Anomalies in QFT

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May 12, 2020

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1 Introduction

Anomalies are by definition symmetries of of a classical field theory which get broken by quantum effects. The existence of anomalies is crucial and at the same time ubiquitous in quantum field theory. There are lots of anomalies, for any kind of symmetries and in any number of dimension: global symmetries, gauge ones, both discrete and continuous, spacetime symmetries, internal ones, bosonic symmetries and supersymmetries.

1.1 Different types of symmetries in Field Theory

Let us now review different types of symmetries in field theory, and their properties:

1. Global symmetry. This is a physical symmetry of a system. One can apply the first Noether theorem to get classically conserved global currents and charges. This symmetry can be spontaneously broken, and as many goldstone bosons are produced as the number of generators of the broken group[1]. We will see that these symmetries can be anomalous.

2. Gauge symmetry. Unphysical symmetry of a system. It is a redundancy of the physical description, crucial to reduce the overdetermination of of-shell degrees of freedom to the on-shell ones. One can apply the second Noether theorem and get Ward identities for large gauge transformations. This symmetry cannot be spontaneously broken (Elitzur theorem), or otherwise the gauge theory is inconsistent. We will see that these symmetries can not be anomalous, or otherwise the gauge theory is inconsistent. It is useful to split the gauge transformations in two subsets:

   - Small gauge transformations. They are all gauge transformations such that they vanish at every point of the boundary of spacetime. They are the usual gauge transformation we typically learn in undergrad.
   - Large gauge transformations. They are all gauge transformations such that they have boundary of spacetime (i.e spacetime infinity). They are the gauge transformations sending one instanton solution to another.

3. Global part of the gauge symmetry. This is a physical symmetry of a system. It is what remains after doing gauge-fixing. One can apply the first Noether theorem and get classically conserved global current and charges, as for example the electric charge in a $U(1)$ gauge theory. This symmetry can be spontaneously broken, wannabe goldstone bosons are produced but immediately eaten up by W-bosons via the Higgs mechanism. We will see that these symmetries can not be anomalous, or otherwise the gauge theory is inconsistent.

1.2 Noether's first theorem

Here we will review a quick way to derive the first Noether's theorem in field theory. Consider for simplicity the case of a scalar field ϕ, and a symmetry

$$\phi \to \phi' = \phi + \delta\phi \tag{1}$$

[1] This holds true only if the spacetime dimension is greater than 2, and if we are discussing about an internal symmetry and not a spacetime one. For spacetime symmetries, typically less Goldstone bosons are produced. The most beautiful example of this I know is Inflation: the inflaton is the unique Goldston boson produced from spontaneously breaking the isometry group of de Sitter. Such group gets broken down to the isometry group of the quasi-de Sitten inflationary background. In particular, the inflaton is the Goldstone boson of the broken time translation, but also the special conformal isometry are broken, and they give no goldstones.

with

$$\delta\phi = \epsilon X(\phi) \tag{2}$$

where ϵ is a constant infinitesiamlly small parameter. By definition, this transformation is a *classical symmetry* if the Lagrangian is invariant under it

$$L \to L' + \delta L, \qquad \delta L = 0. \tag{3}$$

Promote ϵ to a generic continuous function $\epsilon(x)$ of the spacetime coordinates x^μ such that it vanishes fast at infinity. Now let's compute the variation of the Lagrangian density

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\left(\partial_\mu\phi\right) + \frac{\partial\mathcal{L}}{\delta\phi}\delta\phi = \\
&= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu\left(\delta\phi\right) + \frac{\partial\mathcal{L}}{\delta\phi}\delta\phi = \\
&= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu\left(\epsilon X(\phi)\right) + \frac{\partial\mathcal{L}}{\delta\phi}\epsilon X(\phi) = \\
&= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu\epsilon X + \epsilon\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu X + \frac{\partial\mathcal{L}}{\delta\phi}X(\phi)\right]
\end{aligned} \tag{4}$$

Now, we know that if we take ϵ constant, then the lagrangian should be invariant. The first term vanishes for ϵ constant, so one should impose that the second does as well in this limit.

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu X + \frac{\partial\mathcal{L}}{\delta\phi}X(\phi) = 0 \tag{5}$$

Therefore, for non-constant $\epsilon(x)$, we are actually left with

$$\delta\mathcal{L} = (\partial_\mu\epsilon)J^\mu, \qquad J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}X(\phi) \tag{6}$$

and the action then changes as

$$\delta S = \int d^4x\ \delta\mathcal{L} = \int d^4x\ (\partial_\mu\epsilon)J^\mu = -\int d^4x\ \epsilon\partial_\mu J. \tag{7}$$

Now, this equation holds for any field configuration ϕ, but when ϕ is satisfying the equation of motion, then $\delta S = 0$ for any variation $\delta\phi$. This means that when ϕ is satifying the equation of motion,

$$\partial_\mu J^\mu = 0. \tag{8}$$

1.3 Ward Identities in QFT

The discussion above, on the first Noether theorem, regards symmetries of the classical field theory. We would like to derive now a morally analog result, in QFT. Such a thing exist, and it is called *Ward Identities*.

