The Gradient Flow

Lattice Practices 2017

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Overview

Summary of conventions

- Mainly concerned with $SU(3)$ Yang-Mills or QCD

$$S = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] + \int d^4x \sum_{i=1}^{N_f} \overline{\psi}_i(\not{D} + m)\psi_i$$

- The lie algebra $\mathfrak{su}(3)$ is the linear space of all traceless anti-hermitian 3×3 matrices
- We choose a basis in $\mathfrak{su}(3)$, T^a with $a = 1, \dots, 8$ such that

$$\mathrm{Tr}\{T^a T^b\} = -\frac{\delta_{ab}}{2}$$

- Greek indices $\mu, \nu, \dots = 0, \dots, 3$ run over space-time coordinates, while latin indices run over the spatial coordinates $i, j, \dots = 1, 2, 3$.
- We abreviate integrals over momenta

$$\int_p = \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}$$

YANG-MILLS GRADIENT FLOW: BASICS (NARAYANAN, NEUBERGER '06; LÜSCHER '10)

- Add "extra" (flow) time coordinate t ($\neq x_0$). Define gauge field $B_\mu(x,t)$

$$\begin{aligned} G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\ \frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,t=0) = A_\mu(x)\,. \end{aligned}$$

- Important: t has dimensions of length^2. $\sqrt{8t}$ is a new length scale

$$[x] = -1; [t] = -2 \tag{1}$$

- Since

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$

  At large flow time $t \to \infty$ gauge field tends to classical solution of the e.o.m.

$$\lim_{t\to\infty} B_\mu(t,x) = A_\mu^{\mathrm{classical}}(x)\,.$$

- The flow smooths the quantum fluctuations!

How it works

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,0) = A_\mu(x)$$

- In perturbation theory we rescale $A_\mu \to g_0 A_\mu$.
- Expand the flow field in powers of g_0.

$$B_\mu(x,t) = \sum_{n=1}^{\infty} B_{\mu,n}(x,t) g_0^n; \qquad B_{\mu,n}(x,0) = \begin{cases} A_\mu(x) & n=1 \\ 0 & n>1 \end{cases}$$

and insert into flow equation

$$\begin{aligned} \frac{dB_\mu(x,t)}{dt} &= g_0 \frac{dB_{\mu,1}(x,t)}{dt} + \mathcal{O}(g_0^2) \\ G_{\mu\nu}(x,t) &= g_0 \partial_\mu B_{\nu,1}(x,t) - g_0 \partial_\nu B_{\mu,1} + \mathcal{O}(g_0^2) \end{aligned}$$

to obtain

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t) - \partial_\mu \partial_\nu B_{\nu,1}(x,t)$$

- Heat equation
- Gauge dependent part (Think of Landau gauge $\partial_\mu A_\mu = 0$)

How it works

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t) - \partial_\mu \partial_\nu B_{\nu,1}(x,t); \quad B_{\mu,1}(x,0) = A_\mu(x)\,.$$

- Linear equation in $B_{\mu,1}(x,t)$
- ∂-operator is "diagonal" in momentum space. Use

$$B_{\mu,1}(x,t) = \int_p e^{\imath px} \tilde{B}_{\mu,1}(p,t) \qquad \left(\int_p \equiv \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \right)$$

to get

$$\begin{aligned} \frac{dB_{\mu,1}(x,t)}{dt} &= \int_p \frac{d\tilde{B}_{\mu,1}(p,t)}{dt} e^{\imath px} \\ \partial_\nu^2 B_{\mu,1}(x,t) - \partial_\mu \partial_\nu B_{\nu,1}(x,t) &= -\int_p p^2 \tilde{B}_{\mu,1}(p,t) e^{\imath px} + \int_p p_\mu p_\nu \tilde{B}_{\nu,1}(p,t) e^{\imath px} \end{aligned}$$

Flow equation in momentum space

$$\frac{d\tilde{B}_{\mu,1}(p,t)}{dt} = -\left(p^2 \delta_{\mu\nu} - p_\mu p_\nu \right) \tilde{B}_{\nu,1}(p,t); \quad \tilde{B}_{\mu,1}(p,0) = \tilde{A}_\mu(p)\,.$$

How it works

$$\frac{d\tilde{B}_{\mu,1}(p,t)}{dt} = -\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\tilde{B}_{\nu,1}(p,t); \quad \tilde{B}_{\mu,1}(p,0) = \tilde{A}_\mu(p)\,.$$

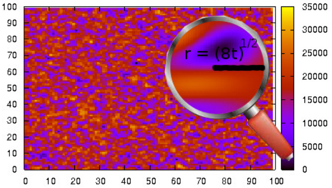
Forget about gauge terms: Solution

$$\tilde{B}_{\mu,1}(p,t) = e^{-tp^2}\tilde{B}_{\mu,1}(p,0) = e^{-tp^2}\tilde{A}_\mu(p)$$

High momentum modes (small scale fluctuations) are exponentially damped

Gradient flow vs. Heat flow

- Heat Flow: Smooths temperature fluctuations at scales shorter than $\sqrt{8t}$
- Gradient flow: Smooths quantum fluctuations at scales shorter than $\sqrt{8t}$



We are "looking" at world with a resolution $\sim \sqrt{8t}$.

HOW IT WORKS

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Exercise 1

Show that the solution of the flow equation in momentum space is

$$\tilde{B}_{\mu,1}(p,t) = e^{-tp^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\tilde{A}_\nu(p)$$

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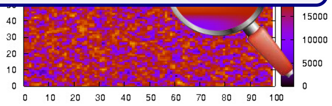

We are "looking" at world with a resolution $\sim \sqrt{8t}$.

