Large-N localization and quiver CFT

KZ, 2003.00993

Two simplest $N=2$ CFTs:

- SQCD

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}^2 + i\,\bar{\Psi}_f \not{D} \Psi^f + \text{superpartners}$$

$N_f = N_c$

$a \neq c$ holographic dual never weakly coupled

- $N=4$ SYM

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}^2 + i\,\mathrm{tr}\,\bar{\Psi} \not{D}_{adj} \Psi + \text{superpartners}$$

$a = c$ has weakly-coupled gravitational description at $\lambda \gg 1$

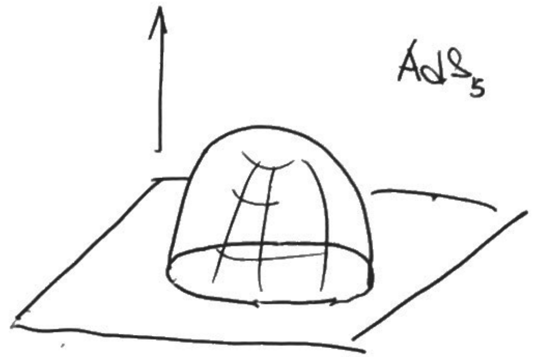
$\lambda = g^2 N_c$

Circular Wilson loop

$$W(C) = \left\langle \frac{1}{N} \operatorname{tr} P \exp \oint_C ds \left(i \dot{x}^\mu A_\mu + |\dot{x}| \Phi \right) \right\rangle$$

SYM:

$$W(C) \overset{\lambda \to \infty}{\simeq} \underbrace{\sqrt{\frac{2}{\pi}}\, \lambda^{-\frac{3}{4}}}_{\text{quantum fluctuations}} \underbrace{e^{\sqrt{\lambda}}}_{\text{Area law}} :$$



$$T = \frac{\sqrt{\lambda}}{2\pi}$$

$$A = -2\pi$$

$$e^{-TA} = e^{\sqrt{\lambda}}$$

SQCD:

$$W(C) \overset{\lambda \to \infty}{\simeq} \text{const} \cdot \frac{\lambda^3}{(\ln \lambda)^{3/2}}$$

Passerini, Z '11

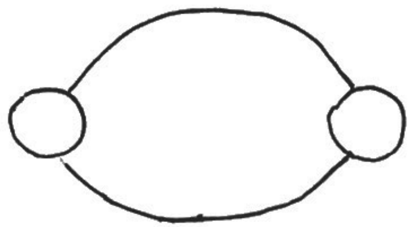
Quiver CFT

Interpolating theory:

- SQCD w. gauged $SU(N)_F$

$$\mathcal{L} = -\frac{1}{4} \operatorname{tr}_c F_{1\mu\nu}^2 - \frac{1}{4} \operatorname{tr}_f F_{2\mu\nu}^2 + i \bar{\Psi} \not{D} \Psi + \text{superpartners}$$

$$D_\mu \psi^{if} = \partial_\mu \psi^{if} + i g_1 A^i_{1\mu j} \psi^{jf} + i g_2 A^f_{2\mu g} \psi^{ig}$$

$\lambda_1 = g_1^2 N$  $\lambda_2 = g_2^2 N$

$g_2 = 0 \implies$ Second $SU(N)$ decouples $\implies$ SQCD

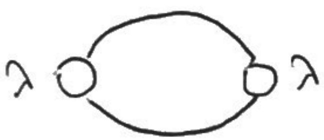
$g_1 = g_2 \implies$ Extra $\mathbb{Z}_2$ symmetry

Orbifold equivalence

$$A_\mu = \begin{bmatrix} A_{1\mu} & \\ & A_{2\mu} \end{bmatrix} \qquad \Psi = \begin{bmatrix} & \psi \\ \psi^\dagger & \end{bmatrix}$$

$$\mathcal{L}^{SU(2N)}_{N=4\ SYM} = -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}^2 + i\,\mathrm{Tr}\,\overline{\Psi}\,\not{D}_{adj}\,\Psi + \ldots = \mathcal{L}^{SU(N)\times SU(N)}_{Quiver}$$

