Flag manifold σ-models and Ricci flow

Dmitri Bykov

Max-Planck-Institut für Physik (München)

Steklov Mathematical Institute (Moscow)

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Based on several papers of the author,
including a recent one with D. Lüst and an upcoming one

In this talk we will consider relativistic σ-models with worldsheet $\mathbb{R}^2$ and target space $\mathcal{M}$. We will be interested in the case when $\mathcal{M}$ is a flag manifold:

$$\mathscr{F}_{n_1,\ldots,n_m} = \frac{SU(N)}{S(U(n_1) \times \cdots \times U(n_S))}, \qquad \sum_{i=1}^{S} n_i = N$$

There are several encounters of such sigma-models in mathematical physics.

First, they arise as effective continuum theories of spin chains with $SU(N)$-symmetry. ($SU(2)$-case: [Haldane '83])

The idea is that the flag manifold is the space of Néel vacua of the classical chain:

– Geometric theory: [DB '11-'12]
– Analysis of spin chains: [Affleck et.al. '17 ($SU(3)$), '19 ($SU(N)$)]
– Discrete 't Hooft anomalies [Tanizaki & Sulejmanpasic '18, Seiberg et.al. '18]

– Flag manifold sigma-models arise in the [Cho '80]-[Faddeev-Niemi '97-'99] approach to $SU(N)$ gauge theories, and in the case of complete flags these admit Hopfion solutions, since $\pi_3\left(\frac{U(N)}{U(1)^N}\right) \simeq \mathbb{Z}$ [Amari, Sawado '18]

– Describe the worldsheet theory of non-Abelian vortices in certain 4D (SUSY) gauge theories [Bolokhov, Shifman, Yung, '09] [Ireson '19]

– In this talk I will discuss mostly the sigma-models that are (conjecturally) integrable. These were considered in:
[Young '06, Beisert & Lücker '12] – $\mathbb{Z}_S$-graded spaces,
[DB '14-19] – General complex homogeneous spaces,
[Costello & Yamazaki '19] – Chern-Simons theory,
[Bytsko '94, Brodbeck & Zagermann '00, Delduc, Kameyama, Lacroix, Magro, Vicedo '19] –
Ultralocality of Poisson brackets: $\{\mathcal{L}_\lambda(x), \mathcal{L}_\mu(y)\} \sim [\mathbf{r}(\lambda-\mu), \mathcal{L}_\lambda \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_\mu]\, \delta(x-y)$

Let $g \in SU(N)$, $g^{-1}dg = i\left(\Phi\, dz + \overline{\Phi}\, d\overline{z}\right)$.

$\overline{\partial}\Phi + i\,[\overline{\Phi}, \Phi] = 0$.   Flatness of Noether current, Principal Chiral Model

Impose the condition $\Phi^S = 0$ (closure of nilpotent orbit in $\mathfrak{gl}_N$)

Assume

$$\Phi^S = 0, \qquad \text{and} \qquad \Phi^{S-1} \neq 0\,.$$

The map $\Phi(z, \overline{z})$ satisfying the E.O.M. defines a flag

$$0 \subset \operatorname{Ker}(\Phi) \subset \operatorname{Ker}(\Phi^2) \subset \cdots \subset \operatorname{Ker}(\Phi^S) \simeq \mathbb{C}^N$$

This is a point in

$$\mathscr{F} := \frac{U(N)}{U(\kappa_1) \times \cdots \times U(\kappa_S)}, \qquad \text{where}$$

$\kappa_j = \dim \operatorname{Ker}(\Phi^j)/\operatorname{Ker}(\Phi^{j-1})$ = number of Jordan blocks of size at least i.

Claim: the map $\Sigma \to \mathscr{F}$ is a solution of a flag manifold sigma-model

This is a map to a single orbit: type does not change due to e.o.m. [DB, 2019]

I will now describe the model.

Complex structures.