How it works

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t); \quad B_{\mu,1}(x,0) = A_\mu(x)\,.$$

- Heat kernels are fundamental solutions of the heat equation

$$K(x,x',t) = \frac{1}{(4\pi t)^2} e^{\frac{(x-x')^2}{4t}} \tag{2}$$

(i.e. solutions with the property $\lim_{t\to 0} K(x,x',t) = \delta^4(x-x')$)

- This is "gaussian smearing" with radius $\sqrt{8t}$!

$$B_{\mu,1}(x,t) = \int d^4y\, K(x,y,t)A_\mu(y) = \frac{1}{(4\pi t)^2}\int d^4y\, e^{\frac{(x-y)^2}{4t}} A_\mu(y)\,.$$

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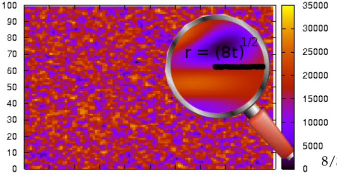

HOW IT WORKS

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t); \quad B_{\mu,1}(x,0) = A_\mu(x)\,.$$

- Heat kernels are fundamental solutions of the heat equation

Exercise 2

Show that

$$B_{\mu,1}(x,t) = \int d^4y\, K(x,y,t) A_\mu(y) = \frac{1}{(4\pi t)^2}\int d^4y\, e^{\frac{(x-y)^2}{4t}} A_\mu(y)\,.$$

is actually a solution to the flow equation to leading order in g_0 (wo. gauge term).

Gradient flow vs. Heat flow

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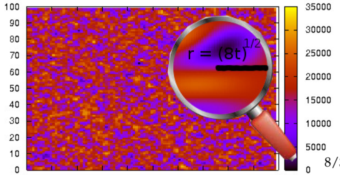

MAIN CHARACTERISTIC OF THE FLOW

- Gauge covariant under gauge transformations that are independent of t

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t)$$

- Composite gauge invariant operators are renormalized observables defined at a scale $\mu = 1/\sqrt{8t}$ (M. Lüscher '10; M. Lüscher, P. Weisz '11).
- Example (Note that at $t=0$ this is terribly divergent $\propto 1/a^4$)

$$\langle E(t)\rangle = -\frac{1}{2}\langle \mathrm{Tr} G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\rangle$$

  finite quantity for $t > 0$.
- Continuum limit to be taken at fixed t.
- The energy density $\langle E(t)\rangle$ will be a main character in this talk!!

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- Gauge covariant under gauge transformations that are independent of t

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t)$$

- Composite gauge invariant operators are renormalized observables defined at

Exercise 3

Show that the flow equation is gauge covariant

$$\langle E(t) \rangle = -\frac{1}{2} \langle \mathrm{Tr} G_{\mu\nu}(x,t) G_{\mu\nu}(x,t) \rangle$$

finite quantity for $t > 0$.
- Continuum limit to be taken at fixed t.
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Finitness of $\langle E(t)\rangle$ at leading order in PT

In this slide " $=$ " means up to higher order terms

- Start with the identity

$$G_{\mu\nu} = g_0 \partial_\mu B_{\nu,1}(x,t) - g_0 \partial_\nu B_{\mu,1}(x,t) = \imath g_0 \int_p \left(p_\mu B_{\nu,1} - p_\nu B_{\mu,1} \right) e^{\imath p x}$$

- Now

$$G_{\mu\nu} G_{\mu\nu} = -2 g_0^2 \int_{p,q} e^{\imath (p+q) x} \left[p_\mu q_\mu \tilde{B}_{\nu,1}(p) \tilde{B}_{\nu,1}(q) - p_\nu q_\mu \tilde{B}_{\mu,1}(p) \tilde{B}_{\nu,1}(q) \right]$$

- Use invariance under translations (add $\int d^4 x$ and get $\delta^4(p+q)$)

$$G_{\mu\nu} G_{\mu\nu} = 2 g_0^2 \int_p \tilde{B}_{\mu,1}(p) \left[p^2 \delta_{\mu\nu} - p_\mu p_\nu \right] \tilde{B}_{\nu,1}(-p)$$

- Use the solution of the flow equation and gluon propagator $\langle \tilde{A}^a_\mu(p) \tilde{A}^b_\nu(-p) \rangle = \frac{\delta_{ab}}{p^2} \delta_{\mu\nu}$

$$\langle E(t) \rangle = -\frac{1}{2} \langle \mathrm{Tr} G_{\mu\nu} G_{\mu\nu} \rangle = \frac{g_0^2}{2} \int_p e^{-2tp^2} \left[p^2 \delta_{\mu\nu} - p_\mu p_\nu \right] \langle \mathrm{Tr}\{ \tilde{A}_\mu(p) \tilde{A}_\nu(-p) \} \rangle$$

$$= \frac{3 \times 8}{4} g_0^2 \int_p e^{-2tp^2} = \frac{3 \times 8}{128 \pi^2 t^2} g_0^2 = \frac{3 \times 8}{128 \pi^2 t^2} g_{\overline{\mathrm{MS}}}^2$$

FINITNESS OF $\langle E(t)\rangle$ AT LEADING ORDER IN PT

IN THIS SLIDE " $=$ " MEANS UP TO HIGHER ORDER TERMS

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$$G_{\mu\nu} = g_0\partial_\mu B_{\nu,1}(x,t) - g_0\partial_\nu B_{\mu,1}(x,t) = \imath g_0 \int_p \left(p_\mu B_{\nu,1} - p_\nu B_{\mu,1}\right) e^{\imath px}$$

- Now

$$G_{\mu\nu}G_{\mu\nu} = -2g_0^2 \int_{p,q} e^{\imath(p+q)x} \left[p_\mu q_\mu \tilde{B}_{\nu,1}(p)\tilde{B}_{\nu,1}(q) - p_\nu q_\mu \tilde{B}_{\mu,1}(p)\tilde{B}_{\nu,1}(q)\right]$$

- Use invariance under translations (add $\int d^4x$ and get $\delta^4(p+q)$)

