One integral to rule them all: an algorithm to compute Feynman integrals

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In collaboration with K. Yan and J.M. Henn

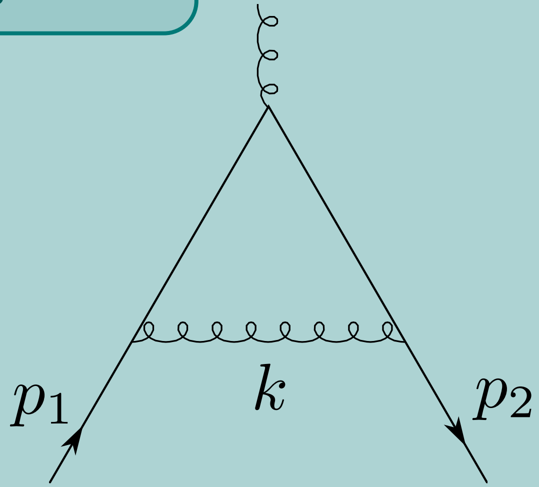
Outline

- About Feynman integrals
- Differential equation method
- New algorithm for finding solvable differential equation
- Applications

Feynman Integrals

Scattering Process → Feynman Diagrams → Numerator Algebra → Feynman Integrals

- Cross-sections



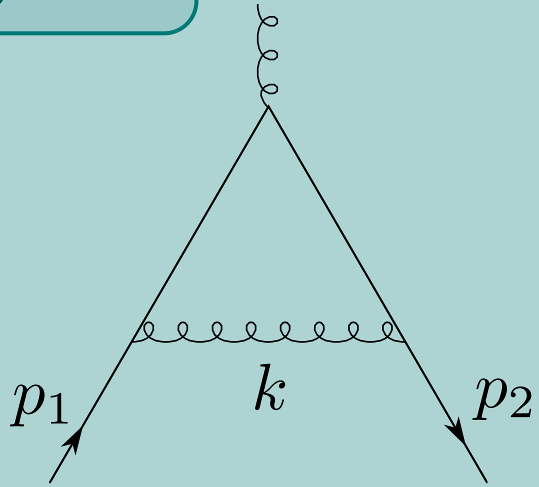
$$I = \int \mathrm{d}^D k \, \frac{1}{k^2 (k+p_1)^2 (k+p_2)^2}$$

Feynman Integrals

Scattering Process → Feynman Diagrams → Numerator Algebra → Feynman Integrals

- Cross-sections

- Interesting mathematical objects:
    - hypergeometric functions, polylogarithms, zeta-values, periods,…



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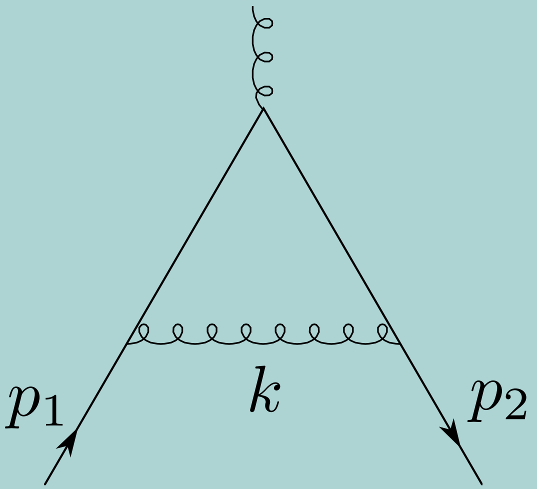
Integral families

- Definition:

$$I_{a_1 a_2 a_3} = \int \mathrm{d}^D k \frac{1}{[k^2]^{a_1} [(k+p_1)^2]^{a_2} [(k+p_2)^2]^{a_3}}$$



Integral families

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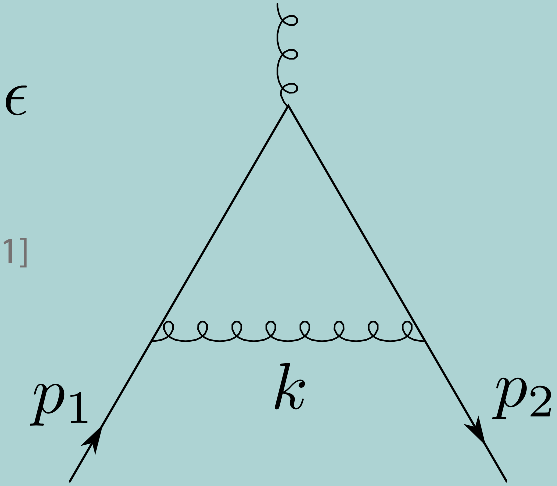
$$I_{a_1 a_2 a_3} = \int \mathrm{d}^D k \frac{1}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_2)^2]^{a_3}}$$

