Flavor Constraints from Unitarity and Analyticity [2004.02885]

Constraints on the coefficients of dim 8 4-fermi Operators of the SMEFT, schematically $\partial^2 (\bar{\psi}_m \psi_n)(\bar{\psi}_p \psi_q)$ m, n, p, q flavor indices.

<u>The method:</u> (hep-th/0602178, 1605.06111, 1405.2960)

constraints for the Wilson coefficients of some operators in an EFT.

Example: take a scalar theory (real) with $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{c}{M^4} (\partial \phi)^4$$

Consider the scattering of two particles ϕ:

$$\mathcal{M}(s,t) = \frac{2c}{M^4} (s^2 + t^2 + u^2)$$

$s = (p_1 + p_2)^2$

$t = (p_1 - p_3)^2$

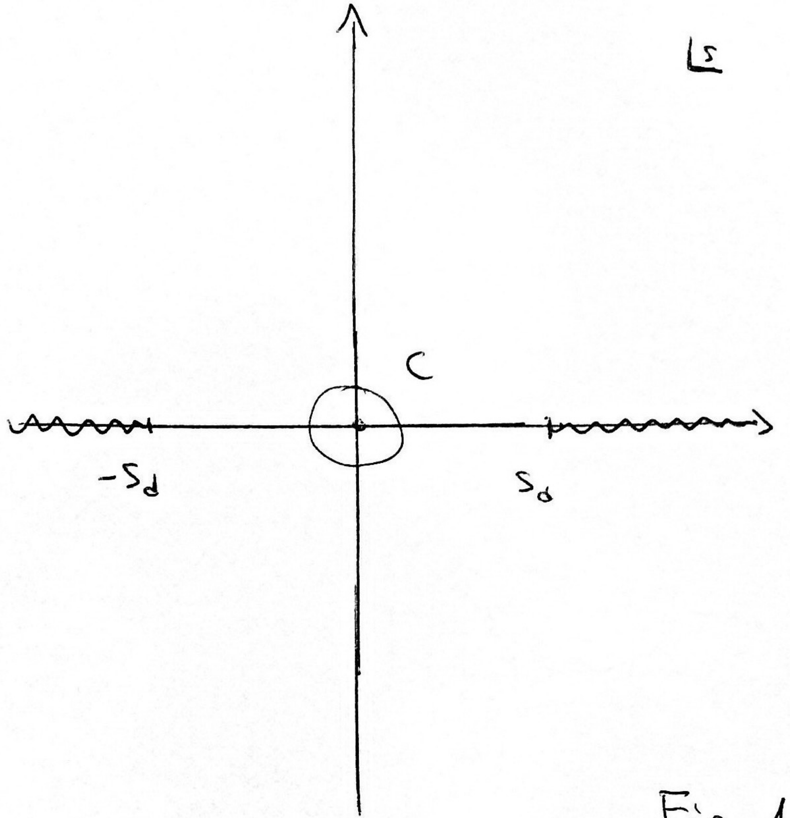
$u = (p_1 - p_4)^2$

Take the forward limit $t \to 0$ $(u \to -s)$

$$A(s) \equiv \mathcal{M}(s, t=0) = \frac{4cs^2}{M^4}$$

Now we consider the analytic continuation of $A(s)$ to the complex plane for complex s.

The structure of $A(s)$ is the following:



Suppose the discontinuities (poles, branch cuts) start at $s = s_d$.

Crossing symmetry imposes

$$A(s) = A(-s)$$

Fig 1.