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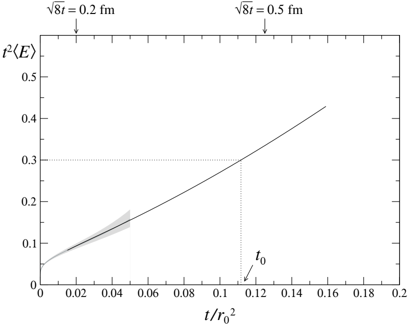
Exercise 4

- Where does the factors 3 and 8 come from?
- Check that the result is independent on the gauge
- Check that if one uses the solution of the flow equation with the gauge term, the solution is still the same

$$= \frac{3\times 8}{4}\, g_0^2 \int_p e^{-2tp^2} = \frac{3\times 8}{128\pi^2 t^2}\, g_0^2 = \frac{3\times 8}{128\pi^2 t^2}\, g_{\overline{\mathrm{MS}}}^2$$

Scale setting and renormalized couplings



- $t^2 \langle E(t) \rangle$ is dimensionless.
- Depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: t_0 (M. Lüscher JHEP 1008 '10).
- Similar quantities: $t_1, w_0, \ldots$ (Borsanyi et. al. '12; R. Sommer Latt. '14).
- Renormalized couplings at scale $\mu = \frac{1}{\sqrt{8t}}$

$$t^2 \langle E(t) \rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu) \left[1 + \mathcal{O}(g^2_{\overline{MS}}) \right]$$

Scale setting and couplings

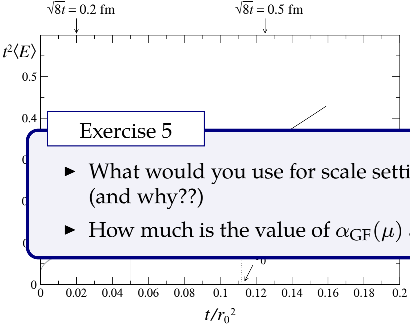
- Define a reference scale (t_0) via

$$t^2 \langle E(t) \rangle \Big|_{t=t_0} = 0.3$$

- Define a renormalized coupling (the gradient flow coupling) via

$$g^2_{\mathrm{GF}}(\mu) = \frac{16\pi^2}{3} t^2 \langle E(t) \rangle \Big|_{\mu = 1/\sqrt{8t}}$$

SCALE SETTING AND RENORMALIZED COUPLINGS



- $t^2\langle E(t)\rangle$ is dimensionless.
- Depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: t_0 (M. Lüscher JHEP 1008 '10).

Exercise 5

- What would you use for scale setting in the case of $SU(2)$? and for $SU(124)$? (and why??)
- How much is the value of $\alpha_{\mathrm{GF}}(\mu)$ at $\mu = 1/\sqrt{8t_0}$

$$t^2\langle E(t)\rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu)\left[1 + \mathcal{O}(g^2_{\overline{MS}})\right]$$

Scale setting and couplings

- Define a reference scale (t_0) via

$$t^2\langle E(t)\rangle\Big|_{t=t_0} = 0.3$$

- Define a renormalized coupling (the gradient flow coupling) via

$$g^2_{\mathrm{GF}}(\mu) = \frac{16\pi^2}{3} t^2\langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

EXERCISES OF THE INTRODUCTION

1. Show that the solution of the flow equation in momentum space is

$$\tilde{B}_{\mu,1}(p,t) = e^{-tp^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\tilde{A}_\nu(p)$$

2. Show that

$$B_{\mu,1}(x,t) = \int d^4y\, K(x,y,t)A_\mu(y) = \frac{1}{(4\pi t)^2}\int d^4y\, e^{\frac{(x-y)^2}{4t}} A_\mu(y)\,.$$

   is actually a solution to the flow equation to leading order in g_0 (wo. gauge term).

3. Show that the flow equation is gauge covariant
4. In the formula for $\langle E(t)\rangle$ to leading order
   - Where does the factors 3 and 8 come from?
   - Check that the result is independent on the gauge
   - Check that if one uses the solution of the flow equation with the gauge term, the solution is still the same
5. Scale setting and couplings
   - What would you use for scale setting in the case of $SU(2)$? and for $SU(124)$? (and why??)
   - How much is the value of $\alpha_{\rm GF}(\mu)$ at $\mu = 1/\sqrt{8t_0}$

RECAP (I)

- The gradient flow

$$\begin{aligned} G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\ \frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,t=0) = A_\mu(x)\,. \end{aligned}$$

  Defines a smooth gauge field $B_\mu(x,t)$ from your fundamental gauge field $A_\mu(x)$.

- Gauge invariant composite operators made of $B_\mu(x,t)$ are automatically renormalized (after renormalization of parameters in the Lagrangian), due to the exponential suppression (e^{-tp^2}) of the high momentum modes of the gauge field $A_\mu(x)$.

- Action density as the prototipical example

$$\langle E(t)\rangle = -\frac{1}{2}\langle \mathrm{Tr} G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\rangle$$

  is finite for $t > 0$

- $t^2\langle E(t)\rangle$ is a dimensionless renormalized quantity (can be computed on the lattice), that depends on a scale ($\mu = 1/\sqrt{8t}$).
  Ideal candidate for reference scale and renormalized coupling definition.

Overview

Beyond leading order

- An explicit calculation shows that actually $t^2\langle E(t)\rangle$ is finite to NLO (Lüscher '10)

$$t^2\langle E(t)\rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu) \left[1 + c_1 g^2_{\overline{MS}}(\mu) + \mathcal{O}(g^4_{\overline{MS}})\right]$$

  with c_1 finite
- Actually this is true to all orders, and for all gauge invariant observables made of the flow field $B_\mu(x,t)$ (Lüscher, Weisz '11)
- How to prove this? We know how to renormalize composite operators, but...
- $B_\mu(x,t)$ is not a local field (i.e. $B_\mu(x,t)$ depends on the fundamental field $A_\mu(y)$ for $y \neq x$)!
- One has to see the the flow field $B_\mu(x,t)$ as living in 5d!!