To this end one needs a complex structure $\mathscr{J}$ on $\mathscr{F}$.
It is defined by an ordering of the factors in the denominator $\frac{SU(N)}{S(U(n_1) \times \cdots \times U(n_S))}$
[Borel & Hirzebruch '58].

$\mathscr{F}$ may then be interpreted as the manifold of embedded linear subspaces:

$$0 \in V_1 \subset \ldots \subset V_S = \mathbb{C}^N, \qquad \dim_{\mathbb{C}} V_k := d_k = \sum_{i=1}^{k} n_i \,.$$

$\mathscr{F} = G/H$, the Lie algebra $\mathfrak{g}$ admits the standard decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$
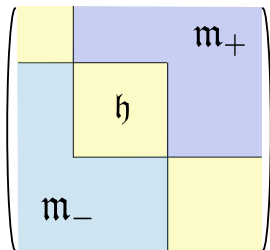
In the presence of $\mathscr{J}$ one has a more detailed decomposition of the Lie algebra:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \qquad \mathscr{J} \circ \mathfrak{m}_{\pm} = \pm i\, \mathfrak{m}_{\pm} \,.$$

$$[\mathfrak{h}, \mathfrak{m}_{\pm}] \subset \mathfrak{m}_{\pm} \,, \qquad \text{(Homogeneity of } \mathscr{J})$$
$$[\mathfrak{m}_{\pm}, \mathfrak{m}_{\pm}] \subset \mathfrak{m}_{\pm} \qquad \text{(Integrability of } \mathscr{J}).$$

The decomposition of the Lie algebra.

Decompose $\mathfrak{m}_+$ into irreps of $\mathfrak{h}$: $\mathfrak{m}_+ = \oplus_{1 \leq i < j \leq S}(\mathfrak{m}_+)_{ij}$,

Maurer-Cartan one-form $J := -g^{-1}dg = \sum\limits_{i,j=1}^{S} J_{ij}$.

$$ds^2 = \mathscr{G}_{ij}dX^i dX^j = \sum_{1 \leq i < j \leq S} a_{ij}\,\mathrm{tr}(J_{ij}J_{ji}), \qquad a_{ij} > 0$$

$$B = \sum_{1 \leq i < j \leq S} b_{ij}\,\mathrm{tr}(J_{ij} \wedge J_{ji})$$

If $b_{ij} = a_{ij}$, B is called the fundamental Hermitian form of the metric $\mathscr{G}$ w.r.t. one of the complex structures $\mathscr{J}$ on $\mathscr{F}$: $B = \mathscr{G} \circ \mathscr{J}$.

- Kähler metric: $a_{ij} = z_i - z_j$ (Kirillov-Kostant-Souriau form $B = \mathrm{Tr}(z\,J \wedge J)$).
- Normal metric: $a_{ij} = 1, \quad ds^2 = \mathrm{Tr}(J_\mathfrak{m}J_\mathfrak{m})$ ('Killing metric')
  Geodesics are homogeneous [Alekseevsky & Arvanitoyeorgos '07].

Action: $\quad \mathcal{S}[\mathscr{G}, \mathscr{I}] := \int_\Sigma d^2z\,\|\partial X\|_\mathscr{G}^2 + \int_\Sigma X^*B \quad \sim \quad \int_\Sigma d^2z\,G_{m\overline{n}}\,\partial U^m \overline{\partial U^n}$

The conjecture of integrability of the models is based on the following evidence:

- The zero-curvature representation

$$A_u = \frac{1+u}{2} K_z dz + \frac{1+u^{-1}}{2} K_{\bar{z}} d\bar{z}, \qquad u \in \mathbb{C}^*,$$

  where K = Noether current (*flat*).
  A more general class – complex hom. spaces [DB '16]:
  toric bundles over flag manifolds [Wang '54].
- Involutivity of the integrals of motion [Delduc et. al. '19]
- Explicit classical solutions $\left(\frac{U(3)}{U(1)^3}\right)$ [DB '16], generalizing [Din, Zakrzewski '80]
- Analogy with the case of symmetric spaces (review: [Zarembo '17])
- Explicit form of anomaly in non-local charge:
  similar to Grassmannian case [DB '19]

Complex <u>symmetric</u> spaces: $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$ ($\Rightarrow [\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$).
Symmetric spaces of $SU(N)$: Grassmannians $G(m, N) := \frac{SU(N)}{S(U(m) \times U(N-m))}$

The gauged linear sigma-model (GLSM).