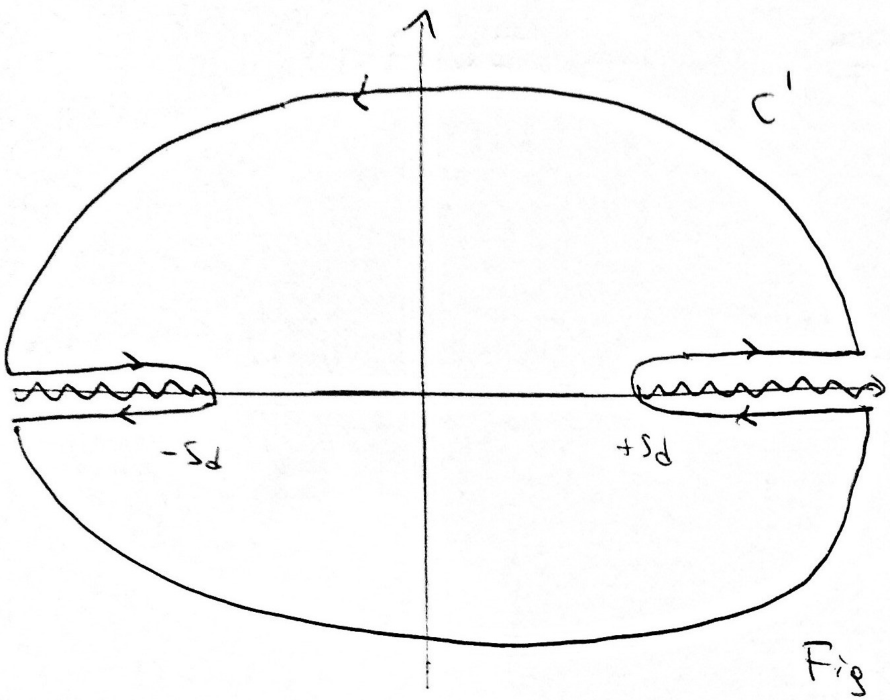


Fig 2.

Expand $A(s)$ as

$$A(s) = \sum_{n=0}^{+\infty} \lambda_n s^n$$

with in particular

$$\lambda_2 = \frac{1}{2\pi i} \oint_C \frac{A(s)}{s^3}$$

since we want the s^2 coeff.

We deform the contour C to C' (Fig 1 → Fig 2)

$$\lambda_2 = \frac{1}{2\pi i} \oint_{C'} \frac{ds}{s^3} A(s) = C_R + \frac{1}{2\pi i} \left(\int_{-\infty}^{-s_d} \frac{ds}{s^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] \right) +$$

$$+ \frac{1}{2\pi i} \left(\int_{s_d}^{+\infty} \frac{ds}{s^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] \right)$$

$$= \cancelto{0}{C_R} + \frac{1}{\pi i}\left(\int_{s_d}^{+\infty} \left[A(s+i\varepsilon) - A(s-i\varepsilon)\right]\right) = \frac{2}{\pi}\int_{s_d}^{+\infty} \frac{ds}{s^3}\, \mathrm{Im}\, A(s)$$

Optical theorem $\quad \mathrm{Im}(A(s)) = s\,\sigma(s)$

$$\Rightarrow \lambda_2 = \frac{2}{\pi}\int_{s_d}^{+\infty} \frac{ds}{s^2}\,\sigma(s) > 0 \quad \text{since } \sigma(s) > 0 \quad \text{(non trivial theory)}$$

$$\lambda_2 > 0 \Rightarrow c > 0$$

Now to the paper: try to constrain the coefficients of dim 8 4-fermi operators, schematically of the form $\partial^2 (\bar{\psi}_m \psi_n)(\bar{\psi}_p \psi_q)$ where m, n, p, q are flavor indices

Label

		SU(3)	SU(2)	U(1)
Q:	left-handed quark doublet	$\underline{3}$	$\underline{2}$	$\frac{1}{3}$
L:	left-handed lepton doublet	1	$\underline{2}$	-1
u_R:	right-handed up-type quark	$\underline{3}$	1	$\frac{4}{3}$
d_R:	right-handed down-type quark	$\underline{3}$	1	$-\frac{2}{3}$
e_R:	right-handed lepton field	1	1	-2

each carrying a generation/flavor index

Work in the unbroken phase $\not{D}\psi = 0$

Define

$$J^\mu[\psi]_{mn} = \bar{\psi}_m \gamma_\mu \psi_n \qquad K_{\mu\nu}[\psi]_{mn} = \bar{\psi}_m \gamma_\mu \partial_\nu \psi_n$$

$$J^\mu[\psi]^a_{mn} = \bar{\psi}_m T^a \gamma_\mu \psi_n \qquad K_{\mu\nu}[\psi]^I_{mn} = \bar{\psi}_m \tau^I \gamma_\mu \partial_\nu \psi_n$$