5D local formulation (Zinn-Justin '86, Zwanziger '88, Lüscher, Weisz '11)

Usually one sees the gradient flow observables in "two steps"

$$\langle O[B_\mu]\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A\, O[B_\mu] e^{-S_{\rm YM}[A_\mu]}$$

where B_μ is the solution of the flow equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,t=0) = A_\mu(x)\,.$$

Alternatively, we can promote $t \in (0,\infty)$ to a fifth coordinate and set-up a weird 5d theory

$$\mathcal{Z} = \int \mathcal{D}B_\mu \mathcal{D}L_\mu\; e^{-S_{\rm 5d}[B_\mu, L_\mu]}$$

with $S_{\rm 5d} = S_{\rm YM} + S_{\rm bulk}$

$$\begin{aligned} S_{\rm YM} &= -\frac{1}{2g_0^2}\int d^4x\, {\rm Tr}[F_{\mu\nu}F_{\mu\nu}] \\ S_{\rm bulk} &= -2\int_0^\infty dt \int d^4x {\rm Tr}\left\{L_\mu(x,t)\left[\partial_t B_\mu - D_\nu G_{\nu\mu}\right]\right\} \end{aligned}$$

5D local formulation (Zinn-Justin '86, Zwanziger '88, Lüscher, Weisz '11)

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}B_\mu \mathcal{D}L_\mu \; e^{-S_{5\text{d}}[B_\mu, L_\mu]}; \quad S_{5\text{d}} = S_{\text{YM}} + S_{\text{bulk}} \\
S_{\text{YM}} &= -\frac{1}{2g_0^2} \int d^4x \, \text{Tr}[F_{\mu\nu}F_{\mu\nu}] \\
S_{\text{bulk}} &= -2 \int_0^\infty dt \int d^4x \text{Tr} \left\{ L_\mu(x,t) \left[\partial_t B_\mu - D_\nu G_{\nu\mu} \right] \right\}
\end{aligned}$$

- $L_\mu(x,t) = L_\mu^a(x,t) T^a$ lives in the adjoint representation, with purely imaginary components
- The path integral $\mathcal{D}L_\mu$ can be done exactly

$$\int \mathcal{D}L_\mu e^{-S_{\text{bulk}}} = \delta \left(\partial_t B_\mu - D_\nu G_{\nu\mu} \right) \,,$$

  and it imposes the flow equation
- Now we have

$$\langle O[B_\mu] \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}B \mathcal{D}L \, O[B_\mu] e^{-S_{5\text{d}}[B_\mu, L_\mu]} = \int \mathcal{D}B \, O[B_\mu] e^{-S_{\text{YM}}[A_\mu]} \delta \left(\partial_t B_\mu - D_\nu G_{\nu\mu} \right) \,,$$

5D local formulation (Zinn-Justin '86, Zwanziger '88, Lüscher, Weisz '11)

$$\mathcal{Z} = \int \mathcal{D}B_\mu \mathcal{D}L_\mu \; e^{-S_{5d}[B_\mu, L_\mu]}; \quad S_{5d} = S_{\rm YM} + S_{\rm bulk}$$

$$S_{\rm bulk} = -2\int_0 dt \int d^4x \, {\rm Tr}\{L_\mu(x,t)[\partial_t B_\mu - D_\nu G_{\nu\mu}]\}$$

Exercise 6

- What are the dimensions of the field $L_\mu(x,t)$?
- Name a few symmetries of the action S_{5d}

- $L_\mu(x,t) = L_\mu^a(x,t)T^a$ lives in the adjoint representation, with purely imaginary components
- The path integral $\mathcal{D}L_\mu$ can be done exactly

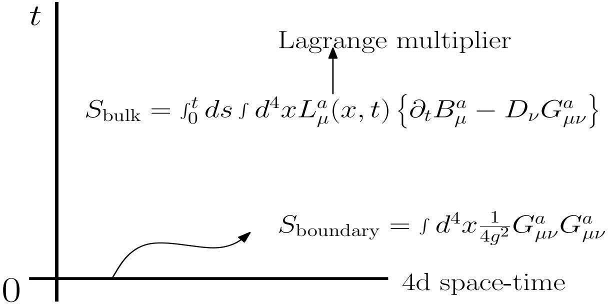
$$\int \mathcal{D}L_\mu e^{-S_{\rm bulk}} = \delta\left(\partial_t B_\mu - D_\nu G_{\nu\mu}\right) \,,$$

  and it imposes the flow equation
- Now we have

$$\langle O[B_\mu]\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}B\mathcal{D}L \, O[B_\mu]e^{-S_{5d}[B_\mu, L_\mu]} = \int \mathcal{D}B \, O[B_\mu]e^{-S_{\rm YM}[A_\mu]}\delta\left(\partial_t B_\mu - D_\nu G_{\nu\mu}\right) \,,$$

5D LOCAL FORMULATION (LÜSCHER, WEISZ '11)

We can see the theory as a 5d local field theory (Zinn-Justin '86, Zinn-Justin, Zwanziger '88)



$$S_{5\mathrm{d}} = S_{\mathrm{bulk}} + S_{\mathrm{boundary}}$$

PROOF OF RENORMALIZABILITY

Key to the proof: in the 5d theory there are no loops in the bulk

Flow for $\lambda - \phi^4 : \partial_t \varphi(x,t) = \partial^2 \varphi(x,t)$. Use momentum space

$$
\begin{aligned}
S_{5\mathrm{d}} &= \frac{1}{2}\int_{p_1,p_2} \delta(p_1+p_2)\tilde{\phi}(p_1)\left[p_1^2 + m^2\right]\tilde{\phi}(p_2) \\
&+ \frac{\lambda}{4}\int_{p_1,p_2,p_3,p_4} \delta(p)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) \\
&+ \int_0^\infty dt \int_{p_1,p_2} \left\{\tilde{L}(p_1,t)\left[\partial_t\tilde{\varphi}(p_2,t) - p^2\tilde{\varphi}(p_2,t)\right]\right\}
\end{aligned}
$$