at $\lambda_1 = \lambda_2$



$= SYM/\mathbb{Z}_2$

- planar diagrams are the same

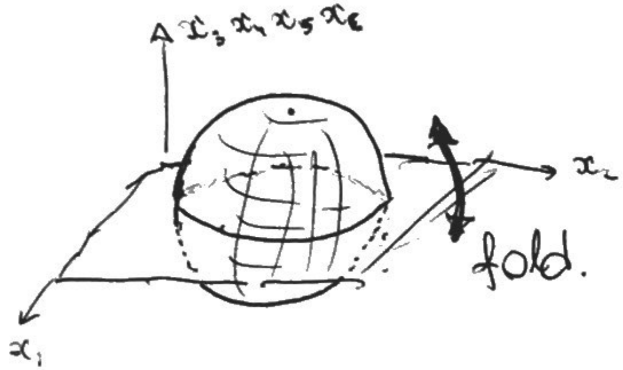
Bershadsky, Johansen '98

Lawrence, Nekrasov, Vafa '98

Holographic dual

$AdS_5 \times (S^5/\mathbb{Z}_2)$

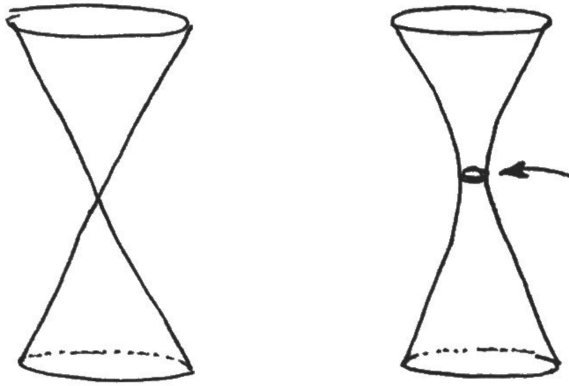
Kachru, Silverstein '98



Effective coupling: $\displaystyle \frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$

String tension: $\displaystyle T = \frac{\sqrt{\lambda}}{2\pi}$

B-flux



orbifold resolution

vanishing cycle : $S_1 \times S_2^2$ for $S^5/\mathbb{Z}_2$

topological term.

$$S_{str} = \int \left(\frac{\sqrt{\lambda}}{4\pi}\, dX^M \wedge * dX^N G_{MN} + \Theta\, dX^M \wedge dX^N \underbrace{B_{MN}}_{\text{const}} \right)$$

$$\boxed{\Theta = \pi - \pi \frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} = \frac{2\pi\lambda_1}{\lambda_1 + \lambda_2}}$$

Lawrence, Nekrasov, Vafa '98
Klebanov, Nekrasov '99
Gadde, Pomoni, Rastelli '09

- Orbifold point $(\lambda_1 = \lambda_2)$: $\Theta = \pi$

String path integral:

$$\langle W(C)\rangle = e^{A\sqrt{\lambda}}\Big(C_0 + \underset{1 \text{ inst.}}{C_1 e^{-\sqrt{\lambda}\varepsilon + i\theta}} + \underset{2 \text{ inst.}}{C_2 e^{-\sqrt{\lambda}2\varepsilon + 2i\theta}} + \dots + c.c.\Big)$$

$$\overset{\varepsilon\to 0}{=} F(\cos\theta)\, e^{A\sqrt{\lambda}}$$

$$\frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

$$\theta = \frac{2\pi\lambda_1}{\lambda_1+\lambda_2}$$

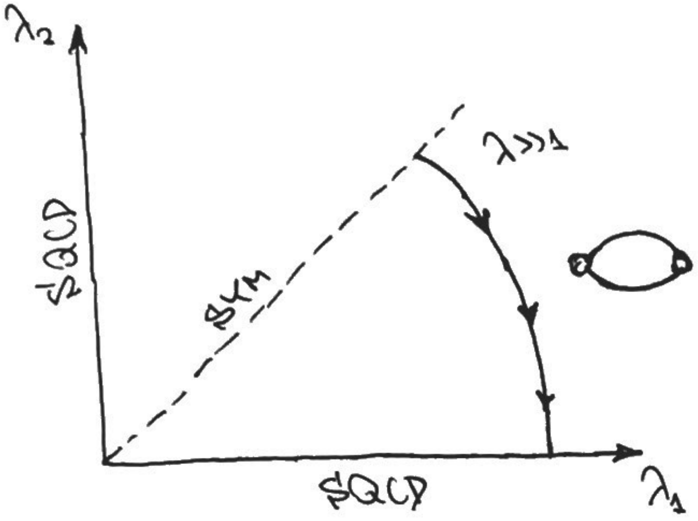
Puzzles:

- In which sense corr. func. are periodic in $\frac{\lambda_1}{\lambda_1+\lambda_2}$?

- Why $\theta\to 0$ and $\theta\to 2\pi$ are singular?

Spectral data: protected & unprotected

Gadde, Pomoni, Rastelli '08, 10
Gadde, Rastelli '10



Goal:

- compute circular Wilson loop at

$$\lambda_1, \lambda_2 \longrightarrow \infty \qquad \frac{\lambda_1}{\lambda_2} \text{ - fixed.}$$

Localization

Exact partition function on S^4:

$$Z=\int\prod_{a=1}^{2}\prod_{i}da_{ai}\,\frac{\prod_a\prod_{i<j}\left(a_{ai}-a_{aj}\right)^2H^2\left(a_{ai}-a_{aj}\right)}{\prod_{ij}H^2\left(a_{1i}-a_{2j}\right)}\,e^{-\sum_a\frac{8\pi^2N}{\lambda_a}\sum_i a_{ai}^2}$$