Let us consider for simplicity a theory of a single scalar field $\phi(x)$, and the path integral with the introduction of a classical source $K(x)$ for $\phi(x)$.

$$Z[K] = \int \mathcal{D}\phi\ \exp\left(-S[\phi] + \int d^4x\ K\phi\right) \tag{9}$$

Again we consider the symmetry

$$\phi \to \phi' = \phi + \epsilon(x) X(\phi) \tag{10}$$

1. This is just a change of variable at the level of the path integral

$$\begin{aligned} Z[K] \to Z'[K] = \int \mathcal{D}\phi' \ \exp\left(-S[\phi'] + \int d^4x \ K\phi'\right) = \\ = Z[K] \end{aligned} \tag{11}$$

so the partition function is actually invariant, as the integration variable is a dummy one.

2. We can nevertheless use the symmetry to do the following manipulation

$$\begin{aligned} Z'[K] &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi'] + \int d^4x \ K\phi'\right) = \\ &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi + \epsilon X] + \int d^4x \ K\phi + K\epsilon X\right) = \\ &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi] + \int d^4x \epsilon\partial_\mu J^\mu + \int d^4x \ K\phi + K\epsilon X\right) = \\ &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \exp\left(\int d^4x \ (\epsilon\partial_\mu J^\mu + K\epsilon X)\right) = \end{aligned} \tag{12}$$

Now, we will Taylor expand the second exponential, and keep only the leading term in ϵ:

$$\begin{aligned} Z[K]' &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \left(1 + \int d^4x \ (\epsilon\partial_\mu J^\mu + K\epsilon X)\right) = \\ &= \int \mathcal{D}\phi' \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) + \\ &+ \int \mathcal{D}\phi' \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \int d^4x \ (\epsilon\partial_\mu J^\mu + K\epsilon X) \end{aligned} \tag{13}$$

3. Now, to proceed, we need to make a crucial assumption. This assumption is that the measure of integration is invariant:

$$\mathcal{D}\phi' = \mathcal{D}\phi \tag{14}$$

4. By assuming that the integration measure is invariant, we get to

$$Z[K]' = Z[K] + \int \mathcal{D}\phi \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \int d^4x \ (\epsilon\partial_\mu J^\mu + K\epsilon X) \tag{15}$$

Now, we use the fact that $Z[K]' = Z[K]$ to get:

$$Z[K] = Z[K] + \int \mathcal{D}\phi \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \int d^4x \ (\epsilon\partial_\mu J^\mu + K\epsilon X) \tag{16}$$

Therefore we must have

$$\int \mathcal{D}\phi \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \int d^4x \ \epsilon(x) \left(\partial_\mu J^\mu + KX\right) = 0 \quad (17)$$

Furthermore, this has to hold true for any function $\epsilon(x)$. Then this implies

$$\int \mathcal{D}\phi \ \exp\left(-S[\phi] + \int d^4x \ K\phi\right) \left(\partial_\mu J^\mu + KX\right) = 0 \quad (18)$$

5. From this equality we can generate an infinite set of identities on the correlation functions, known as *Ward identities.* For example, sekking $K = 0$ we get

$$\int \mathcal{D}\phi' \ \exp\left(-S[\phi]\right) \partial_\mu J^\mu = 0 \quad (19)$$

which is simply

$$\langle \partial_\mu J^\mu \rangle = 0 \quad (20)$$

By taking n derivaties with respect of K and then setting $K = 0$ we get to

$$\langle \phi\phi \cdots \phi \partial_\mu J^\mu \rangle = 0 \quad (21)$$

We conclude that $\partial_\mu J^\mu$ when this is sticked inside a correlation function. When this happens, we say that $\partial_\mu J^\mu = 0$ *in an operatorial way.*

2 Chiral anomaly

Consider a 4 theory of a Dirac fermion ψ in a background (non dynamical) $U(1)$ gauge field

$$S = \int d^4x \ i\bar{\psi}\not{D}\psi \quad (22)$$

This action has two classical global $U(1)$ symmetries

1. Vector rotation. $U(1)_V$. Action in the field is $\psi \to \psi' = e^{i\alpha}\psi$
2. Axial rotation. $U(1)_A$ Action in the field is $\psi \to \psi' = e^{i\alpha\gamma_5}\psi$

For each of these symmetries we can derive the conserved currents by using the first Noether theorem.

1. Vector rotation. $j_V^\mu = \bar{\psi}\gamma^\mu\psi$
2. Axial rotation. $j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$

However, we want to ask now if these currents are also conserved in the quantum theory. Therefore, we need to derive the Ward identities as before. The only non-trivial step is looking wether the measure of integration is invariant or not. To do so, it is first of all useful to write the integration measure in a nicer way.