If we study correlation functions

$$
\begin{aligned}
\langle\tilde{\varphi}(p,t)\tilde{\varphi}(-p,t)\rangle &= \frac{1}{\mathcal{Z}}\int \mathcal{D}\varphi\mathcal{D}L\, e^{-S_0[\tilde{\phi}]}e^{-S_{\mathrm{fl}}[\tilde{L},\tilde{\varphi}]}\tilde{\varphi}(p,t)\tilde{\varphi}(-p,t) \\
\times &\left[1 + \frac{\lambda}{4}\int_{p_1,p_2,p_3,p_4} \delta(p)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) + \dots\right]
\end{aligned}
$$

In this case flow equation can be exactly solved $\tilde{\varphi}(p,t) = e^{-tp^2}\tilde{\phi}(p)$, and the path integral $\int \mathcal{D}L$ can be analitically done

All loops comes from the "usual" interaction at $t = 0$

Proof of renormalizability

Key to the proof: in the 5d theory there are no loops in the bulk

Flow for $\lambda - \phi^4 : \partial_t \varphi(x,t) = \partial^2 \varphi(x,t)$. Use momentum space

$$
\begin{aligned}
S_{5\mathrm{d}} \quad &= \quad \frac{1}{2} \int_{p_1,p_2} \delta(p_1+p_2)\tilde{\phi}(p_1) \left[p_1^2 + m^2 \right] \tilde{\phi}(p_2) \\
&+ \quad \frac{\lambda}{4} \int_{p_1,p_2,p_3,p_4} \delta(p)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) \\
&+ \quad \int_0^\infty dt \int_{p_1,p_2} \left\{ \tilde{L}(p_1,t) \left[\partial_t \tilde{\varphi}(p_2,t) - p^2 \tilde{\varphi}(p_2,t) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\langle \tilde{\varphi}(p,t)\tilde{\varphi}(-p,t) \rangle \quad &= \quad \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi \, e^{-S_0[\tilde{\phi}]} e^{-2tp^2} \tilde{\phi}(p)\tilde{\phi}(-p) \\
\times \quad & \quad \left[1 + \frac{\lambda}{4} \int_{p_1,p_2,p_3,p_4} \delta(p)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) + \dots \right]
\end{aligned}
$$

Proof of renormalizability

Key to the proof: in the 5d theory there are no loops in the bulk

$\partial_t \varphi(x,t) = \partial^2 \varphi(x,t)$. Use momentum space

Exercise 7

- Why we do not define the flow in Yang-Mills by the equation

$$\frac{dB_\mu(x,t)}{dt} = \partial_\nu^2 B_\mu(x,t)$$

  ??
- Why we did not define the flow in $\lambda - \phi^4$ by the equation

$$\partial_t \varphi(x,t) = (\partial^2 - m^2)\varphi(x,t)$$

- Does the flow correlator $\langle \tilde{\varphi}(p,t)\tilde{\varphi}(-p,t)\rangle$ still have divergences? How is this possible?
- What divergences in $\langle \tilde{\varphi}(p,t)\tilde{\varphi}(-p,t)\rangle$ have been killed by the flow?

Proof of renormalizability (Lüscher, Weisz '11)

$$\mathcal{Z} = \int \mathcal{D}B_\mu \mathcal{D}L_\mu \; e^{-S_{5\mathrm{d}}[B_\mu, L_\mu]}; \quad S_{5\mathrm{d}} = S_{\mathrm{YM}} + S_{\mathrm{bulk}}$$

$$S_{\mathrm{YM}} = -\frac{1}{2g_0^2} \int d^4x \, \mathrm{Tr}[F_{\mu\nu} F_{\mu\nu}]$$

$$S_{\mathrm{bulk}} = -2 \int_0^\infty dt \int d^4x \mathrm{Tr} \left\{ L_\mu(x,t) \left[\partial_t B_\mu - D_\nu G_{\nu\mu} \right] \right\}$$

- Key new element: No loops in the bulk $\Longrightarrow$ only divergences at the boundary $t = 0$. For $t > 0$ is like a classical theory!
- At $t = 0$ we have the usual YM action. Common lore applicable: only coupling is g (dimensionless) $\Longrightarrow$ renormalizable
- We still need to show that we have included all $d \leq 4$ operators at the boundary $t = 0$. **Note that we have a new field** $L_\mu(x,t)$

Exercises

7. What are the dimensions of the field $L_\mu(x,t)$?
8. Name a few symmetries of the action $S_{5\mathrm{d}}$
9. Why we do not define the flow in Yang-Mills by the equation

$$\frac{dB_\mu(x,t)}{dt} = \partial_\nu^2 B_\mu(x,t)$$

??

10. Why we did not define the flow in $\lambda - \phi^4$ by the equation

$$\partial_t \varphi(x,t) = (\partial^2 - m^2)\varphi(x,t)$$

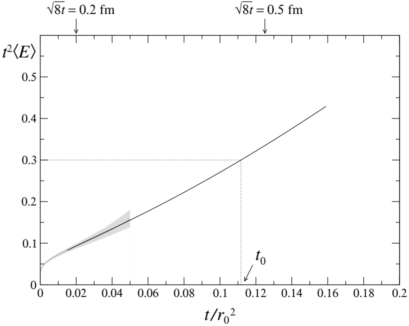
11. Does the flow correlator $\langle \tilde{\varphi}(p,t)\tilde{\varphi}(-p,t)\rangle$ still have divergences? How is this possible?
12. What divergences in $\langle \tilde{\varphi}(p,t)\tilde{\varphi}(-p,t)\rangle$ have been killed by the flow?