Kähler case: GLSM $\leftrightarrow$ Kähler quotients.

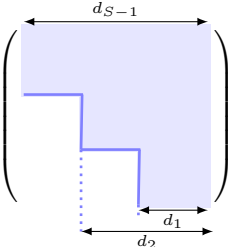Grassmannian: $G(m, N) = \operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^N)//U(m)$. Lagrangian

$$\mathscr{L} = \operatorname{Tr}((\overline{D}U)^\dagger(\overline{D}U)), \qquad \overline{D}U := \overline{\partial}U - i\,U\,\overline{\mathcal{A}}, \qquad U^\dagger U = \mathbb{1}_m\,.$$

[Cremmer, Scherk '78, D'Adda, Lüscher, di Vecchia '78].

Flag manifold with Kähler metric: GLSM $\leftrightarrow$ Nakajima (quiver) varieties

[Nakajima '94, Nitta '03, Donagi & Sharpe '08].

Flag manifold with 'Killing metric' (not Kähler for $S > 2$): a 'gauge field' [DB '17]

$$\overline{\mathcal{A}} = (\mathcal{A})^\dagger\,, \qquad \text{A 'reduced' gauge field!}$$

[Costello-Yamazaki '19]: a semi-holomorphic 4D Chern-Simons theory on $\Sigma \times \mathscr{C}$, where
Σ = 'topological plane' $(z, \overline{z}) \to$ worldsheet to-be
$\mathscr{C}$ = complex curve $(w, \overline{w})$ with a holomorphic differential $\omega = dw \neq 0$.
$K_{\mathscr{C}} = 0$ implies $\mathscr{C} \simeq \mathbb{C}, \mathbb{C}^*, E_\tau$. The action:

$$S_{\mathrm{CS}} = \frac{1}{\hbar} \int\limits_{\Sigma \times \mathscr{C}} \omega \wedge \mathrm{Tr} \left(A \wedge \left(dA + \frac{2}{3} A \wedge A \right) \right),$$

where $A = A_z dz + A_{\overline{z}} d\overline{z} + A_{\overline{w}} d\overline{w}$. One couples this theory to two $\beta\gamma$ systems, with target space $T^* \mathcal{M}$, where $\mathcal{M}$ is a complex homogeneous space:
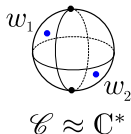
$$S_{def} = \int\limits_{\Sigma} d^2 z \, (p_i D_{\overline{z}}^{(w_1)} q^i + \overline{p}_i D_z^{(w_2)} \overline{q}^i),$$

$$\text{where} \qquad D_{\overline{z}}^{(w_1)} q^i = \partial_{\overline{z}} q^i - \sum_a (A_{\overline{z}}^{(w_1)})_a v_a^i.$$

'Light-cone' gauge $A_{\overline{w}} = 0$, solve for A_z, $A_{\overline{z}}$. In this gauge the equations are

$$\partial_{\overline{z}} A_z - \partial_z A_{\overline{z}} + [A_z, A_{\overline{z}}] = 0,$$
$$\partial_{\overline{w}} A_z = \delta^{(2)}(w - w_1) \sum_a p_i v_a^i \, \tau_a$$
$$\partial_{\overline{w}} A_{\overline{z}} = \delta^{(2)}(w - w_2) \sum_a \overline{p}_{\overline{i}} v_a^{\overline{i}} \, \tau_a \, .$$

Family $A_z(w)$, $A_{\overline{z}}(w)$ of flat connections, depending meromorphically on w!