2.1 The integration measure

Consider the Dirac operator $i\not{D}$ acting on a fermion in a background electric field. We will consider the eigenspinors of this operator

$$i\not{D}\phi_n = \lambda_n \phi_n \tag{23}$$

where the entries of the eigenspinors ϕ_n are complex numbers.

We will expand any spinor in the base of the eigenspinors of $i\not{D}$.

$$\psi(x) = \sum_n a_n \phi_n(x), \qquad \bar{\psi}(x) = \sum_n \bar{b}_n \bar{\phi}_n(x) \tag{24}$$

where now a_n and $\bar{b}_n$ are Grassmann numbers. We recall that different eigenspinors are orthogonal

$$\int d^4x \phi_n \bar{\phi}_m = \delta_{mn} \tag{25}$$

We now want to write the Path integral in terms of the eigenfunction expansion. The integration measure in this context reads

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_n \int d\bar{b}_n da_n \tag{26}$$

while the action reads

$$\begin{aligned} S = \int d^4x \; i\bar{\psi}\not{D}\psi &= \int d^4x \; i(\sum_n \bar{b}_n \bar{\phi}_n)\not{D}(\sum_n a_n \phi_n) = \\ = \int d^4x \; i \sum_{n,m} \bar{b}_n a_n \lambda_m \bar{\phi}_n \phi_n &= \sum_n \lambda_n a_n \bar{b}_n \end{aligned} \tag{27}$$

Putting these two together we can write the path integral

$$\begin{aligned} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{-S} &= \prod_n \int d\bar{b}_n da_n e^{-\sum_n \lambda_n a_n \bar{b}_n} = \\ &= \prod_n \int d\bar{b}_n da_n \left(1 - \sum_n \lambda_n a_n \bar{b}_n\right) = \\ &= \prod_n \lambda_n \end{aligned} \tag{28}$$

where we have used the fact that squares of Grassmann variables vanish, and the usual integration rules over Grassmann variables. We learn then that the Path Integral is the product of all the eigenvalue of the dirac operator. We will define such quantity as the determinant of the Dirac operator.

$$\det i\not{D} := \prod_n \lambda_n \tag{29}$$

Rewriting the measure in this way is very useful to see how it behaves under the chiral symmetry.

2.2 Transformation of the measure

Recall that infinitesimally the chiral rotation is

$$\delta\psi = i\epsilon(x)\gamma^5\psi \tag{30}$$

By substituting the expansion of *psi* in the base of the eigenspinors we get

$$\sum_n \delta a_n \phi_n = i\epsilon(x)\sum_m a_m \gamma^5 \phi_m \tag{31}$$

Now we can explicitate δa_n by using the ortogonality relation.

$$\delta a_n = \left(i \int d^4x \epsilon(x) \bar{\phi}_n \gamma^5 \phi_m \right) a_m := X_{n,m} a_m \tag{32}$$

We would like to compute the Jacobian of the transformation $a_n' = a_n + X_{n,m} a_m$. This is a linear transformation, so the Jacobian will be independent on the value of a_n. If a_n was a c-number, then the Jacobian would be

$$J = \det(1 + X) \tag{33}$$

however, we are now considering a linear transformation of grassmann variables. Then the Jacobian is the inverse of the determinant.

$$J = \det{}^{-1}(1 + X) \tag{34}$$

At leading order in ϵ we can expand this and get

$$J = \det{}^{-1}(1 + X) \approx \det(1 - X) \approx \det e^{-X} \approx e^{-\operatorname{tr} X} \tag{35}$$

where the trace means a trace over the spinor indices, and an integration over spacetime. We can finally write the form of the Jacobian in detail

$$J = \exp\left(-i \int d^4x \; \epsilon(x) \sum_n \bar{\phi}_n(x) \gamma^5 \phi_n(x) \right) \tag{36}$$

2.3 Evaluating the Jacobian

We need to evaluate the following jacobian of equation (36). We will do this in steps.

1. First of all, regularize the expression.

$$\begin{aligned}\int d^4x \; \epsilon(x) \sum_n \bar{\phi}_n \gamma^5 \phi_n &= \lim_{\Lambda\to\infty} \int d^4x \; \epsilon(x) \sum_n \bar{\phi}_n \gamma^5 e^{-\lambda_n^2/\Lambda^2} \phi_n = \\ &= \lim_{\Lambda\to\infty} \int d^4 \; x\epsilon(x) \sum_n \bar{\phi}_n \gamma^5 \phi_n e^{-(i\not{D})^2/\Lambda^2} \phi_n\end{aligned} \tag{37}$$