Recap (II)

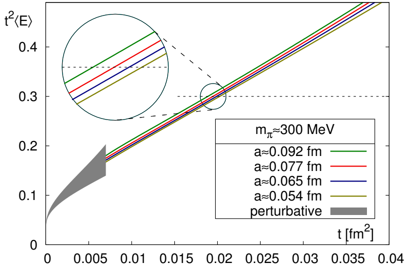
- The Yang-Mills flow can be seen as a 5d local quantum field theory.
- The flow equation is imposed by adding "Lagrange multipliers" fields $L_\mu(x,t)$. The path integral over $L_\mu(x,t)$ gives a delta function.
- Usual tools of quantum field theory can be applied to this theory.
- Very similar to stochastic quantization (Book by Zinn-Justin)

Overview

Scale setting



(M. Lüscher JHEP 1008 (2010) 071).



(Borsanyi et. al JHEP 1209 (2012) 010).

- Reference scale (t_0) via

$$t^2\langle E(t)\rangle\Big|_{t=t_0} = 0.3$$

- Reference scale w_0

$$t\frac{d}{dt}t^2\langle E(t)\rangle\Big|_{t=w_0^2} = 0.3$$

Topological susceptibility

Topological charge

$$Q = \int_x q(x); \qquad q(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(x)$$

has important role in non-perturbative physics

- Q: Why is the η meson not a pseudo-goldstone boson ($U_A(1)$ puzzle)?
- A: $U_A(1)$ is only a symmetry at the classical level, destroyed by quantum fluctuations

$$\partial_\mu \overline{\psi} \gamma_\mu \gamma_5 \psi(x) \propto q(x)\,.$$

  In fact in the large N limit

$$m_\eta \propto \chi_t + \ldots; \qquad \chi_t = \int_x \langle q(x) q(0) \rangle = \frac{\langle Q^2 \rangle}{V}$$

- How to compute χ_t on the lattice?

$$\langle q(0) q(0) \rangle \sim 1/a^4$$

Topological susceptibility

- Q can be rigorously defined on the lattice via the index

$$Q = \mathrm{index}(D) = n_+ - n_-$$

  of the Neuberger Dirac operator. This can be used to compute χ_t (Del Debbio '05)
- Q and topological structure of the vacuum defined via smearing/cooling in many works in the literature.
- Gradient Flow (and 5d) provides the framework to show that the flow and the index definitions agree (Cè et al. '15)

$$q(x,t) = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}G_{\mu\nu}(x,t)G_{\rho\sigma}(x,t)$$

Running couplings

- Use the gradient flow coupling definition in finite volume

$$\alpha(\mu) = \# t^2 \langle E(t) \rangle \Big|_{\mu = 1/\sqrt{8t} = 1/(cL)}$$

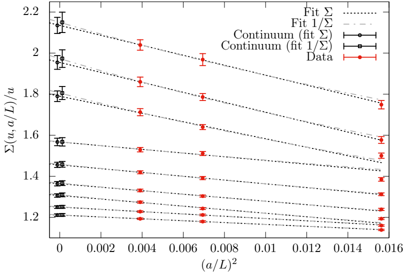
Constant # depends on the choice of boundary conditions (periodic, twisted, dirichlet (SF), ...)

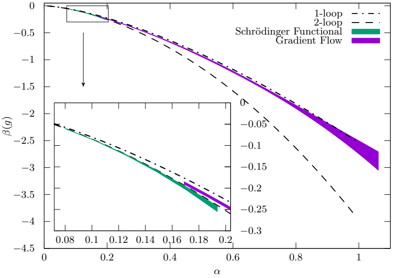
- We can measure how much changes the coupling when we change $\mu \to \mu/2$ by changing our lattice size L
- Step scaling function

$$\sigma(u) = \alpha(\mu/2) \Big|_{\alpha(\mu) = u}$$

RUNNING COUPLING

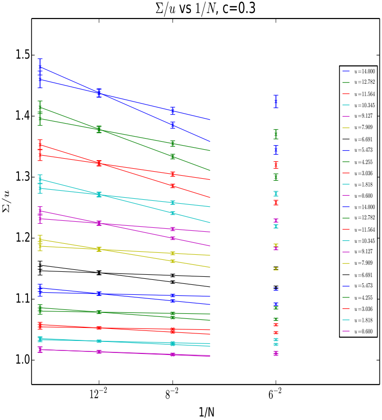
$SU(3)$ with $N_f = 3$. The determination of $\alpha_s(m_Z)$ (ALPHA '16)

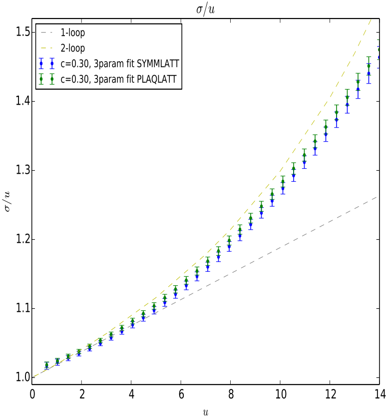




Running coupling

$SU(N)$ in the limit $N \to \infty$ and Reduction (M. Garcia Perez et. al '14).





Simulations done in a lattice with **One point**. Change of scale by $N \to N'$

What happens when $t \to 0$

Any operator at positive flow time has an expansion in terms of renormalized fields

$$O(x,t) = \sum_\alpha c_\alpha(t)\{O^\alpha\}_R(x) + \mathcal{O}(t)$$

Mixing pattern determined by continuum symmetries!