Pestun'07

$$H(x)=\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}$$

Exact circular Wilson loop:

$$W_a=\left\langle\frac{1}{N}\sum_i e^{2\pi a_{ai}}\right\rangle$$

- Matrix Model is not Gaussian even at $\lambda_1=\lambda_2$

Saddle-point equations

$$\rho_a(x) = \left\langle \frac{1}{N} \sum_i \delta(x - a_{ai}) \right\rangle$$

$$\int_{-\mu_1}^{\mu_2} dy\, \rho_1(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_2}^{\mu_2} dy\, \rho_2(y)\, K(x-y) = \frac{8\pi^2}{\lambda_1} x$$

$$\int_{-\mu_2}^{\mu_2} dy\, \rho_2(y) \left(\frac{1}{x-y} - K(x-y) \right) + \int_{-\mu_1}^{\mu_2} dy\, \rho_1(y)\, K(x-y) = \frac{8\pi^2}{\lambda_2} x$$

$$K(x) = -\frac{H'(x)}{H(x)} = x\left(\psi(1+ix) + \psi(1-ix) + 2\gamma \right)$$

- At $\lambda_1 = \lambda_2$: ansatz $\rho_1(x) = \rho_2(x)$ goes thru.

$$\Downarrow$$

orbifold equivalence : $\rho_1 = \rho_2 = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2}$

Strong coupling

"Locking":

- at $\lambda_1, \lambda_2 \gg 1$ w. $\lambda_1/\lambda_2 \sim \mathcal{O}(1)$: $\rho_1(x) \approx \rho_2(x)$

Approximate solution:

$$\rho_{12}(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2}$$

$$\mu = \frac{\sqrt{\lambda}}{2\pi}$$

$$\frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Rey, Suyama '10

Mitev, Pomoni '14

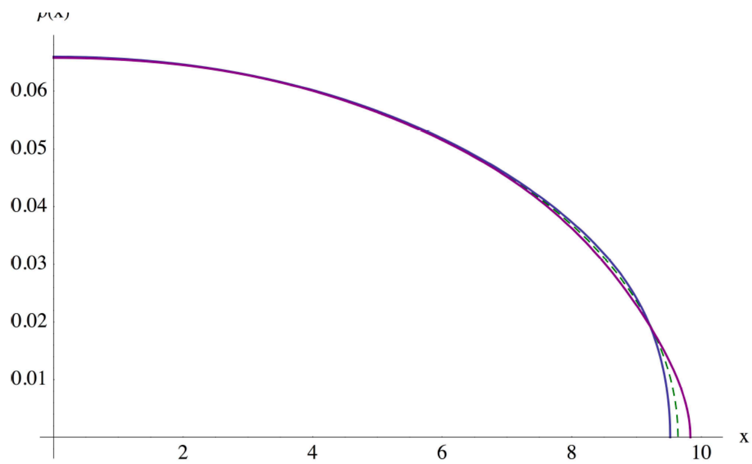


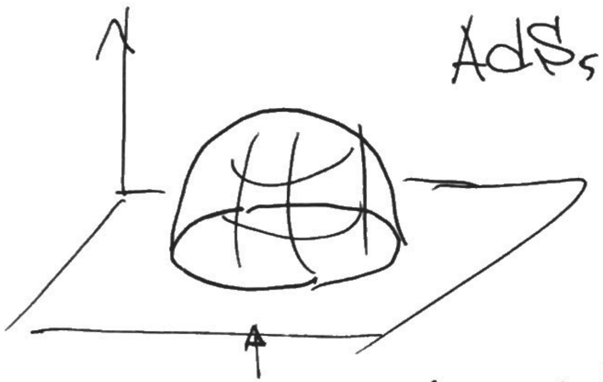
Figure 2: The eigenvalue densities ρ_1 (purple line) and ρ_2 (blue line) obtained by numerically solving (2.6), (2.7) for $\lambda_1 = 5320$, $\lambda_2 = 2797$. The dashed line is the Wigner distribution with the effective coupling $\lambda = 3667$. The density for the gauge

Wilson loops:

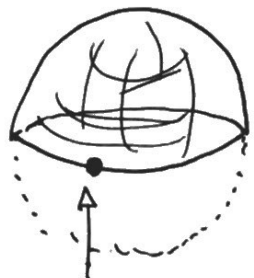
$$W_a = \int_{-\mu_a}^{\mu_a} dx\, \rho(x)\, e^{2\pi x} \simeq e^{2\pi \mu_a}$$

$$\mu_a \simeq \frac{\sqrt{\lambda}}{2\pi}$$

$$W_a \simeq e^{\sqrt{\lambda}}$$



AdS_5 $\times$ $S^2/\mathbb{Z}_2$



unaffected by orbifold projection

insensitive to b-flux

$$\Downarrow$$

$$W_{holographic} \simeq e^{\sqrt{\lambda}} \quad \checkmark$$

NLO

$$(H-K)*\rho_1 + K*\rho_2 = \frac{8\pi^2}{\lambda_1}x$$

$$(H-K)*\rho_2 + K*\rho_1 = \frac{8\pi^2}{\lambda_2}x$$

Hilbert $\qquad\qquad K \approx x\ln x^2 + 2\gamma x + \frac{1}{6x} + \dots$

$$H * \sqrt{\mu^2 - x^2} = \pi x \qquad K^\infty * \sqrt{\mu^2-x^2} = \frac{\pi}{3}x^3 + \left(\pi\mu^2 \ln\frac{\mu e^{\gamma+\frac{1}{2}}}{2} + \frac{\pi}{6}\right)x$$

$$H * \frac{1}{\sqrt{\mu^2-x^2}} = 0 \qquad K^\infty * \frac{1}{\sqrt{\mu^2-x^2}} = 2\pi \ln\frac{\mu e^{\gamma+1}}{2}\, x$$

$$H * \frac{1}{(\mu^2-x^2)^{3/2}} = 0 \qquad K^\infty * \frac{1}{(\mu^2-x^2)^{3/2}} = -\frac{2\pi}{\mu^2}x$$