- Relations between integrals: IBP

$$D = 4 - 2\epsilon$$

$$0 = p_1^{\mu} \int \mathrm{d}^D k \frac{\partial}{\partial k^{\mu}} \frac{1}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_2)^2]^{a_3}}$$
$$= c_1 I_{a_1 a_2 a_3} + c_2 I_{(a_1+1) a_2 a_3} + \dots$$

[Tkachov, Chetyrkin, '81]



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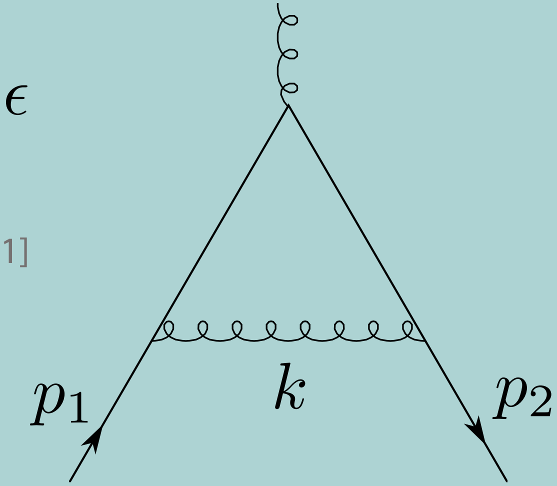

$\Longrightarrow$ finite number of basis or "master" integrals

[Smirnov, Petukhov, '10]

e.g. $$\vec{f} = \begin{pmatrix} I_{111} \\ I_{101} \end{pmatrix}$$

Differential equations method

- Derivative w.r.t. external kinematics

[Kotikov, '91 / Remiddi, '97 / Gehrmann, Remiddi, '00 / Argeri, Mastrolia, '07]

$$\begin{aligned}\frac{\partial}{\partial s} I_{111} &= c_1 I_{121} + c_2 I_{112} + \ldots \\ &= c_1' I_{111} + c_2' I_{101}\end{aligned}$$

$$\frac{\partial}{\partial s} \vec{f} = A(s, \epsilon) \vec{f}$$

$$\vec{f} = \begin{pmatrix} I_{111} \\ I_{101} \end{pmatrix}$$

$$s = (p_1 + p_2)^2$$

$$D = 4 - 2\epsilon$$

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- Solve by transforming to canonical form: $\vec{g} = T \vec{f}$

[Henn, '13]

$$\frac{\partial}{\partial s} \vec{g} = \epsilon \, \tilde{A}(s) \vec{g} \qquad \vec{g} = \mathrm{P}\, e^{\epsilon \int \tilde{A} \, \mathrm{d}s} \, \vec{g}^{(0)}$$

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$$\vec{g} = \sum_{k=0}^{\infty} \epsilon^k \vec{g}^{(k)}$$

Uniform weight integrals

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- Power in $\epsilon \sim$ nr. of integrations

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E.g. $\epsilon^0 \times 1, \quad \epsilon^1 \times \log(1-x), \quad \epsilon^2 \times \mathrm{Li}_2(x), \ldots$

$$\int_0^x \frac{1}{t-1}\mathrm{d}t \qquad -\int_0^x \frac{1}{t}\log(1-t)\mathrm{d}t$$

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assign weight -1

$$\int_0^x \frac{1}{t-1}\mathrm{d}t \qquad -\int_0^x \frac{1}{t}\log(1-t)\mathrm{d}t$$

assign weight +1

$\Longrightarrow$ Canonical basis consists of uniform weight integrals

Finding a uniform weight basis

- Various algorithms and methods exist:
  - Canonica, Epsilon, Fuchsia
  - Unitarity Cuts, Power-counting, Intuition, ...

$$\begin{tikzcd} \frac{\partial}{\partial s}\vec{f} = A(s,\epsilon)\vec{f} \arrow[d] \\ \frac{\partial}{\partial s}\vec{g} = \epsilon\,\tilde{A}(s)\vec{g} \end{tikzcd}$$

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- Compute canonical form from initial integral

[Höschele, Hoff, Ueda, '14]

[CD, Henn, Yan, '20]

n 1^{st} - order DEs

$$\frac{\partial}{\partial s}\vec{f} = A(s,\epsilon)\vec{f}$$

$$f_1 = g_1$$

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n 1^{st} - order DEs

one n^{th} - order DE

$$\frac{\partial}{\partial s}\vec{f} = A(s,\epsilon)\vec{f} \longrightarrow f_1 = \Psi \begin{pmatrix} f_1' \\ \vdots \\ f_1^{(n)} \end{pmatrix}$$