Example: $E(x,t) = G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)$

$$E(x,t) = c_1(t)\mathbf{1} + c_2(t)\{F_{\mu\nu}F_{\mu\nu}\}_R(x) + \mathcal{O}(t)$$

What is the renormalization condition for $\{F_{\mu\nu}F_{\mu\nu}\}_R(x)$?

$$\langle\{F_{\mu\nu}F_{\mu\nu}\}_R(x)\rangle = 0$$

So we can determine

$$c_1(t) = \langle E(x,t)\rangle \tag{3}$$

And this can be used to determine the trace of the EM tensor

$$T_{\mu\mu}(x) = \{F_{\mu\nu}F_{\mu\nu}\}_R(x) = \lim_{t\to 0} c_2^{-1}(t)\left[E(t,x) - \langle E(t,x)\rangle\right]$$

But we need $c_2(t)$!!

The fermion flow and condensates

Flow for fermion fields (M. Lüscher, '13)

$$\partial_t \chi(x,t) = D_\mu D_\mu \chi(x,t); \quad D_\mu = \partial_\mu + B_\mu$$

with initial condition $\chi(x,t)|_{t=0} = \psi(x)$.

- Composite operators O made of $\chi(x,t), \overline{\chi}(x,t)$ renormalize multiplicatively ($t > 0$)

$$\langle O_{\mathrm{R}} \rangle = (Z_\chi)^{(n+n')/2} \langle O \rangle; \quad n \text{ and } n' \text{ number of } \chi \text{ and } \overline{\chi} \text{ fileds.}$$

- Chiral condensate does not mix for $t > 0$ (M. Lüscher, '13)

$$\Sigma(t) = \langle \overline{u}(t,x) u(t,x) \rangle$$

- Compute proton strange content (A. Shindler '13).

$$m_s \langle N | \bar{s}s(t) | N \rangle_{\mathrm{c}} = c_3(t) m_s \langle N | \bar{s}s(0) | N \rangle_{\mathrm{c}} + \mathcal{O}(t)$$

but chiral symmetry relates $c_3(t)$ with the $G_\pi(t) = |\langle 0 | \pi(t) \rangle|^2$

$$c_3(t) = \frac{G_\pi(t)}{G_\pi(0)}$$

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- Chiral condensate does not mix for $t > 0$ (M. Lüscher, '13)

Exercise 8

- Write the 5d local action that includes the fermion flow

- Compute proton strange content (A. Shindler '13).

$$m_s \langle N | \bar{s}s(t) | N \rangle_{\mathrm{c}} = c_3(t) m_s \langle N | \bar{s}s(0) | N \rangle_{\mathrm{c}} + \mathcal{O}(t)$$

but chiral symmetry relates $c_3(t)$ with the $G_\pi(t) = |\langle 0 | \pi(t) \rangle|^2$

$$c_3(t) = \frac{G_\pi(t)}{G_\pi(0)}$$

Overview

Solving the flow equation on the lattice

The continuum equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad \left(\sim -g_0^2 \frac{\delta S_{\rm YM}[B]}{\delta B_\mu} \right)$$

How do the links $V_\mu(x,t) = \exp[B_\mu(t,x)]$ change with the t?

Simplest solution: Wilson flow

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S^{\rm Wilson}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

where

$$S^{\rm Wilson} = \frac{1}{g_0^2} \sum_{\rm pl} {\rm Tr} \left(1 - \square \right)$$

Wilson flow equation solved by setting

$$\begin{aligned} Z_\mu(x,t) &= -\epsilon g_0^2 \frac{\delta S^{\rm Wilson}[V]}{\delta V_\mu(x,t)} && \in \mathfrak{su}(3) \\ V_\mu(x,t+a^2\epsilon) &= \exp\left\{ Z_\mu(x,t) \right\} V_\mu(x,t) && \in SU(3) \end{aligned}$$

Compute observables from $V_\mu(x,t)$ (i.e. the average plaquette)

HIGHER ORDER INTEGRATORS

$$
\begin{aligned}
Z_\mu(x,t) &= -\epsilon g_0^2 \frac{\delta S^{\mathrm{Wilson}}[V]}{\delta V_\mu(x,t)} \quad \in \mathfrak{su}(3) \\
V_\mu(x,t+a^2\epsilon) &= \exp\left\{Z_\mu(x,t)\right\} V_\mu(x,t) \quad \in SU(3)
\end{aligned}
$$

This Euler scheme is very slow and inefficient. In practice integrating the flow equations is numerically expensive!

3rd order Runge-Kutta

Define $Z_i = \epsilon Z(W_i) =$ "Force" (Lüscher '10)

$$
\begin{aligned}
W_0 &= V_\mu(x,t)\,, \\
W_1 &= \exp\left\{\frac{1}{4} Z_0\right\} W_0\,, \\
W_2 &= \exp\left\{\frac{8}{9} Z_1 - \frac{17}{36} Z_0\right\} W_1\,, \\
V_\mu(x,t+a^2\epsilon) &= \exp\left\{\frac{3}{4} Z_2 - \frac{8}{9} Z_1 + \frac{17}{36} Z_0\right\} W_2\,,
\end{aligned}
$$

Very tricky: Exponentials do not commute! Any Runge-Kutta <u>not</u> valid

Adaptive step size integrators: Define $Z_i = \epsilon Z(W_i)$ (Fritzsch et al '12)

3rd and 2nd order Runge-Kutta nested

$$
\begin{aligned}
W_0 &= V_\mu(x,t)\,, \\
W_1 &= \exp\left\{\frac{1}{4}Z_0\right\} W_0\,, \\
W_2 &= \exp\left\{\frac{8}{9}Z_1 - \frac{17}{36}Z_0\right\} W_1\,, \\
V_\mu(x,t+a^2\epsilon) &= \exp\left\{\frac{3}{4}Z_2 - \frac{8}{9}Z_1 + \frac{17}{36}Z_0\right\} W_2\,,
\end{aligned}
$$