$$J^\mu[\psi]^I_{mn} = \bar{\psi}_m \tau^I \gamma_\mu \psi_n \qquad K_{\mu\nu}[\psi]^a_{mn} = \bar{\psi}_m T^a \gamma_\mu \partial_\nu \psi_n$$

$$J^\mu[\psi]^{Ia}_{mn} = \bar{\psi}_m \tau^I T^a \gamma_\mu \psi_n$$

Basis of operators

$$\left.\begin{aligned}
\mathcal{O}_1[\psi] &= C^{\psi,1}_{mnpq}\, \partial_\mu J_\nu[\psi]_{mn}\, \partial^\mu J^\nu[\psi]_{pq} && \psi = \text{any} \\
\mathcal{O}_2[\psi] &= C^{\psi,2}_{mnpq}\, \partial_\mu J_\nu[\psi]^I_{mn}\, \partial^\mu J^\nu[\psi]^I_{pq} && \psi = L, Q \\
\mathcal{O}_3[\psi] &= C^{\psi,3}_{mnpq}\, \partial_\mu J_\nu[\psi]^a_{mn}\, \partial^\mu J^\nu[\psi]^a_{pq} && \psi = d, u, Q \\
\mathcal{O}_4[Q] &= C^{Q,4}_{mnpq}\, \partial_\mu J_\nu[Q]^{Ia}_{mn}\, \partial^\mu J^\nu[Q]^{Ia}_{pq} &&
\end{aligned}\right]\ \text{self-quartic}$$

$$\left.\begin{aligned}
\mathcal{O}_{J1}[\psi,\chi] &= b^{\psi\chi,1}_{mnpq}\, \partial_\mu J_\nu[\psi]_{mq}\, \partial^\mu J^\nu[\chi]_{np} && \psi,\chi = \text{any} \\
\mathcal{O}_{J2}[Q,L] &= b^{QL,2}_{mnpq}\, \partial_\mu J_\nu[Q]^I_{mq}\, \partial^\mu J^\nu[L]^I_{np} && \\
\mathcal{O}_{J3}[\psi,\chi] &= b^{\psi,\chi,3}_{mnpq}\, \partial_\mu J_\nu[\psi]^a_{mq}\, \partial^\mu J^\nu[\chi]^a_{np} && \psi,\chi \in \{d,u,Q\} \\
\mathcal{O}_{K1}[\psi,\chi] &= -a^{\psi\chi,1}_{mnpq}\, K_{\mu\nu}[\psi]_{mq}\, K^{\nu\mu}[\chi]_{np} && \psi,\chi = \text{any} \\
\mathcal{O}_{K2}[Q,L] &= -a^{QL,2}_{mnpq}\, K_{\mu\nu}[Q]^I_{mq}\, K^{\nu\mu}[L]^I_{np} && \\
\mathcal{O}_{K3}[\psi,\chi] &= -a^{\psi\chi,3}_{mnpq}\, K_{\mu\nu}[\psi]^a_{mq}\, K^{\nu\mu}[\chi]^a_{np} && \psi,\chi = \{d,u,Q\}
\end{aligned}\right]\ \text{cross-quartic}$$

Now it's a matter of choosing the right states for the $2\to 2$ scattering. In particular we pick

$$|\psi_1\rangle = \alpha_m |\bar{e}_m\rangle \qquad |\psi_2\rangle = \beta_m |e_m\rangle$$

$$|\psi_3\rangle = \gamma_m |\bar{e}_m\rangle \qquad |\psi_4\rangle\ \delta_m |e_m\rangle$$

to constrain $c^{e,1}_{mnpq}$.

Impose $\gamma_m = \beta_m^*$ and $\delta_m = \alpha^*_m$ so $A(s) = A(-s)$ under $|\psi_1\rangle \leftrightarrow |\psi_4\rangle$

to get $A(s) = 4\, c^{e1}_{mnpq}\, \alpha_m \beta_n \beta^*_p \alpha^*_q\, s^2$ and

$$\boxed{c^{e,1}_{mnpq}\, \alpha_m \beta_n \beta^*_p \alpha_q > 0 \qquad \forall \alpha, \beta}$$

main result (this and the ones corresponding to the other operators)