Ansatz:
$$\rho_a(x) = A\sqrt{\mu_a^2-x^2} + \frac{2\mu_a A B_a}{\sqrt{\mu_a^2-x^2}} + \frac{4\mu_a^2 A C_a}{(\mu_a^2-x^2)^{3/2}} + \dots$$

$$\mathcal{O}(\mu) = \mathcal{O}(\sqrt{\lambda}) \qquad \mathcal{O}(1) \qquad \mathcal{O}(1/\sqrt{\lambda})$$

Bulk solution

$$\rho_a(x) = A\sqrt{\mu_a^2 - x^2} + \frac{2\mu_a A B_a}{\sqrt{\mu_a^2 - x^2}} + \frac{4\mu_a^2 A C_a}{(\mu_a^2 - x^2)^{3/2}}$$

7 parameters: A, μ_a, B_a, C_a

Normalization:

$$B_a = \frac{1}{2\pi A \mu_a} - \frac{\mu_a}{4}$$

2 eqs

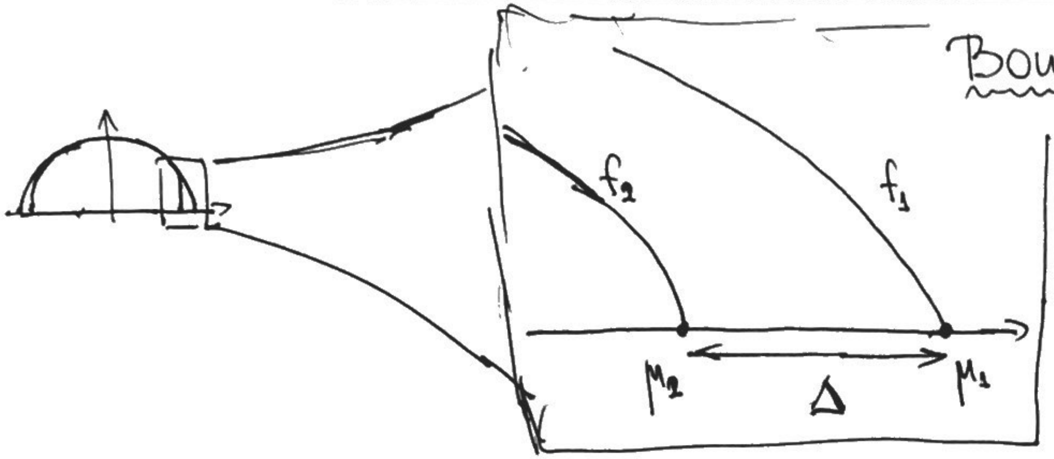
Saddle-point eqs:

$$1 + \frac{\mu_{1,2}^2 - \mu_{2,1}^2}{2} - \frac{2}{\pi A} \ln \frac{\mu_{1,2}}{\mu_{2,1}} + 8\left(C_{1,2} - C_{2,1}\right) = \frac{8\pi}{A\lambda_{1,2}}$$

2 eqs

- # unknowns $= 7 - 2 - 2 = 3$
- wrong boundary conditions!

Boundary



$$\mu_{1,2} = \frac{\sqrt{\lambda}}{2\pi} + \alpha \pm \frac{\Delta}{2}$$

$$\rho_a(x) \simeq A\sqrt{2\mu_a}\, f_a(\mu_a - x)$$

$$f(\xi) \simeq \sqrt{\xi} + \frac{B_a}{\sqrt{\xi}} + \frac{C_a}{\xi^{3/2}}$$

from near-endpoint expansion of bulk solution

- long-distance ($\xi \to \infty$) asymptotics from boundary point of view

matching

Boundary eqs:

$$R * f = R^{\infty} * f^{\infty}$$

$$R(\xi) = \begin{bmatrix} \frac{1}{\xi} - K(\xi) & K(\xi+\Delta) \\ K(\xi+\Delta) & \frac{1}{\xi} - K(\xi) \end{bmatrix}$$

$$f_a^{\infty}(\xi) = \sqrt{\xi} + \frac{B_a}{\sqrt{\xi}} + \frac{C_a}{\xi^{3/2}}$$

$$K^{\infty}(\xi) = \xi \ln \xi^2 + 2\gamma\xi + \frac{1}{6\xi}$$

$$K * f(\xi) = \int_0^{\infty} d\eta \; K(\xi-\eta)\, f(\eta)$$

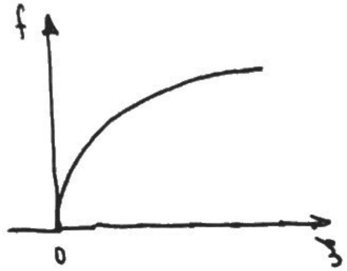
- Wiener-Hopf problem.