Key observation: Green's function $\overline{\partial}_{\overline{w}}^{-1} = $ classical r-matrix [Belavin, Drinfeld '80]

Rational case: $r(w) = \frac{\sum \tau_a \otimes \tau_a}{w} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r(w) = \frac{\mathbb{1}}{w} \in \mathrm{End}(\mathfrak{g})$.

Trigonometric case: let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (complex structure on G, Manin triple, etc.), then
$r_{\mathrm{comp.}}(u) = \frac{\sum \tau_a^+ \otimes \tau_a^-}{1-u} - \frac{\sum \tau_a^- \otimes \tau_a^+}{1-u^{-1}} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r_{\mathrm{comp.}}(z) = \frac{\Pi_+}{1-u} - \frac{\Pi_-}{1-u^{-1}} \in \mathrm{End}(\mathfrak{g})$.

Upon integrating out $A_z, A_{\overline{z}}$, we get (rational case)

$$S = \int d^2 z \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \mathbf{r}_{w_1 - w_2} \left(p_i v_a^i \tau_a, \overline{p}_{\overline{i}} v_a^{\overline{i}} \tau_a \right) \right) =$$

$$= \int d^2 z \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \frac{1}{w_1 - w_2} \sum |p_i v_a^i|^2 \right) =$$

$$= \text{integrate out} \quad p, \overline{p} \quad (\text{the fiber of} \quad T^* \mathcal{M}) \sim$$

$$\sim \int d^2 z \left(G_{i\overline{j}} \partial_{\overline{z}} q^i \partial_z \overline{q}^{\overline{j}} \right) \qquad \text{with} \qquad G_{i\overline{j}} = \left(\sum_a v_a^i v_a^{\overline{j}} \right)^{-1}$$

Invertibility $\leftrightarrow$ Homogeneous space

Rational case: the flag manifold σ-model described earlier. [DB, 2019]

But this also provides deformations of those models, trigonometric and elliptic. We pass over to this topic, starting with deformations of the $\mathbb{CP}^{n-1}$ model.

Simplest deformation: 2D target space with a $U(1)$-isometry [DB-Lüst, 2020]

$$ds^2 = \sum_{i,j=1}^{2} G_{ij}\, dX^i dX^j = \frac{1}{4\, g(\mu)}\, d\mu^2 + g(\mu)\, d\phi^2\,.$$

For what $g(\mu)$ is the model integrable?
Mechanical reduction always integrable (2 integrals of motion).
Generalized Pohlmeyer map: set $G_{ij}\, \partial X^i \overline{\partial} X^j = \cosh\chi$.

Sinh-Gordon equation replaced by $\quad \overline{\partial}\partial\chi - 2g''(\mu)\, \sinh\chi = 0\,.$
Add the equation for μ: $\qquad\qquad \partial\overline{\partial}\mu - 2\, g'(\mu)\, \cosh\chi = 0\,.$

The two eqs. follow from a single Lagrangian if $g(\mu) = b + a\, \cosh\mu$:

$$\mathscr{L} = \frac{1}{2}\partial\mu\overline{\partial}\mu + \frac{1}{2}\partial\chi\overline{\partial}\chi + 2a\, \cosh\mu\, \cosh\chi =$$
$$= \left(\frac{1}{2}\partial\widetilde{\mu}\, \overline{\partial}\widetilde{\mu} + a\, \cosh\left(\sqrt{2}\widetilde{\mu}\right)\right) + \left(\frac{1}{2}\partial\widetilde{\chi}\, \overline{\partial}\widetilde{\chi} + a\, \cosh\left(\sqrt{2}\widetilde{\chi}\right)\right)$$

The (Kähler) metric of the 'sausage' [Fateev, Onofri, Zamolodchikov, 1994]

$$ds^2 = \left(\frac{1}{s} - s\right) \frac{|dW|^2}{(s + |W|^2)(\frac{1}{s} + |W|^2)}, \qquad 0 < s < 1.$$

$\mathbb{CP}^{N-1}$ also has a generalized Kähler deformation [Demulder et.al. 2020], constructed along the lines of [Delduc, Magro, Vicedo 2013]. The B-field has the form $B = \sum_i b_i \wedge d\phi_i$, so T-dualizing all angles we get rid of it.