2. Make a change of base. In order to get a feeling of it, recall first a change of base we typically do in QM.

$$\begin{aligned}\sum_n \phi_n^\dagger(x) \mathcal{O} \phi_n(x) &= \sum_n \langle \phi_n | x \rangle \langle x | \mathcal{O} \phi_n \rangle = \sum_n \langle x | \mathcal{O} \phi_n \rangle \langle \phi_n | x \rangle = \\ &= \sum_n \langle x | \mathcal{O} | x \rangle = \int \frac{dk}{2\pi} \langle x | \mathcal{O} | k \rangle \langle k | x \rangle = \\ &= \int \frac{dk}{2\pi} \langle k | x \rangle \langle x | \mathcal{O} | k \rangle = \int \frac{dk}{2\pi} e^{-ikx} \mathcal{O} e^{ikx}\end{aligned} \tag{38}$$

We want to do an analog change of variable.

$$\sum_n \bar{\phi}_n(x)\gamma^5 e^{\not{D}^2/\Lambda^2}\phi_n(x) = \int \frac{d^4k}{(2\pi)^4}\ \mathrm{tr}\left(\gamma^5 e^{-ik\cdot x} e^{\not{D}^2/\Lambda^2} e^{ik\cdot x}\right) \tag{39}$$

where now the trace is only over spinor indices.

3. Regularize this inner integral.

$$\begin{aligned} &\int d^4x\ \epsilon(x)\sum_n \bar{\phi}_n(x)\gamma^5 e^{\not{D}^2/\Lambda^2}\phi_n(x) = \\ &= \lim_{\Lambda\to\infty}\int d^4x\ \epsilon(x)\int^\Lambda \frac{d^4k}{(2\pi)^4}\ \mathrm{tr}\left(\gamma^5 e^{-ik\cdot x} e^{\not{D}^2/\Lambda^2} e^{ik\cdot x}\right) = \\ &= \int d^4x\ \epsilon(x)\lim_{\Lambda\to\infty}\int^\Lambda \frac{d^4k}{(2\pi)^4}\ \mathrm{tr}\left(\gamma^5 e^{-ik\cdot x} e^{\not{D}^2/\Lambda^2} e^{ik\cdot x}\right) \end{aligned} \tag{40}$$

In the next steps, we will focus on the most interior integral now:

$$\lim_{\Lambda\to\infty}\int^\Lambda \frac{d^4k}{(2\pi)^4}\ \mathrm{tr}\left(\gamma^5 e^{-ik\cdot x} e^{\not{D}^2/\Lambda^2} e^{ik\cdot x}\right) \tag{41}$$

this still seems a bit hard to evaluate, so we will make use of some identities.

4. Now, we use to identities for the covariant derivative.

 (a)

$$\begin{aligned} \not{D}^2 = \gamma^\mu\gamma^\nu D_\mu D_\nu &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu D_\nu + \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]D_\mu D_\nu = \\ &= D^2 + \frac{1}{4}\left[\gamma^\mu,\gamma^\nu\right]\left[D_\mu, D_\nu\right] = \\ &= D^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \end{aligned} \tag{42}$$

 (b)

$$e^{-ik\cdot x}D_\mu e^{ik\cdot x} = D_\mu + ik_\mu \tag{43}$$

5. Combining the two identities above, we can rewrite

$$\begin{aligned} e^{-ik\cdot x}e^{\not{D}^2/\Lambda^2}e^{ik\cdot x} &= e^{-ik\cdot x}e^{D^2/\Lambda^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2}e^{ik\cdot x} \\ &= e^{(D_\mu+ik_\mu)^2/\Lambda^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2} = \\ &= e^{(D_\mu+ik_\mu)^2/\Lambda^2}e^{-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2}e^{\cdots}\ldots \end{aligned} \tag{44}$$

where in the last step we used the Becker-Campbell-Hausdorff formula. We are here neglecting the other terms, as they will be irrelevant for the computation, as we will shortly see.

6. Now, the next step is expanding in Taylor series the exponents. The term we are mostly interested is

$$e^{-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2} = 1 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\frac{1}{\Lambda^2} - \frac{e^2}{8}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F_{\rho\sigma}\frac{1}{\Lambda^4} + o\left(\frac{1}{\Lambda^6}\right) \tag{45}$$

The other one expands as

$$e^{(D_\mu+ik_\mu)^2/\Lambda^2} = 1 + \frac{(D_\mu+ik_\mu)^2}{\Lambda^2} + \frac{(D_\mu+ik_\mu)^4}{2\Lambda^4} + o\left(\frac{1}{\Lambda^6}\right) \tag{46}$$

7. Now we will take the product, keeping only terms up to $o\left(\frac{1}{\Lambda^6}\right)$. We get

$$\gamma^5 e^{(D_\mu+ik_\mu)^2/\Lambda^2} e^{-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} = \gamma^5 \left[1 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\frac{1}{\Lambda^2} - \frac{e^2}{8}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F_{\rho\sigma}\frac{1}{\Lambda^4} + \right.$$
$$\left. + \frac{(D_\mu+ik_\mu)^2}{\Lambda^2} - \frac{(D_\mu+ik_\mu)^2}{\Lambda^4}\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \frac{(D_\mu+ik_\mu)^4}{2\Lambda^4} + o\left(\frac{1}{\Lambda^6}\right)\right] \tag{47}$$