Solved by Fourier transform:

$$f_a(\xi) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\xi}\, f_a(\omega)$$

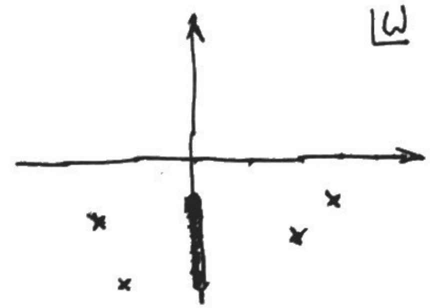
Wiener – Hopf

$f(z)$:



$\Longleftrightarrow$

$f(\omega)$:



Projection:

$$\mathcal{F}_\pm(\omega) = \pm\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi i}\;\frac{\mathcal{F}(\nu)}{\nu-\omega\mp i\varepsilon}$$

↑ analytic in upper/lower half-plane

Wiener-Hopf eqn:

$$R * f = r$$

Solution:

$$f(\omega) = G_+^{-1}(\omega)\left[G_-(\omega)\, r(\omega)\right]_+$$

Analytic factorization of kernel:

$$\boxed{G_- R = G_+}$$

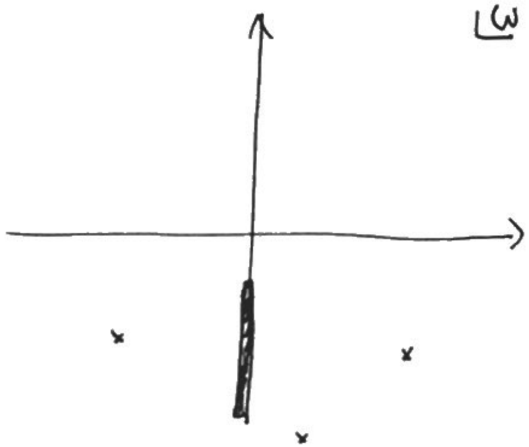
Riemann - Hilbert

$$R(\omega) = 2\pi i \,\operatorname{sign}\omega \,\coth\frac{\omega}{2} \begin{bmatrix} \coth\omega & -\dfrac{e^{i\Delta\omega}}{\sinh\omega} \\ -\dfrac{e^{-i\Delta\omega}}{\sinh\omega} & \coth\omega \end{bmatrix}$$

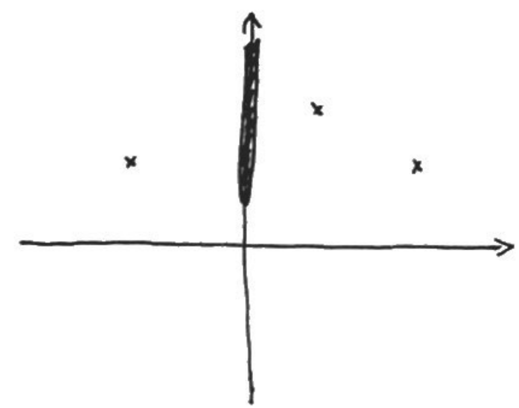
Find G_-, G_+ :

$$G_-(\omega)\, R(\omega) = G_+(\omega)$$

G_+ :



G_- :

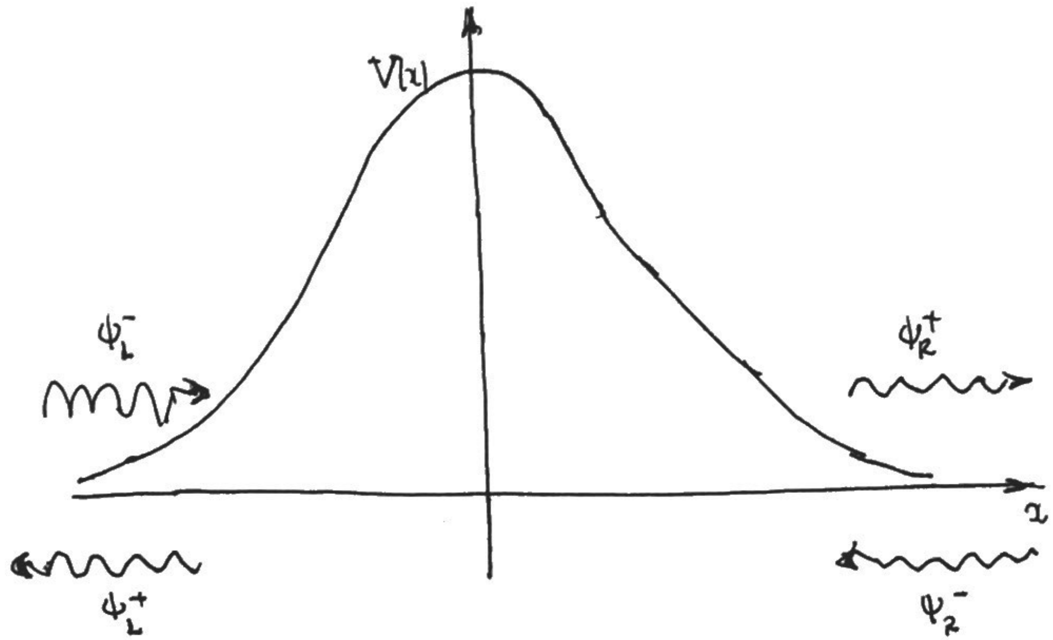


Jost functions

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = k^2\psi$$

$$\psi_L^{\pm} \simeq e^{\mp ikx} \qquad x \to -\infty$$

$$\psi_R^{\pm} \simeq e^{\pm ikx} \qquad x \to +\infty$$



$$\chi_{L,R}^{+} = \psi_{L,R}^{+}\, e^{\pm ikx} \qquad \text{upper}$$

half-plane analytic in k.