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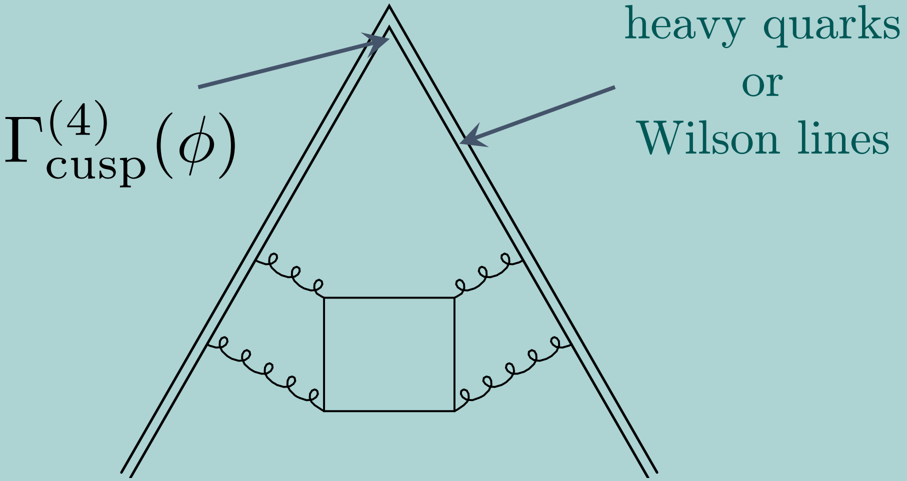
make ansatz

fix by comparing

Applications: large scales

- QCD four-loop cusp anomalous dimension
  - important for IR structure of scattering processes
  - HQET, SCET, ...
  - Resummation
- 519 sectors
- Largest sector has 17 master integrals
  - Algorithm takes 2 min.
- Initial integral $f_1 = g_1$ easy to find



[Brüser, CD, Henn, Yan, '20]

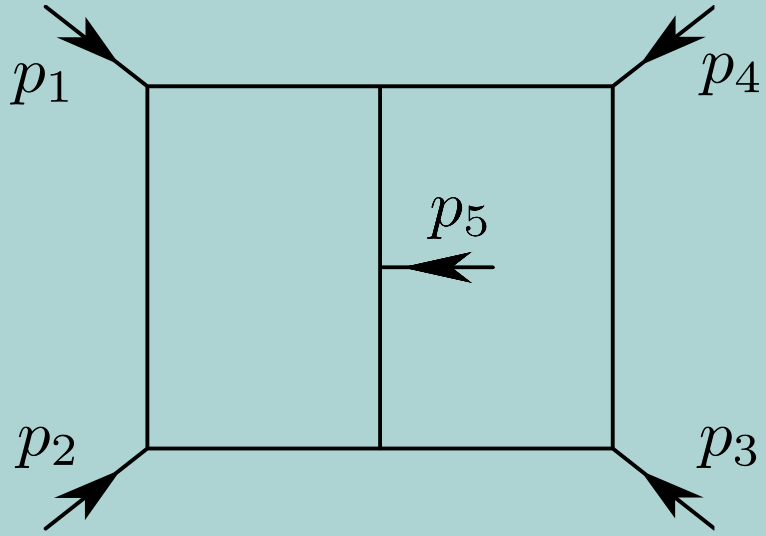
Applications: multi-variable

- (Non-planar) two-loop five-particle scattering

[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, ’19 /
Abreu, Dixon, Herrmann, Page, Zeng, ‘19]