T-dual geometry is Kähler with potential [DB-Lüst, 2020]

$$\mathcal{K} = \sum_{j=1}^n (Z_j \overline{Z}_{j-1} - \overline{Z}_j Z_{j-1}) + 2 \sum_{j=1}^n P(t_j - t_{j-1} - 2\tau), \qquad t_j = Z_j + \overline{Z_j},$$

where $P(t) = \mathrm{Li}_2(e^{-t}) + \frac{t^2}{4}$.

$\mathcal{K}$ not invariant under $Z_i \to Z_i + \delta\alpha_i$. T-duality does not preserve the Kähler property (otherwise T: chiral $\leftrightarrow$ twisted chiral [Rocek, Verlinde 1991]).

The metric satisfies the (simple) Ricci flow equation $-\frac{dg_{i\bar{j}}}{d\tau} = 4 R_{i\bar{j}}$ with $s = e^{N\tau}$.

The $\beta\gamma$ deformed Lagrangian (s = def. parameter, $\Phi = A_z$): [DB, 2019]

$$\mathscr{L} = \mathrm{Tr}\left(V\overline{\mathscr{D}}U\right) + \mathrm{Tr}\left(V\overline{\mathscr{D}}U\right)^{\dagger} + \mathrm{Tr}\left(r_s^{-1}(\Phi)\overline{\Phi}\right)$$
$$\overline{\mathscr{D}}U = \overline{\partial}U + iU\overline{\mathcal{A}} + i\,\overline{\Phi}U\,.$$

r_s is the classical r-matrix: $r_s = \frac{s}{1-s}\,\pi_+ + \frac{1}{1-s}\,\pi_- + \frac{1}{2}\frac{1+s}{1-s}\,\pi_0$ (solution of CYBE)

Φ enters quadratically $\rightarrow$ integrate it out.

Very fruitful approach: introduce a 'Dirac boson'

$$\Psi_a = \begin{pmatrix} U_a \\ \overline{V}_a \end{pmatrix}, \qquad a = 1,\ldots,N\,.$$
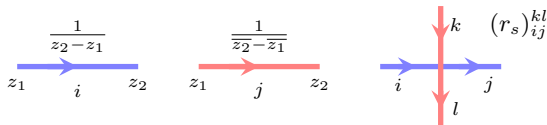
Then

$$\mathscr{L} = \overline{\Psi_a}\,\slashed{D}\Psi_a + (r_s)_{ab}^{cd}\left(\overline{\Psi_a}\frac{1+\gamma_5}{2}\Psi_c\right)\left(\overline{\Psi_d}\frac{1-\gamma_5}{2}\Psi_b\right)\,.$$

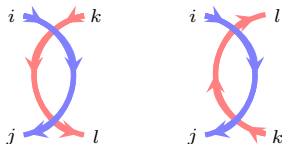σ-model = chiral gauged Gross-Neveu model (in bosonic incarnation)!

– Chirality: integrate out V-variables $\rightarrow$ return to the geometric form of the model.

– The deformation is manifestly Hermitian

Feynman rules:

Diagrams contributing to the β-function at one loop:

β-function:

$$\beta_{ij}^{kl} = \sum_{p,q=1}^{N} \left((r_s)_{ip}^{kq}(r_s)_{pj}^{ql} - (r_s)_{ip}^{ql}(r_s)_{pj}^{kq} \right)$$

The Ricci flow equation $\dot{r}_{ij}^{kl} = \beta_{ij}^{kl}$ has a remarkably simple solution $s = e^{N\tau}$ (was conjectured in [Costello-Yamazaki 2019]).