8. Now, we need to take the trace over the spinor indices. For this, we will use the following identities among the gamma matrices

   (a) $\operatorname{tr}\gamma^5 = 0$

   (b) $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu = 0$

   (c) $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\epsilon^{\mu\nu\rho\sigma}$

   therefore in the above expression, only the last term in the first line survives. We then get to

$$\begin{aligned}\sum_n \bar\phi_n\gamma^5 e^{\not{D}^2/\Lambda^2}\phi_n &= \lim_{\Lambda\to\infty}\int^\Lambda \frac{d^4k}{(2\pi)^4}\operatorname{tr}\left(\gamma^5 e^{-ik\cdot x}e^{\not{D}^2/\Lambda^2}e^{ik\cdot x}\right) = \\ &= \lim_{\Lambda\to\infty}\int^\Lambda \frac{d^4k}{(2\pi)^4}\left(\frac{e^2}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\frac{1}{\Lambda^4} + o\left(\frac{1}{\Lambda^6}\right)\right) = \\ &= \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\end{aligned} \tag{48}$$

   where the integration measure gave a Λ^4 term, compensing the one in the integrand. Very crucially, in the limit all the other terms that we didn't write come with higher inverse powers of Λ, so they don't contribute. Finally noticed I have cheated a bit, as a factor of π^2 disappears in thin air. I still don't know where it got lost. But the final answer should be the one given above.

2.4 Anomalous Ward Identity

We have leant how the measure of integration changes:

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi \to \int \mathcal{D}\psi\mathcal{D}\bar\psi\ \exp\left(-\frac{ie^2}{16\pi^2}\int d^4x\epsilon(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right). \tag{49}$$

It is now easy to re-do the derivation of the Ward identity that we did above, but now instead of assuming the measure is invariant, we write specifically the way it transforms. After some easy steps essentially identical to those of the previous section one gets the anomalous Ward identity:

$$\partial_\mu j^\mu_A = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{50}$$

Again, this is meant to be an operatorial equation, meaning it holds true when inside a correlation function.

2.5 A puzzling $\eta - \eta'$ story

One big puzzle back in the days was that the η' meson has the same quark composition as the η meson, but its mass is double.

This thing made no sense. Why should the mass be double, if it is a bound state of the same constituents? The solution is coming from the ABJ chiral anomaly. Let us schematically see this.

1. Consider first a limit in which up, down and strange are massless, and charm, bottom and top are massive. There is a $U(3)$ global symmetry rotating the massless quarks. At energies below chiral symmetry breaking, if we wrongly assume that the $U(3)$ was non-anomalous, then we get that this is spontaneously broken. We then expect 9 goldstone massless bosons. Those are the mesons. They are 3 pions, two ηs, and four Kaons.

2. In the real world, u, d and s are not massless. But they are still much less massive than c, b, t. Therefore the chiral symmetry $U(3)$ is broken since the beginning, but broken very mildly. Therefore at energies smaller then chiral symmetry beraking, there are massive pseudo-goldstone bosons produced. Again, they are the real world mesons. And we expect them to be of the same mass. But we see the η' meson to be much more massive.

3. Actually $U(3) \simeq SU(3) \times U(1)_A$ (disregarding a global $\mathbb{Z}_3$ factor). And we know now that $U(1)_A$ is anomalous. Therefore badly broken in the quantum theory, not just mildly. Therefore, in the quantum theory only $SU(3)$ is an approximate symmetry, producing goldstone bosons of approximeately the same mass. They are the 8 mesons, excluding the η'. The η' would be the pseudogoldstone boson associated to $U(1)_A$ and the fact this is anomalous explains its bigger mass.

4. One can go further (and in fact Witten did) and use the anomaly to experimentally predict the mass of the η'. Essentially we saw that the non-conservation of the chiral current receives contribution from instanton configurations. Witten evaluated this contribution (altough in a large-N expansion) and explained the physical origin of the anomalous mass of the η' meson.

2.6 A decadent pion story

The neutral pion π^0 is the lightst of the meson. It decays as

$$\pi^0 \to \gamma + \gamma \tag{51}$$

the vast majority of times (above 99% branching ration). The second preferred channel is the Dalitz decay. However, this experimental fact was a huge puzzle back in the days. Indeed, this decay of the pion into two photons is extremely suppressed if the chiral symmetry is quantum exact.

1. If we assume masseless u and d limit, then the decay of the neutral pion in teo gammas is actually forbidden, as the only decay process mediating them is by coupling the pion to the chiral current, and this vanishes under this (wrong) assumption.

2. In the real world quarks are not massless, and even if the axial symetry was exact, this decay would be possible but super suppressed. Too suppressed to match the exact result.

3. The solution to this puzzle is that the axial anomaly is of course anomalous. So the neutral pion can decay happily into photons, and indeed it does.