$$\chi_{L,R}^{-} = \psi_{L,R}^{-}\, e^{\mp ikx} \qquad \text{lower}$$

S-matrix

$$\begin{bmatrix} \psi_L^+ & \psi_R^+ \end{bmatrix} = \begin{bmatrix} \psi_R^- & \psi_L^- \end{bmatrix} \begin{bmatrix} t(k) & r(k) \\ -\bar{r}(k) & t(k) \end{bmatrix}$$

$$\begin{bmatrix} \chi_L^+ & \chi_R^+ \\ \frac{d\chi_L^+}{dx} - ik\,\chi_L^+ & \frac{d\chi_R^+}{dx} + ik\,\chi_R^+ \end{bmatrix} = \begin{bmatrix} \chi_R^- & \chi_L^- \\ \frac{d\chi_R^-}{dx} - ik\,\chi_R^- & \frac{d\chi_L^-}{dx} + ik\,\chi_L^- \end{bmatrix} \begin{bmatrix} t(k) & r(k)\,e^{-2ikx} \\ -\bar{r}(k)\,e^{2ikx} & t(k) \end{bmatrix}$$

$$G_+ \qquad\qquad\qquad G_- \qquad\qquad\qquad R$$

Pöschl-Teller potential

$$V(x) = \frac{1}{4\cosh^2 x}$$

Jost functions:

$$\phi_R^{\pm} = e^{\pm ikx}\sqrt{1+e^{-2x}}\; {}_2F_1\left(\tfrac{1}{2} \mp ik, \tfrac{1}{2}; 1 \mp ik; -e^{-2x}\right)$$

$$\phi_L^{\pm} = e^{\mp ikx}\sqrt{1+e^{2x}}\; {}_2F_1\left(\tfrac{1}{2} \mp ik, \tfrac{1}{2}; 1 \mp ik; -e^{2x}\right)$$

S-matrix:

$$S(k) = i\,\frac{B\left(\frac{1}{2}+ik, \frac{1}{2}\right)}{B\left(\frac{1}{2}-ik, \frac{1}{2}\right)} \begin{bmatrix} \tanh \pi k & -\frac{i}{\cosh \pi k} \\ -\frac{i}{\cosh \pi k} & \tanh \pi k \end{bmatrix}$$

$$\Big\downarrow \quad k \to \frac{\omega}{\pi} + \frac{i}{2} \qquad x \to -\frac{\pi\Delta}{2}$$

$$R(\omega) = \begin{bmatrix} \coth \omega & -\frac{e^{i\Delta\omega}}{\sinh \omega} \\ -\frac{e^{-i\Delta\omega}}{\sinh \omega} & \coth \omega \end{bmatrix}$$

Solution of Boundary problem

$$f_{1,2}(\omega)=\frac{i^{3/2}\,B\!\left(\frac{1}{2}-\frac{i\omega}{2\pi},\frac{\Delta}{2}\right)}{2\sqrt{\pi}\,(\omega+i\varepsilon)^{3/2}}\int_0^1 du\left(\frac{1+e^{\pm\pi\Delta}u^2}{1-u^2}\right)^{\frac{i\omega}{\pi}}\left(1-\frac{2i\omega}{\pi}\,\frac{1}{1+e^{\pm\pi\Delta}u^2}\right)$$

Boundary conditions and matching to the bulk:

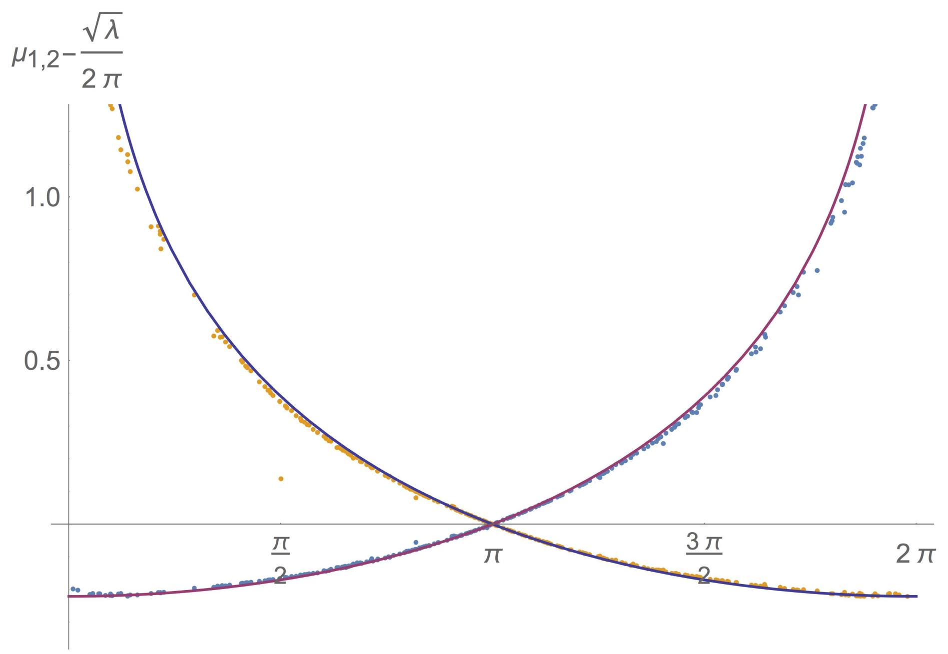
$$\alpha=-\frac{1}{\pi}\ln\sin\frac{\theta}{2}\qquad\qquad \Delta=\frac{2}{\pi}\ln\tan\frac{\theta}{2}$$

$$\theta=\frac{2\pi\lambda_1}{\lambda_1+\lambda_2}$$

- trigonometric functions!

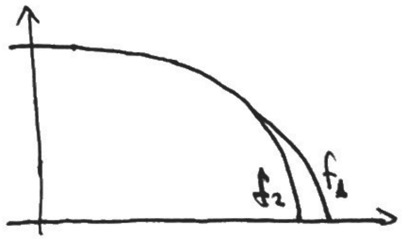
$$\mu_1=\frac{\sqrt{\lambda}}{2\pi}-\frac{1}{\pi}\ln\left(2\cos^2\frac{\theta}{4}\right)$$

$$\mu_2=\frac{\sqrt{\lambda}}{2\pi}-\frac{1}{\pi}\ln\left(2\sin^2\frac{\theta}{4}\right)$$



$\mu_{1,2}-\frac{\sqrt{\lambda}}{2\pi}$
1.0
0.5
θ
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Wilson loops



$$W_a = \int_{-\mu_a}^{\mu_a} dx\, \rho_a(x)\, e^{2\pi x} \underset{x \simeq \mu_a}{\simeq} A\sqrt{2\mu_a}\, e^{2\pi\mu_a} \int_0^\infty d\xi\, f_a(\xi)\, e^{-2\pi\xi}$$

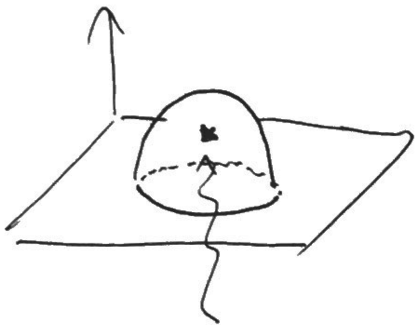
$$W_{1,2} = 8\sqrt{\pi}\, f_{1,2}(2\pi i)\, \lambda^{-3/4}\, e^{\sqrt{\lambda} + 2\pi\alpha \pm \pi\Delta}$$

determines the pre-factor

($\equiv$ one loop in string theory)

Twisted:

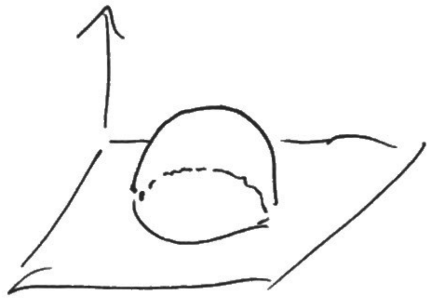
$$W_- = \frac{W_1 - W_2}{2 W_{SYM}}$$



twist operator

Untwisted:

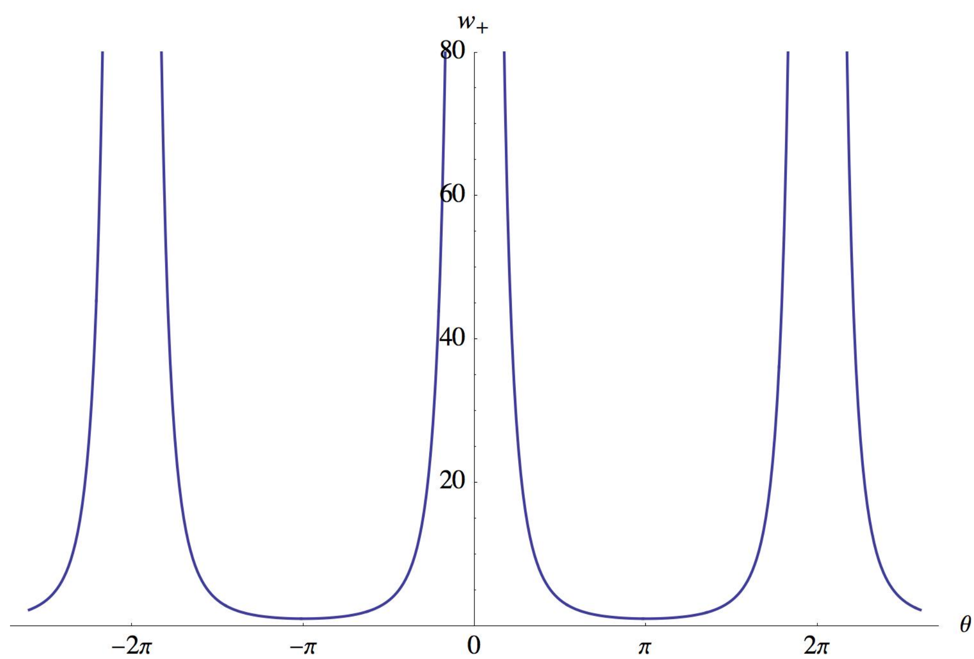
$$W_+ = \frac{W_1 + W_2}{2 W_{SYM}}$$



From localization:

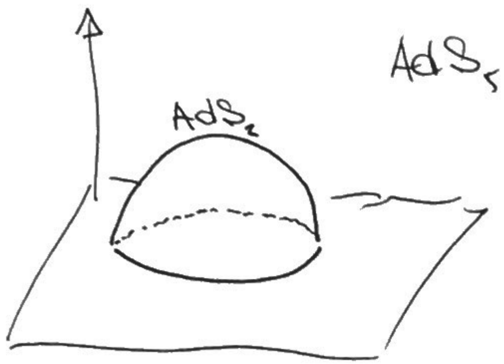
$$\boxed{W_- = -\frac{\pi \cos\frac{\theta}{2}}{2 \sin^3\frac{\theta}{2}}}$$

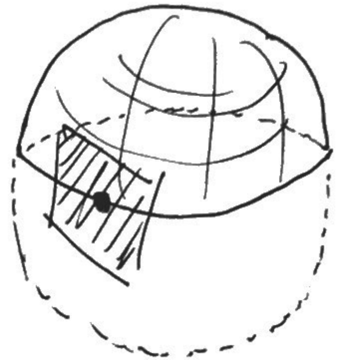
$$\boxed{W_+ = \frac{1 + \frac{\pi-\theta}{2}\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}}$$



w_+
80
60
40
20
θ
-2π
$-\pi$
0
π
2π

String interpretation





tangent space $= \mathbb{R}^1 \times \mathbb{C}^2/\mathbb{Z}_2$

$$W_+ = \frac{W_1 + W_2}{2 W_{SYM}} = \frac{Z_{\mathbb{C}^2/\mathbb{Z}_2}(\theta)}{Z_{\mathbb{C}^2/\mathbb{Z}_2}(\pi)}$$

worldsheet : AdS_2

target : $\mathbb{C}^2/\mathbb{Z}_2$

$$Z_{\mathbb{C}^2/\mathbb{Z}_2}(\theta) = \lim_{\varepsilon \to 0} \sum_k \mathcal{A}_k \, e^{-\sqrt{\lambda}\, \varepsilon |k| + i\theta k}$$

$\uparrow$ k-instanton amplitude

$$W_+ = \sum_k A_k \, e^{ik\theta}$$

Localization prediction:

$$W_+ = \frac{1 + \frac{\pi-\theta}{2}\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

$$A_k \overset{?}{=} \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{-ik\theta} \, \frac{1 + \frac{\pi-\theta}{2}\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \infty$$

- ill-defined, if taken at face value

(problem with $\varepsilon \to 0$ on the string side?)

Decoupling limit

$\theta \to 0$:

$$W_+ \simeq \frac{4\pi}{\theta^3} \qquad \text{non-integrable singularity}$$

But expected from SQCD.

$$\theta = \frac{2\pi\lambda_1}{\lambda_1+\lambda_2} \ll 1 \iff \lambda_1 \ll \lambda_2$$

- $SU(N)_1$ decouples leaving SQCD w. $\lambda = \lambda_2$

but effective coupling $\lambda = \dfrac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} \approx 2\lambda_1$

$T = \dfrac{\sqrt{\lambda}}{2\pi}$ becomes small

$$\theta \approx 2\pi\frac{\lambda_1}{\lambda_2}$$

$W_+ \sim (\lambda_2)^3$ recovers the λ^3 scaling of Wilson loop in SQCD.

Conclusions

$$\longleftrightarrow \quad AdS_5 \times \left(S_5/\mathbb{Z}_2\right)_\theta \qquad \frac{2}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

$$T = \frac{\sqrt{\lambda}}{2\pi} \qquad \theta = \frac{2\pi\lambda_1}{\lambda_1+\lambda_2}$$

- Lots of field-theory data:
  - spectrum Gadde, Pomoni, Rastelli '09, 10; Gadde, Rastelli '10
  - Wilson loops this talk
- Severe singularity at $\theta \to 0$ and $\theta \to 2\pi$. Why?

  Seen at "one loop" $\Longleftrightarrow$ $\mathbb{C}^2/\mathbb{Z}_2$ orbifold σ-model on AdS_2
- On field-theory side:
  - generalization to $\mathbb{Z}_n$ quiver

in progress.