Define $\rho^{\alpha}_{mn} \equiv \alpha_m \alpha^*_q$ $\quad \rho^{\beta}_{np} \equiv \beta_n \beta^*_p$ to recast into

$$c^{e1}_{\alpha\beta} \equiv c^{e,1}_{mnpq}\, \rho^{\alpha}_{mq}\, \rho^{\beta}_{np} > 0 \qquad \forall\, \alpha, \beta$$

Let's see another one $\quad |\psi_1\rangle = \alpha_{mi} L_{mi} \quad |\psi_2\rangle = \beta_{mi} L_{mi}$

$$\underset{\text{flavor}}{\underset{\downarrow}{m}}\qquad \underset{SU(2)}{i}$$

$$A = 4 g^2 \left[\left(C^{L,1}_{mnpq} - \frac{1}{4} C^{L,2}_{mnpq} \right) \alpha^*_{mi} \beta_{ni} \beta^*_{pj} \alpha_{qj} + \frac{1}{2} C^{L,2}_{mnpq} \alpha^*_{mi} \beta_{nj} \beta_{pj} \alpha_{qi} \right]$$

pick alternatively $\begin{cases} \alpha_{mi} = \alpha_m \delta_{1i} \\ \beta_{mi} = \beta_m \delta_{1i} \end{cases}$ or $\begin{cases} \alpha_{mi} = \alpha_m \delta_{1i} \\ \beta_{ni} = \beta_n \delta_{2i} \end{cases}$ to get

$$C^{L,1}_{\alpha\beta} + \frac{1}{4} C^{L,2}_{\alpha\beta} \geq 0 \qquad \text{and} \qquad C^{L,2}_{\alpha\beta} \geq 0$$

and so on.

What do they look like? ~~Take~~ E.g. take $c^{e,1}_{\alpha,\beta} > 0$

pick

$$\begin{cases} \alpha_m = \delta_{1m} \\ \beta_n = \delta_{2n} \end{cases} \Rightarrow c^{e,1}_{1221} > 0 \qquad \begin{cases} \alpha_m = \delta_{1m} \\ \beta_n = \delta_{1n} \end{cases} \Rightarrow c^{e,1}_{1111} > 0$$

More interestingly, if ρ_α and ρ_β have off-diagonal terms

e.g.

$$\begin{cases} \alpha_m = \delta_{1m} \\ \beta_n = \delta_{2n} \cos\theta + \delta_{3n} e^{i\varphi} \sin\theta \end{cases} \Rightarrow C_{1221}\, C_{1331} > |C_{1231}|^2$$

so an interaction that violates lepton number $(\Delta L_\mu, \Delta L_\tau) = (+1, -1)$ is ~~allowed~~ allowed only if the analogous flavor conserving operators are non-zero

Examples of UV completion respect these bounds. E.g.

$$\mathcal{L} \supset -m^2 \phi_{\mu\nu,mn} \phi^{*\mu\nu}_{mn} + \left(m y \phi^{\mu\nu}_{mn} \partial_\mu J_\nu [e]_{mn} + h.c\right)$$

for $\phi^{\mu\nu}_{mn}$ complex, SM singlet

$$\Rightarrow \mathcal{L}_{EFT} \supset |y|^2 \delta_{mq} \delta_{np} \partial_\mu J_\nu [e]_{mn} \partial^\mu J^\nu [e]_{pq}$$ whence

$$C^{e,1}_{\alpha\beta} = |y|^2 \, Tr[\rho^\alpha] \, Tr[\rho^\beta] = |y|^2 > 0$$

Phenomenology

e.g if $Br(\mu^+ \to e^+ e^+ e^-)$ measured at Mu3e

shows $C^{e,1}_{1112} \neq 0$ then by these bounds

$C^{e,1}_{1111} \neq 0$
$C^{e,1}_{2112} \neq 0$ and $C^{e,1}_{1111} C^{e,1}_{2112} > |C^{e,1}_{1112}|^2$

Explicit computation for the integration on the complex field.