3 Local gauge anomalies

They make the theory inconsistent.

There is a easy way to make a theory free of local gauge anomaly. Only use Dirac fermions, and gauge field which couple in the same way to the right and left chirality Weyl fermions composing the Dirac one. Such theories are called *vectorlike*.

However, in general, left and right Weyl fermion can be coupled differently to the gauge field. When this happens, we say the theory is *chiral*. One example of chiral theory is the Standard Model, as left and right chirality fermion couple differently under the weak interaction. Notice that a theory can be chiral only if the fermions are massless. In order to write a mass term one must use both right and left chirality Weyl fermions, and in order for it to be gauge invariant, the left and right fermions must couple to the gauge field in the same way. Said in other word, fermion masses are only possible for vectorlike matter. This is precisely the reason for which the Higgs field had to be added to Standard model.

3.1 Abelian gauge anomaly

Suppose we have N_L left-fermions with electric charges Q_a^L and N_R left-fermions with electric charges Q_a^R. In order for the triangle diagram to vanish, we need to have

$$\sum_{a=1}^{N_L}(Q_a^L)^3 = \sum_{a=1}^{N_R}(Q_a^L)^3 \tag{52}$$

One obvious solution ot this is taking $N_L = N_R$ and $Q_a^L = Q_a^R$. This is the case of vectorlike theories discussed above. We are interested into more non-trivial cases.

In 4d, charge conjugation changes particles with antiparticles, and inverts the chirality. Therefore we can take N left-handed Weyl fermions with charges $Q_a = \{Q_i^L, -Q_i^R\}$ and rewrite the equation (52) as:

$$\sum_{a=1}^{N} Q_a^3 = 0 \tag{53}$$

We would like to find now what are the possibilities for having an anomaly free theory. What is the simplest chiral field theory?

1. $N = 1$. If there is a single charged Weyl fermion, the theory is anomalous.

2. $N = 2$. We have
$$Q_1^2 + Q_2^2 = 0 \tag{54}$$
The only solution to this equation is $Q_1 = \pm 1,\ Q_2 = \mp 1$. This is the vectorlike case discussed above.

3. $N = 3$. We have
$$Q_1^2 + Q_2^2 + Q_3^2 = 0 \tag{55}$$
It is clear that the Q_i cannot be all positive, or all negative. Then suppose Q_1 and Q_2 are positive, and Q_3 is negative. Define $Q_a = (x, y, -z)$ with now $x, y, z > 0$. We need to find the solutions of
$$x^3 + y^3 = z^3 \tag{56}$$
Due to Fermat's last theorem, this equation has no solution. If we would have taken two negative charges and a positive one, we would have found the same result.

4. $N = 4$. We now have two sub-possibilities

   (a) Three charges are positive, one is negative
   $$x^2 + y^2 + z^2 = w^2 \tag{57}$$
   The simplest solution to this equation are the integers $3, 4, 5, 6$. Apart from this one, infinite sets of solutions were found. Ramanujan found the following solutions parametrized as
   $$\begin{aligned} x &= 3n^2 + 5nm - 5m^2, \qquad & y &= 4n^2 - 4nm + 6m^2 \\ z &= 5n^2 - 5nm - 3m^2, \qquad & w &= 6n^2 - 4nm + 4m^2 \end{aligned} \tag{58}$$
   However, the full solution is unknown.

   (b) Two charges are positive, two are negative
   $$x^2 + y^2 = z^2 + w^2 \tag{59}$$
   The most general solution to this equation is completely known.

3.2 Non-abelian gauge anomaly

Define the group theory factor as:

$$d^{abc}(\mathcal{R}) = \mathrm{tr}[T^a\{T^b, T^c\}] \tag{60}$$

For the non-abelian gauge anomaly, we will have contribution from two distint triangle diagrams where A^b_ν and A^c_λ are flipped. The anomaly must then be symetric under the exchange of them. Long story short, the contribution of a Weyl fermion to the anomaly is completely fixed by group theory.

Consider N_L left fermions in a representation $\mathcal{R}_{Li}$, and N_R right ones in a representation $\mathcal{R}_{Ri}$. In order to cancel the triangle diagrams, we need

$$\sum_{i=1}^{N_L} d^{abc}(\mathcal{R}_{Li}) = \sum_{i=1}^{N_R} d^{abc}(\mathcal{R}_{Ri}) \tag{61}$$

For example, let us compute the group theory factor for the fundamental representation of $SU(2)$. We take as generators $T^i = \frac{1}{2}\sigma^i$, with σ^i the Pauli matrices. Then we have

$$d^{abc}([1]) = \mathrm{tr}[T^a\{T^b, T^c\}] = \frac{1}{8}\,\mathrm{tr}[\sigma^a\{\sigma^b, \sigma^c\}] = \frac{1}{4}\,\mathrm{tr}[\sigma^a\delta^{b,c}] = 0 \tag{62}$$

so, the anomaly vanishes. Therefore, up to the discussion so far, we conclude that $SU(2)$ with any number of Weyl fermions in the fundamental representation does not have local gauge anomalies.