- Five kinematic variables

$$s_{ij} = (p_i + p_j)^2$$



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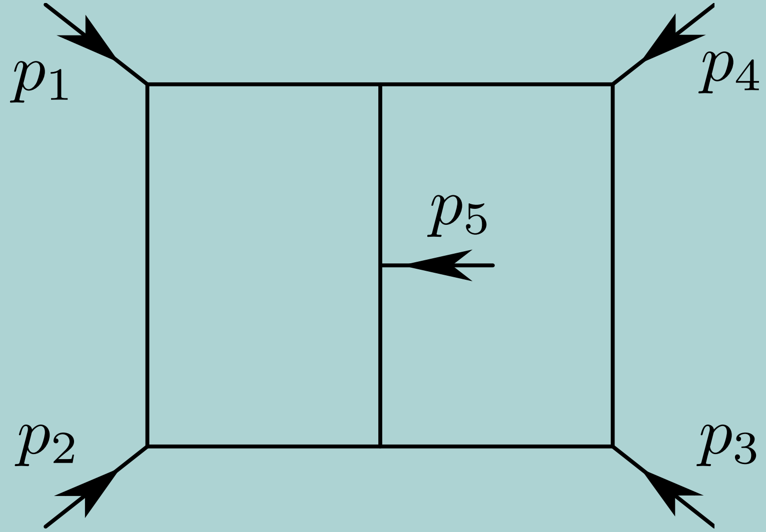
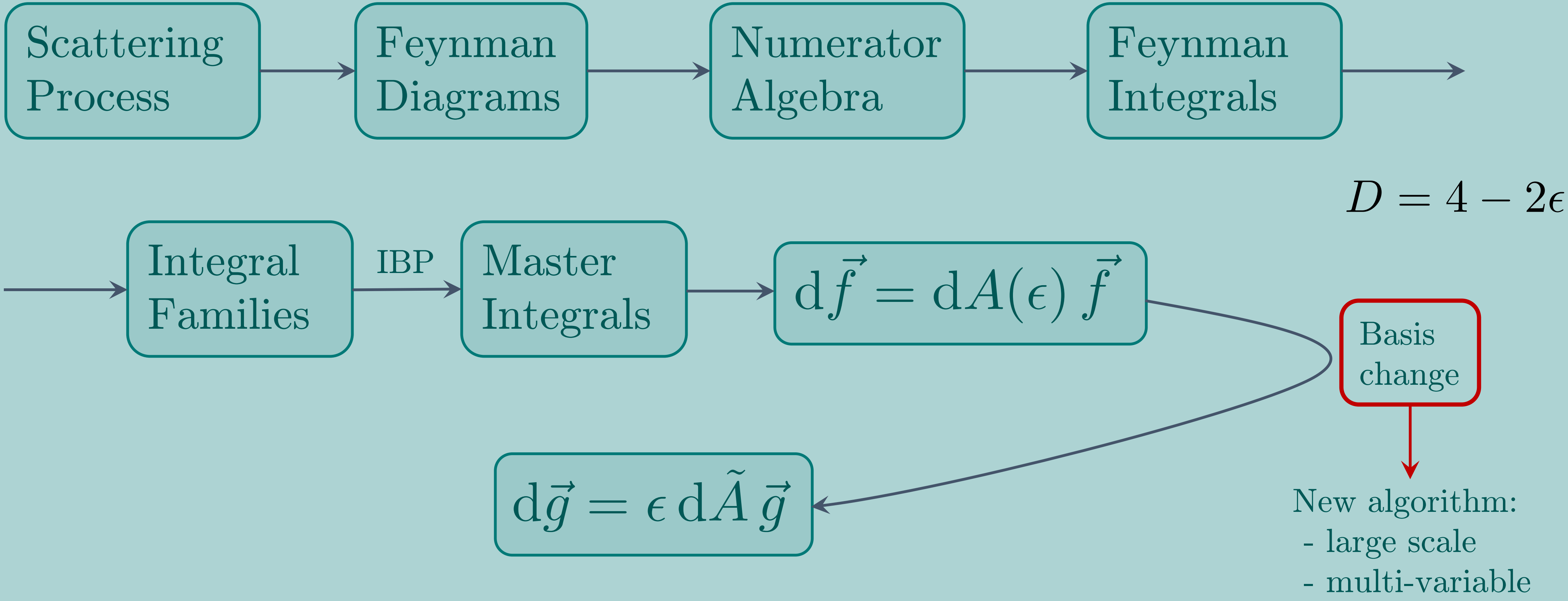


- Five kinematic variables

$$s_{ij} = (p_i + p_j)^2$$

- Methods for multivariate case
  - one variable at a time $(f_1', \ldots, f_1^{(n)})$
  - use partial derivatives $(\partial_1 f_1, \ldots, \partial_5 f_1, \partial_1^2 f_1, \partial_1 \partial_2 f_1, \ldots)$
- Algorithm takes 5 min.

Workflow



Scattering Process

Feynman Diagrams

Numerator Algebra

Feynman Integrals

$D = 4 - 2\epsilon$

Integral Families

IBP

Master Integrals

$\mathrm{d}\vec{f} = \mathrm{d}A(\epsilon)\,\vec{f}$

Basis change

$\mathrm{d}\vec{g} = \epsilon\,\mathrm{d}\tilde{A}\,\vec{g}$

New algorithm:
- large scale
- multi-variable