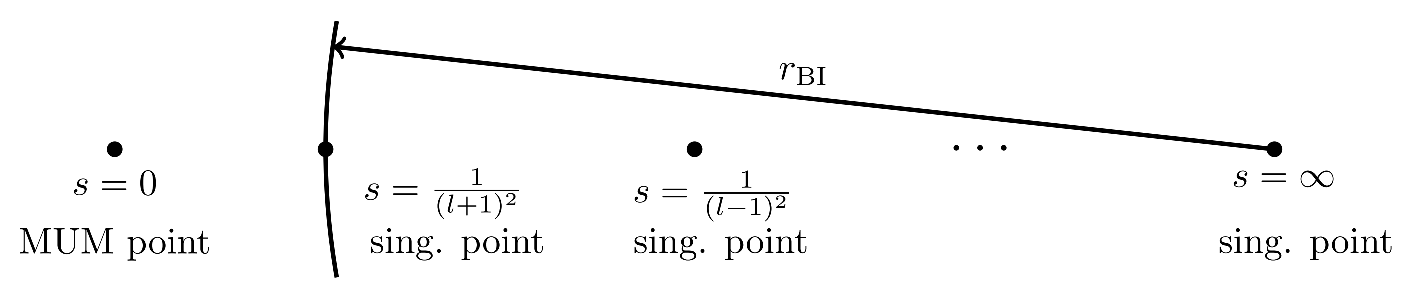
Again for $l = 4$ we have:

$$\begin{aligned}
\varpi_0 &= s + 5s^2 + +45s^3 + 545s^4 + 7885s^5 + \cdots \\
\Sigma_1 &= 8s^2 + 100s^3 + \frac{4148}{3}s^4 + \frac{64\,198}{3}s^5 + \cdots \\
\Sigma_2 &= 2s^2 + \frac{197}{2}s^3 + \frac{33\,637}{18}s^4 + \frac{2\,402\,477}{72}s^5 + \cdots \\
\Sigma_3 &= -12s^2 - \frac{267}{2}s^3 - \frac{19\,295}{18}s^4 - \frac{933\,155}{144}s^5 + \cdots \\
\Sigma_4 &= 1830s^3 + \frac{112\,720}{3}s^4 + \frac{47\,200\,115}{72}s^5 + \cdots
\end{aligned}$$

Singularity structure of PF equation determines radius of convergence

$\longrightarrow$ Discriminant: $\Delta(\mathcal{L}_l) = s \prod_{j=0}^{\lfloor \frac{l+1}{2} \rfloor} \left(1 - s(l+1-2j)^2\right)$

Moduli space: $\mathbb{P}^1 \setminus \left(\bigcup_{j=0}^{\lfloor \frac{l+1}{2} \rfloor} \left\{ \frac{1}{(l+1-2j)^2} \right\} \cup \{0\} \cup \{\infty\} \right)$



Equal Mass Case: λ -Coefficients and $\widehat{\Gamma}$ -Conjecture

Actual Feynman integral as linear combination of Frobenius basis:

$$\mathcal{F}_{\sigma_l} = \sum_k \lambda_k^{(l),\, loc} \varpi_k^{loc} \quad \text{with } \lambda_k^{(l),\, loc} \in \mathbb{C}$$

Numerical computation of $\mathcal{F}_{\sigma_l}$ yields λ-coefficients (Bessel function representation, analytic continuation)

We could guess their analytic form ($l \leq 20$, ~ 300 digits), e.g. for $l = 4$ around the MUM point $s = 0$:

$$\lambda_0^{(4)} = -450\zeta(4) - i\pi \cdot 80\zeta(3) \qquad \lambda_1^{(4)} = 80\zeta(3) - i\pi \cdot 120\zeta(2)$$
$$\lambda_2^{(4)} = 180\zeta(2) \qquad \lambda_3^{(4)} = i\pi \cdot 20$$

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Actually, we find a generating functional for it:

$$\sum_{l=0}^{\infty} \lambda_0^{(l)} \frac{x^l}{(l+1)!} = -\frac{\Gamma(1-x)}{\Gamma(1+x)} \mathrm{e}^{-2\gamma x + i\pi x}$$

and

$$\lambda_k^{(l)} = (-1)^k \binom{l+1}{k} \lambda_0^{(l-k)}$$

Geometric argument/interpretation of it? $\longrightarrow$ Yes! They follow from a (modified) $\widehat{\Gamma}$-conjecture

$$\mathrm{Im}(\mathcal{F}_{\sigma_l}) = \int_{W_{l-1}} \mathrm{e}^{\omega\cdot \mathrm{t}}\, \widehat{\Gamma}(TW_{l-1}) + \mathcal{O}(\mathrm{e}^{\mathrm{t}})$$

and

$$\mathrm{Re}(\mathcal{F}_{\sigma_l}) = \int_{F_l} \mathrm{e}^{\omega\cdot \mathrm{t}}\, \widehat{\Gamma}(1-c_1)^2 \frac{\sin(2\pi c_1)}{2\pi c_1} + \mathcal{O}(\mathrm{e}^{\mathrm{t}})$$

Mirror map: $\mathrm{t} = \frac{1}{2\pi i} \frac{\varpi_1}{\varpi_0}$

Non Equal Mass Case

Similar structure as in equal mass case:

- (inhomogeneous) PF equations computed from GKZ method

$\mathcal{L}_l, S_l \longrightarrow$ $\mathcal{D}$-modul $\{\mathcal{D}_l^{(k)}\}$, inhomogeneities $\{S_l^{(k)}\}$

- Larger Frobenius basis (logarithmic solutions split)

$\varpi_k \longrightarrow$ ϖ_k^s for $s = 1, \ldots$ (primitive vertical Hodge numbers)

- symmetric splitting of λ-coefficients

$\lambda_k^{(l)} \longrightarrow$ $\lambda_{r,s}^{(l)}$ (" $\sim \lambda_k^{(l)}/\mathrm{Hodge}/\mathrm{komb}$")

Computed explicitly the non equal mass banana Feynman integral up to $l \leq 4$

Conclusions

Found full analytic structure of l-loop banana Feynman integrals

- Equal mass case explicitly for $l \leq 20$ and general results
- Non equal mass case explicitly for $l \leq 4$ and understanding of splitting

$\widehat{\Gamma}$-conjecture allows to proof mathematically our results (Iritani)

Guiding principle: Search for associated CY motive of Feynman graph

$\longrightarrow$ whole machinery from algebraic geometry, number theory

Possible extension to other Feynman graphs

- Extension to other CY period integrals (traintracks, ice cream cone, kite, …)
- General non CY graphs? What structure survives? Underlying motive?

Thank you for your attention

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