Alternatively, return to the σ-model and solve the geometric Ricci flow equations

$$-\dot{g}_{ij} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + 2\,\nabla_i \nabla_j \Phi\,,$$

$$-\dot{B}_{ij} = -\frac{1}{2}\,\nabla^k\,H_{kij} + \nabla^k \Phi\,H_{kij}\,,$$

$$-\dot{\Phi} = \text{const.} - \frac{1}{2}\,\nabla^k\nabla_k\,\Phi + \nabla^k\Phi\nabla_k\Phi + \frac{1}{24}\,H_{kmn}H^{kmn}$$

$\mathbb{CP}^1$: the solution $s = e^{2\tau}$ ($N = 2$) found in [Fateev, Onofri, Zamolodchikov, 1994]
$\mathbb{CP}^{N-1}$: Ricci flow interpolates between a cylinder $(\mathbb{C}^\times)^{N-1}$ in the UV (asymptotic freedom) and a 'round' projective space of vanishing radius in the IR.

Consider the Ricci flow for the metric

$$-\dot{g}_{ij} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + \nabla_i \mathcal{D}_j \Phi + \nabla_j \mathcal{D}_i \Phi, \qquad \mathcal{D}\Phi = d\Phi - \mathcal{E}.$$

The structure of the blow-up (strong coupling) is as follows: $g_{ij} \sim (1-s) (g_{\text{hom.}})_{ij}$.

For the homogeneous metric $\mathcal{D}_i \Phi = 0$. Since $s = e^{N\tau}$, in the limit $-\dot{g}_{ij} \to N (g_{\text{hom.}})_{ij}$
Homogeneous (Killing) metric $g_{\text{hom.}}$ satisfies a generalized Einstein condition

$$R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g_{\text{hom.}}^{mm'} g_{\text{hom.}}^{nn'} = N (g_{\text{hom.}})_{ij}.$$

For Grassmannians $H = 0$, i.e. $R_{ij} = N (g_{\text{hom.}})_{ij}$.
$N =$ first Chern number of the tangent bundle
$c_1(G(m,N)) = N [\mathscr{C}]$ ($\mathscr{C} =$ generator) – independent of m!

β-function for a symmetric space = dual Coxeter number (independent of H in $\frac{G}{H}$).
We observe this for non-symmetric spaces – can be proven directly!

Euclidean chiral $\mathbb{C}^\times$-symmetry [Zumino, 1977; Mehta 1990] is crucial:

$$U \to \lambda U, \quad V \to \lambda^{-1} V, \qquad \text{where} \quad \lambda \in \mathbb{C}^\times.$$

Remember $U \in \mathbb{C}^N$, so to pass to $\mathbb{CP}^{N-1}$ we need to gauge the chiral symmetry. However the symmetry is typically anomalous: recall Schwinger's effective action

$$\mathscr{S}_{\text{eff.}} = \frac{\xi}{2} \int dz\, d\overline{z}\, F_{z\overline{z}} \frac{1}{\triangle} F_{z\overline{z}}, \qquad F_{z\overline{z}} = i\,(\partial\overline{\mathcal{A}} - \overline{\partial}\mathcal{A})$$

Not invariant under the complexified gauge transformations
$\mathcal{A} \to \mathcal{A} + \partial\alpha$, $\overline{\mathcal{A}} \to \overline{\mathcal{A}} + \overline{\partial}\overline{\alpha}$. To cancel the anomaly one can add fermions minimally:

$$\mathscr{L} = \overline{\Psi_a}\slashed{D}\Psi_a + (r_s)_{ab}^{cd} \left(\overline{\Psi_a}\frac{1+\gamma_5}{2}\Psi_c\right) \left(\overline{\Psi_d}\frac{1-\gamma_5}{2}\Psi_b\right) + \overline{\Theta_a}\slashed{D}\Theta_a,$$

$$\Psi, \Theta \in \operatorname{Hom}(\mathbb{C}, \mathbb{C}^2 \otimes \mathbb{C}^N).$$

Incidentally these are the same fermions that cancel the anomaly in Lüscher's nonlocal charge [Abdalla et.al., 1981-84]!