$$\frac{1}{2\pi i}\oint_{C'}\frac{ds}{s^3}A(s) = C_\Lambda + \frac{1}{2\pi i}\left(\int_{-\Lambda}^{-s_d}\frac{ds}{s^3}\left[A(s+i\varepsilon)\right.\right.$$

$$\left.\left.-A(s-i\varepsilon)\right]\right) + \frac{1}{2\pi i}\left(\int_{s_d}^{\Lambda}\frac{ds}{s^3}\left[A(s+i\varepsilon)-A(s-i\varepsilon)\right]\right)$$

$$\xrightarrow{\Lambda\to+\infty}\ \frac{1}{2\pi i}\left(\int_{+\infty}^{+s_d}\frac{ds}{s^3}\left[A(-s+i\varepsilon)-A(-s-i\varepsilon)\right]\right)+$$

$$+\frac{1}{2\pi i}\left(\int_{s_d}^{+\infty}\frac{ds}{s^3}\left[A(s+i\varepsilon)-A(s-i\varepsilon)\right]\right)=$$

$$= \cancel{\oint} - \frac{1}{2\pi i}\left(\int_{s_d}^{+\infty}\frac{ds}{s^3}\left[A(-s+i\varepsilon)-A(-s-i\varepsilon)\right]\right)+$$

$$+\frac{1}{2\pi i}\left(\int_{s_d}^{+\infty}\frac{ds}{s^3}\left[A(s+i\varepsilon)-A(s-i\varepsilon)\right]\right)$$

Now use $A(-s)=A(s)$ from crossing symmetry

$$\Rightarrow \frac{1}{\pi i}\left(\int_{s_d}^{+\infty}\frac{ds}{s^3}\left[A(s+i\varepsilon)-A(s-i\varepsilon)\right]\right)=$$

$$=\frac{2}{\pi}\int_{+s_d}^{+\infty}\frac{ds}{s^3}\,\mathrm{Im}\,A(s)$$

If I introduce a mass

$$\mathcal{M}(s,t) = 2(s^2+t^2+u^2) - 8m^4$$

which in the forward limit $t \to 0$ becomes

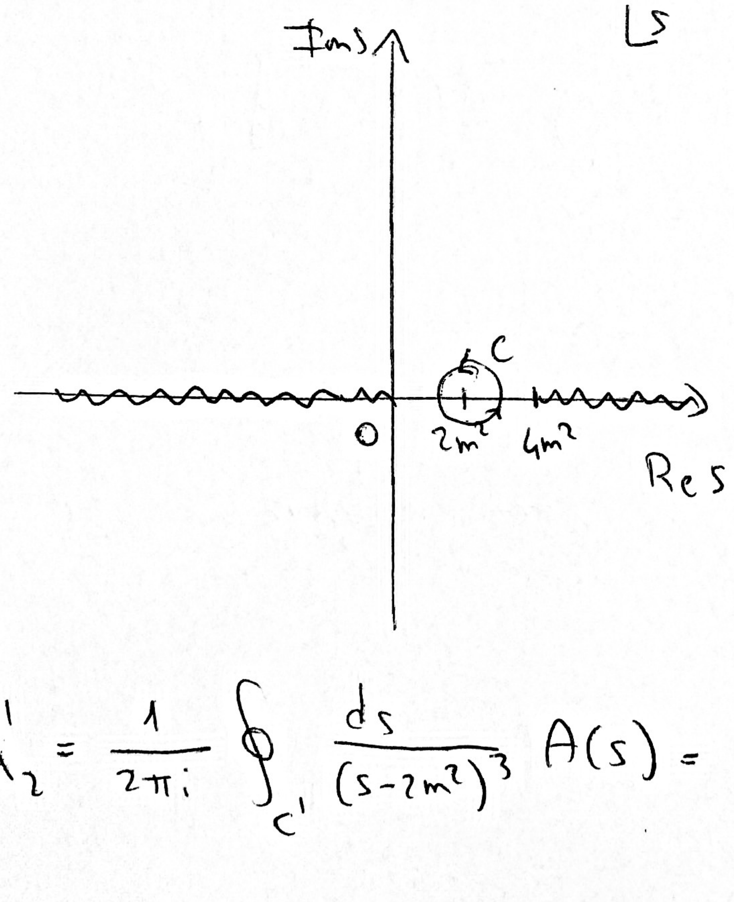
$$A(s) \equiv \mathcal{M}(s, t=0) = 8m^4 + 4(s-2m^2)^2$$