One can do a similar computation for any other group and any other representation, provided that an explicit expression of the generators is given. We will however discuss now a simple result, stating that most of these group theory factors vanish.

1. If a representation $\mathcal{R}$ is real or pseudoreal, then $d^{abc}(\mathcal{R}) = 0$. We will now prove this statement.

   (a) First, recall the definitions of real representation and pseudoreal representation: A representation is called *real* if $\bar{T}^a = T^a$. A representation is called *pseudoreal* if there exist a unitary matrix U such that
   $$\bar{T}^a = UT^aU^{-1}. \tag{63}$$

   (b) We have the following identity of the generators
   $$\bar{T}^a = -T^{a\star} = -(T^a)^t \tag{64}$$

   (c) The proof then goes as follows
   $$\begin{aligned}\mathrm{tr}\left[T^a\{T^b, T^c\}\right] &= \mathrm{tr}\left(T^aT^bT^c + T^aT^cT^b\right) = \mathrm{tr}\,\bar{T}^a\{\bar{T}^b, \bar{T}^c\} = \\ &= \mathrm{tr}\left[-(T^a)^t\{(T^b)^t, (T^c)^t\}\right] = -\,\mathrm{tr}\left[T^a\{T^b, T^c\}\right]\end{aligned} \tag{65}$$
   so we see that $\mathrm{tr}\left[T^a\{T^b, T^c\}\right] = 0$

   Now, we learnt that only gauge groups admitting complex representations can contribute non-trivially to the anomaly. So, which groups do admit complex representations?

2. First look at when a non-abelian complex simple Lie algebra admits complex representation. One necessary condition is that it must have a $\mathbb{Z}_2$ outher authomorphism, namely charge conjugation, sending one representation into the conjugated one. The presence of this outher authomorphism can be seen easily from the Dynkin diagram. We conclude it is possible only for $SU(N), SO(2N), E_6$. However, this condition is necessary but not sufficient. It turns out that not all among the $SO(2N)$ have complex representations. Only $SO(4N+2)$ do. So we are left with the list $SU(N), SO(2N), E_6$ as the potential groups which could be anomalous.

3. One can go further, and show that $d^{abc}(\mathcal{R}) = 0$ for any representation of $SO(2N+2)$ and E_6. So we find a very simple result: only $SU(N),\ N > 2$ can suffer local gauge anomalies[2].

We would like now to have an easy and practical way to compute the group theory factor for $SU(N)$. It turns out that

1. All group theory factors have the same tensor structure as the one for the fundamental representation. Infact, we have.

$$d^{abc}(R) = A(R)d^{abc}([1, 0, ..., 0]) \tag{66}$$

so we just need to know $d^{abc}([1, 0, ..., 0])$ and the coefficient $A(R)$. This coeffient is called *anomaly of the representation.*

2. The anomaly of the fundamental representation is one:

$$A([1, 0, ..., 0]) = 1 \tag{67}$$

3. Some simple rules for direct sums and tensor products:

$$\begin{aligned} A(R_1 \oplus R_2) &= A(R_1) + A(R_2) \\ A(R_1 \otimes R_2) &= \dim(R_2)A(R_1) + \dim(R_1)A(R_2) \end{aligned} \tag{68}$$

4. Complex conjugation flips the sign of the anomaly:

$$A(R_1) = -A(\bar{R}_1) \tag{69}$$

3.2.1 The simplest chiral $SU(N)$ theory

We want now to find what is the simplest chiral theory with $SU(N)$ gauge group and some matter. If we only use fundamental and antifundamental matter, then the only way to be anomaly-free is to be chiral. So we need to look at other representations. The simples irreducible representations of $SU(N)$ are the second rank symmetric one, and the second rank antisymmetric one.

By using the sules above, one can easily compute

$$A(symm) = N + 4, \qquad A(asymm) = N - 4 \tag{70}$$

So the easiest possibility with using these representations is taking $SU(5)$ with a Weyl fermion in the antisymmetric and 1 in the antifundamental. This is $SU(5)$ with one 10 and one $\bar{5}$. We quicky recognize this theory as the $SU(5)$ GUT.

[2]Beware tough of the accidental isomorphism $SO(6) \simeq SU(4)$

4 Global gauge anomaly

Also called Witten anomaly.

The statement is that there cannot be an odd number of Weyl fermions in the fundamental representation of $SU(2)$. This extends also to $Sp(n)$ gauge groups.

Note that is a theory is vectorlike, then it is also automatically free of Witten anomaly. In particular, if we do $\mathcal{N}=2$ QFT without half-hypermultiplets, we are always sure the theory is free of Witten anomaly.

ZZZZZ FINISH THIS PART

5 Anomaly cancellation in Standard Model

In table 1 we give the matter content of Standard Model.