Conjecture: all such flag manifold models with fermions are quantum integrable

Instead of working in a standard gauge like $\overline{U}U = 1$, we can choose

$$U_N = 1 \qquad \text{(inhomogeneous gauge)}$$

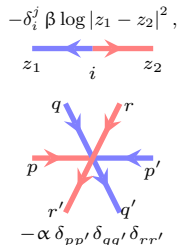Varying the action w.r.t. $\overline{\mathcal{A}}$, we get $UV = 0$, i.e. $V_N = -\sum_{j=1}^{N-1} U_j V_j$.

Substituting in the Lagrangian, we get

$$\mathscr{L} = \sum_{k=1}^{N-1} \left(V_k \overline{\partial} U_k - \overline{V}_k \partial \overline{U}_k + \beta |V_k|^2 \right) +$$

$$+ \underbrace{\sum_{l,m=1}^{N-1} a_{lm} |U_l|^2 |V_m|^2 + \gamma \left| \sum_{p=1}^{N-1} U_p V_p \right|^2}_{\text{quartic vertices}} + \underbrace{\alpha \left(\sum_{k=1}^{N-1} |U_k|^2 \right) \left| \sum_{p=1}^{N-1} U_p V_p \right|^2}_{\text{sextic vertices}}$$

$$-\delta_i^j \, \beta \log |z_1 - z_2|^2 \,,$$

$$-\alpha \, \delta_{pp'} \delta_{qq'} \delta_{rr'}$$

Instead of a σ-model we obtained a theory
with polynomial interactions! Parallel with Ashtekar variables:

- Direct derivation for $\frac{SL(2,\mathbb{R})}{SO(2)}$ [Brodbeck & Zagermann '00]
- Interactions are polynomial
- Degenerations are allowed (compare with nilpotent orbits)

Conclusion and outlook.

- Integrable sigma-models beyond symmetric target spaces
  [DB '14[+], Costello-Yamazaki 2019]
- Related to PCM through nilpotent orbits [DB '19]
- GLSM formulation beyond Kähler target spaces [DB '17]
- Hermitian deformations of these models [Costello-Yamazaki 2019, DB - to appear]
- The anomaly of the bosonic model similar to the one for symmetric spaces
  [Abdalla, Abdalla, Gomes '81-'84, DB '19]
- Cancel it by adding fermions (minimally/ supersymmetrically?),
  bosonized version [Basso & Rej '12]
- Pohlmeyer reduction [Pohlmeyer '76, Grigoriev-Tseytlin '08, DB-Lüst '20]
  Toda theories with additional linear fields. Example: $\frac{U(3)}{U(1)^3}$

$$2\,\partial\overline{\partial}X_1 + e^{2(X_1-X_2)} - e^{2(X_3-X_1)} + e^{2(X_1-X_3)}\|(U_{13})_z\|^2 - e^{2(X_2-X_1)}\|(U_{21})_z\|^2 = 0$$

$$\cdots$$

$$\overline{\partial}(U_{13})_z + e^{2(X_3-X_2)}\,\overline{(U_{32})_z} - e^{2(X_2-X_1)}\,\overline{(U_{21})_z} = 0$$

$$\cdots$$

General case?

- σ-models = gauged chiral Gross-Neveu models [DB, to appear]
- The one-loop β-function is universal for all of these models (one-loop exact?)
- (Complicated) Ricci flow eqs. have a simple (ancient) solution, presumably interpolating between the homogeneous metric and a cylinder

- Relation to deformations of $\mathbb{Z}_k$-graded spaces, i.e. analogues of symmetric space deformations of [Fateev '96, Klimčík '02+, Delduc, Magro, Vicedo '13]
- Dual Toda-like theories, $\mathfrak{gl}(N|N)$ symmetry [Fateev '17], [Litvinov '19]
- Construct the full quantum theory: S/R-matrix, thermodynamic Bethe ansatz. Possibly using the ODE/IQFT approach
  [Bazhanov, Lukyanov, Zamolodchikov 98+, Bazhanov, Kotousov, Lukyanov '17]
- $\cdots$