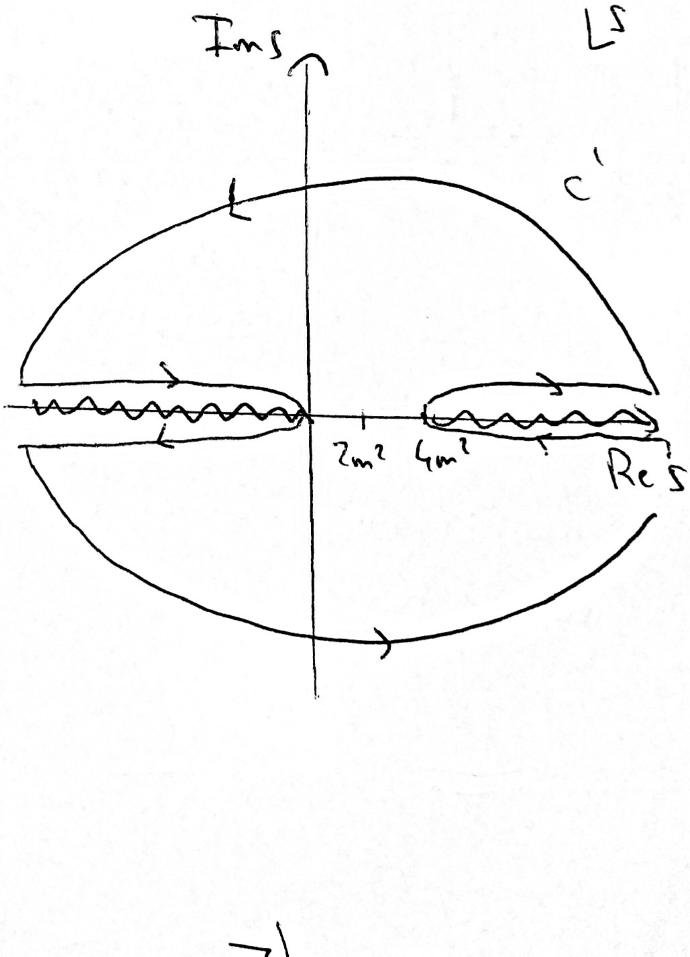
so we just need to expand around $s = 2m^2$

~~on~~ $A(s) = \sum_n \lambda'_n (s-2m^2)^n$

and look for λ'_2

$$\lambda'_2 = \frac{1}{2\pi i} \oint_C \frac{ds}{(s-2m^2)^3} A(s)$$





$$\lambda'_2 = \frac{1}{2\pi i} \oint_{C'} \frac{ds}{(s-2m^2)^3} A(s) =$$

$$= C_1 + \frac{1}{2\pi i} \left(\int_{-\Lambda}^{0} \frac{ds}{(s-2m^2)^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] \right) +$$

$$+ \frac{1}{2\pi i} \left(\int_{4m^2}^{\Lambda} \frac{ds}{(s-2m^2)^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] \right)$$

Sending $\Lambda \to +\infty$ and substituting $s \to -s + 4m^2$ in the second term of the r.h.s.

$$\lambda_2' = \frac{1}{2\pi i} \int_{+\infty}^{+4m^2} \frac{ds}{(s-2m^2)^3} \left[A(-s+4m^2+i\varepsilon) - A(-s+4m^2-i\varepsilon) \right] +$$

$$+ \frac{1}{2\pi i} \Bigg(\int_{4m^2}^{+\infty} \frac{ds}{(s-2m^2)^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] =$$

$$= \frac{1}{2\pi i} \int_{4m^2}^{+\infty} \frac{ds}{(s-2m^2)^3} \left[A(-s-i\varepsilon+4m^2) - A(-s+i\varepsilon+4m^2) \right] +$$

$$+ \frac{1}{2\pi i} \Bigg(\int_{4m^2}^{+\infty} \frac{ds}{(s-2m^2)^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] =$$

$$= \frac{1}{\pi i} \int_{4m} \frac{ds}{(s-2m^2)^3} \left[A(s+i\varepsilon) - A(s-i\varepsilon) \right] =$$

$$= \frac{2}{\pi} \int_{4m^2}^{+\infty} \frac{ds}{(s-2m^2)^3} \cdot s \cdot \sigma(s) > 0$$