Multiplet	$SU(3)$	$SU(2)$	$U(1)$
$l_L = (\nu_e, e)$	1	2	-3
$q_L = (u_L, d_L)$	3	2	$+1$
$l_R = (e_R)$	1	1	-6
$u_R = (u_R)$	3	1	$+4$
$d_R = (d_R)$	3	1	-2

Table 1: Fermionic matter content of the Standard Model

A couple of notes are due:

1. We neglet the vectors and the Higgs doublet, as only fermions contribute to the anomaly.

2. Sometimes people use a different normalization for the hypercharges. This is totally ok, as long as we rescale all of the same number. In life, we are always allowed to rescale the generator of $U(1)$. I chose here to have integer hypercharges.

3. We give the table just for a single family. As we will see, anomalies cancel family-wise.

4. There are essentially two conventions to write down the matter content of Standard Model, and labelling the fields:

   (a) In the first convention, all particles are left-handed Weyl fermions. The label L or R has no meaning related to chirality. It only tells if the multiplet is a $SU(2)$ doublet or a singlet. All anti-particles, of course, will be right-handed Weyl spinors. In this convention, some of the multiplets (those with L label) will couple to $SU(2)$ as doublets, and some of the multiples (those with R-label) will couple to $SU(2)$ as singlets. Of course, antiparticles with L-label (resp R-label) will also couple to $SU(2)$ as doublets (resp. singlets).

   (b) In the second convention, some particles are left-handed Weyl fermions, and some are right-handed. In this convention the label L (resp R) really means that the particle is a left-handed (resp. right handed)

Weyl fermion. Then in this convention, all left-handed fermions couple to $SU(2)$ as doublets, and all the right-handed fermions as singlets. Antiparticles, of course, have flipped chirality. So now all the right-handed particles couple to $SU(2)$ as doublets, and all the left-handed antiparticles couple to $SU(2)$ as singlets.

Of course, the two conventions are completely equivalent. It is just a matter of namings. The isomorphism among the two conventions goes as follows: the sector with L label is identical. For the sector with R label instead, what is called "particle" in the first convention is called "antiparticle" in the second.

I personally find the first convention better, as I like my particles to be all of a definite chirality. It is completely natural and OK for me to have different particles transforming under different representations of the gauge group. I find it a bit more confusing to have some particles described with one kind of spinor, and other described with another one. But still, it is totally consistent and ok. You can take the convention you want. Just be warned that half of the literature is written with one, the other half with the other, so to understand what people mean when they write e_R, for example, you need to be careful about the convention they choose.

Let us now check explicitly that the anomalies cancel.

5.1 Local gauge anomaly cancellation

Let us start with the non-abelian anomaly.

1. For $SU(2)^3$ anomaly all is good: we was $SU(2)$ is always free of local gauge anomalies.
2. The $SU(3)^3$ anomaly could be potentially dangerous, but a quick look at Table (1) tell us it is ok: SM is vectorlike with respect to $SU(3)$. Infact we see that there are two left handed Weyl fermions in the fundamental, and also 2 right-handed in the fundamental. So we are good.

Let us now look at the abelian anomaly $U(1)^3$.

$$\left[2\times(-3)^3+6\times(1)^3\right]-\left[(-6)^3+3\times(4)^3+3\times(-2)^3\right]=0 \tag{71}$$

This completes the discussion for non-mixed anomalies. Let us now discuss mixed anomalies.

1. Turns out that non-abelian factors must all come in pairs, otherwise the contribution vanished.
2. So we have the possibilities of $SU(2)^2\times U(1)$ and $SU(3)^2\times U(1)$.
   (a)
   $$SU(2)^2\times U(1): \qquad -3+3\times(+1)=0 \tag{72}$$
   (b)
   $$SU(2)^3\times U(1): \qquad 2\times(+1)-[4-2]=0 \tag{73}$$

5.2 Witten anomaly cancellation

For this, we simply count how many Weyl fermions transform in the fundamental of $SU(2)$. There are in total four: ν_e, e, u_L, d_L. So Witten anomaly is satisfied.

As a nice final comment, consider what would happen to this counting is we supersymmetrize Standard Model. For every fermion we add a scalar partner, and for every scalar we add a fermion partner. For every gauge field we add a fermion, the gaugino. There is a problem immediately arising. Altough not written in table (1) there exist the Higgs field, which is also in the fundamental of $SU(2)$. Then the Higgsino would be another fermion in the fundamental of $SU(2)$. We therefore get to 5 fermions in total, in the fundamental (or 13 if we count all the families). We see that this naive supersymmetrization of Standard Model is anomalous.

This is precisely one of the two reasons[3] for which in MSSM there are 2 different chiral multiplets in order to describe the Higgs sector. One contains the SM Higgs and the Higgsino, the other contains a second Higgs and a second Higgsino.

[3]Other reason being writing a holomorphic superpotential reproducing the Yukawa couplings