One can have a second estimate of (order 2 integrator)

$$V'_\mu(x,t+a^2\epsilon) = \exp\left\{-Z_0 + 2Z_1\right\} W_0\,.$$

And use the "difference"

$$d = \max_{x,\mu}\left\{\mathrm{dist}(V_\mu(x,t+a^2\epsilon), V'_\mu(x,t+a^2\epsilon))\right\}\,.$$

to tune ϵ to obtain a target precision δ

$$\epsilon \longrightarrow \epsilon 0.95\sqrt[3]{\frac{\delta}{d}}$$

Step size vs. t_{flow} (Better keep a maximum ϵ)

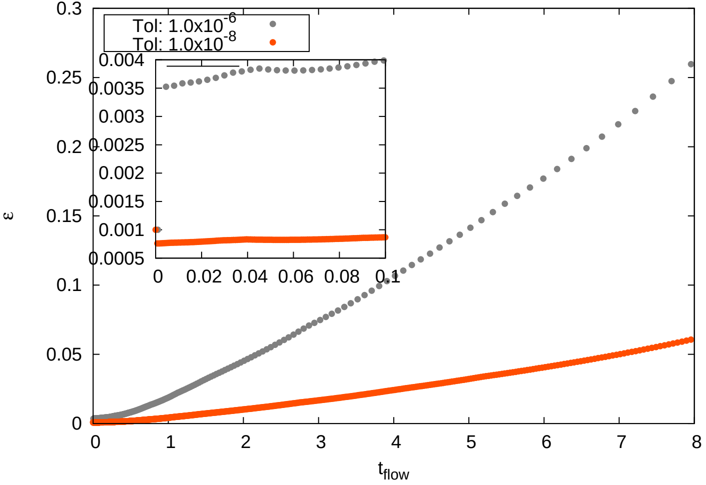

Figure: $L = 8$, 240 steps for $\delta = 10^{-6}$, 1042 for $\delta = 10^{-8}$

Integrator scaling

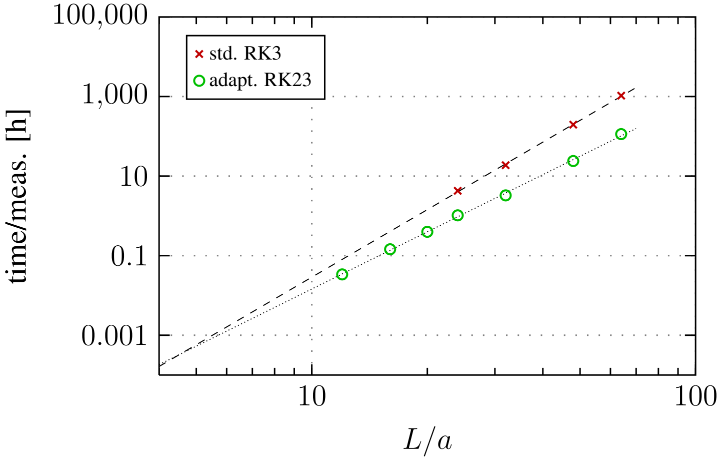

Figure: Scaling behaviour of standard Runge-Kutta integrator (RK3) versus adaptive step-size integrator (RK23) for an equivalent setup integrated up to cmax = 0.5

Solving the flow equation on the lattice

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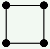
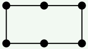
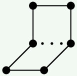
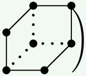
How do the links $V_\mu(x,t) = \exp[B_\mu(t,x)]$ change with the t?

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S^{\rm latt}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

where

$S^{\rm latt}(c_i^{(a)}) = \frac{1}{g_0^2} \sum_x \mathrm{Tr} \Big(1 - c_0^{(a)}$  $- c_1^{(a)}$  $- c_2^{(a)}$  $- c_3^{(a)}$  $\Big)$

- Is this the best option?
- Which lattice action $S^{\rm latt}$? What coeficients $c_i^{(a)}$??

The Zeuthen flow

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \left(1 + \frac{a^2}{12} D_\mu D_\mu^*\right) \frac{\delta S^{\rm LW}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

Symanzik improvement and the Zeuthen flow

- Symanzik effective action describes cutoff effects of all (improved) observables

$$S^{\mathrm{latt}} = S^{\mathrm{cont}} + a^2 S^{(2)} + \mathcal{O}(a^4)$$
$$\langle O \rangle_{\mathrm{latt}} = \langle O \rangle_{\mathrm{cont}} + a^2 \langle O S^{(2)} \rangle_{\mathrm{cont}} + \mathcal{O}(a^4)$$

- Aim: Choose S^{latt} so that $S^{(2)} = 0$.
- 5D local field theory. Lagrange multiplier imposes flow equation on the bulk.

$$S^{\mathrm{cont}} = -\frac{1}{2g_0^2} \int \mathrm{d}^4 x \, \mathrm{Tr}\,\{F_{\mu\nu} F_{\mu\nu}\} - 2 \int_0^\infty \mathrm{d}t \int \mathrm{d}^4 x \, \mathrm{Tr}\,\{L_\mu(x,t) [\partial_t B_\mu(x,t) - D_\nu G_{\nu\mu}]\} \, .$$

- Ansatz for improved action: boundary ($c_i^{(a)}$ and c_4) and bulk ($c_i^{(f)}$) parameters.

$$S^{\mathrm{latt}} = S^{\mathrm{g}}(c_i^{(a)}) + c_4 a^4 \sum_x \mathrm{Tr}\left\{ L_\mu(0,x) \left[g^2 \partial^a_{x,\mu} S^{\mathrm{w}} \right] \right\}$$
$$+ a^4 \sum_x \int_0^\infty \mathrm{d}t \, \mathrm{Tr}\left\{ L_\mu(x,t) \left[\partial_t V_\mu(x,t) V_\mu^{-1}(x,t) + g^2 \partial_{x,\mu} S^{\mathrm{g}}(c_i^{(f)}) \right] \right\} \, .$$

- Bulk improvement coefficients can not depend on g^2: non-perturbative improvement.