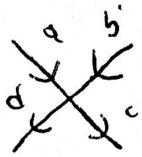
In the case of massless spin-$\frac{1}{2}$ particles, the scattering matrix of a $2 \to 2$ process obeys $(t \to 0)$

$$\mathcal{M}\left(1^{\sigma_1}_{a_1}\, 2^{\sigma_2}_{a_2} \to 1^{\sigma_1}_{a_4}\, 2^{\sigma_2}_{a_3}, s\right) =$$

$$= \mathcal{M}\left(1^{-\sigma_1}_{\bar{a}_4}\, 2^{\sigma_2}_{a_2} \to 1^{-\sigma_1}_{\bar{a}_1}\, 2^{\sigma_2}_{a_3}, -s\right)$$

which, for the states we have chosen, implies $A(s) = A(-s)$

The vertex is:



$$4\Big(-p_{\mu m d}\,\gamma^{\nu}_{da}\,p^{\mu}_{cp}\,\gamma^{\nu}_{cb} + \gamma_{da\nu}\,p_{\mu an}\,p^{\mu}_{cp}\,\gamma^{\nu}_{cb}\Big\} +$$

$$+\, p_{\mu m d}\,\gamma_{da\nu}\,\gamma^{\nu}_{cb}\,p^{\mu}_{qb} - \gamma_{\nu da}\,p_{\mu an}\,\gamma^{\nu}_{cb}\,p^{\mu}_{qb}\Big)$$

The amplitude reads

$$4\left(\alpha_m\,\alpha_q^*\,\beta_n\,\beta_p^*\right)\left(\bar v_{md}\,\gamma^{\nu}_{da}\,u_{na}\cdot\bar u_{cq}\,\gamma^{\nu}_{cb}\,v_{bp}\right)\cdot$$

$$\cdot\left(-p_{\mu 1}\,p^{\mu 3} + p_{2\mu}\,p^{3\mu} + p_{1\mu}\,p^{\mu 4} - p_{2\mu}\,p^{\mu 4}\right)$$

$$= -8\,(p_1\cdot p_2)\,\alpha_m\,\alpha_q^*\,\beta_n\,\beta_p^*\cdot\left(\bar v_{md}\,\gamma^{\nu}_{da}\,u_{na}\cdot\bar u_{cq}\,\gamma^{\nu}\,v_{bp}\right)\cdot$$

$$= -4s\,\alpha_m\,\alpha_q^*\,\beta_p^*\,\beta_n\cdot[1\,\gamma^{\nu}\,2\rangle\,[4\,\gamma_{\nu}^*\,3\rangle =$$

$$= -4s\,\alpha_m\,\alpha_q^*\,\beta_p^*\,\beta_n\cdot\left(2\,\langle 2 1\rangle\,[1 2]\right) =$$

$$= 8s^2\,\alpha_m\,\alpha_q^*\,\beta_p^*\,\beta_n$$

Picking $\alpha_m = \delta_{1m}$, $\beta_m = \delta_{2m} \cos\theta + \delta_{3m} e^{i\varphi} \sin\theta$

$C^{e,1}_{mnpq} \alpha_m \beta_n \beta^*_p \alpha^*_q > 0 \Rightarrow$

$\Rightarrow C_{1221} \cos^2\theta + (C_{1231} e^{-i\varphi} + C^*_{1231} e^{i\varphi}) \cos\theta \sin\theta +$

$+ C_{1331} \sin^2\theta > 0$ Ⓐ

Now minimizing in φ :

$\Rightarrow C_{1231} e^{-i\varphi} - C^*_{1231} e^{i\varphi} = 0$

and we can bring Ⓐ in the form

$C_{1221} + (C_{1231} e^{-i\varphi} + C^*_{1231} e^{i\varphi}) \tan\theta +$

$+ C_{1331} \tan^2\theta > 0$

This is always > 0 if

$\Rightarrow C^2_{1231} e^{-2i\varphi} + C^{*}_{1231} e^{2i\varphi} + 2|C_{1231}|^2 - 4 C_{1221} C_{1331} < 0$

now using that we are at the minimum for φ:

$C^2_{1231} e^{-2i\varphi} = C^{*2}_{1231} e^{2i\varphi} = |C_{1231}|^2$

$\Rightarrow C_{1221} C_{1331} \geq |C_{1231}|^2$