LECTURE NOTES

Chiral Lagrangians

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ABSTRACT: In these lecture notes, we discuss the construction of chiral Lagrangians using a bottom-up effective field theory approach by utilizing the properties of the symmetry breaking pattern. We describe the linear sigma model to demonstrate key features in a toy model, explain in detail the CCZW formalism to build effective Lagrangians in models with spontaneous symmetry breaking, and finally apply it to construct the chiral Lagrangian for QCD.

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1 Introduction

In previous workshop seminars, we have seen how the effective field theory (EFT) techniques can be used to obtain effective Lagrangians from the full theory in a systematic perturbative way. However, EFT methods can also be used in cases where one does not know all the details about the full theory, and this is where EFT really shines. A perfect example for this is using *chiral Lagrangians* to study pion interactions in Quantumchromodynamics (QCD), which is the subject of these lecture notes.

QCD is an *asymptotically free* theory meaning that its coupling constant goes to the zero at large energies (UV). Therefore, at high energies QCD becomes perturbative, and one can use the machinery of perturbative quantum field theory to study scattering amplitudes, decay rates, etc... However at low energies, the coupling constant grows and at some energy scale $E \sim \Lambda_{\text{QCD}} \sim 300\,\text{MeV}$, QCD becomes *non-perturbative*. As a result, we cannot use a top-down approach to construct and effective Lagrangian for QCD at low energies, as we did with the Fermi's theory last week. We need to find a way to build a low-energy effective theory, without necessary knowing how to go from the full theory to the effective theory.

Luckily there is a way to accomplish this in models where spontaneous symmetry breaking occurs, and it only requires the knowledge of the symmetry breaking pattern. During preparing these notes, I heavily inspired by the following excellent resources:

- "Chiral effective Lagrangians" by Heinrich Leutwyler [1].

- "Effective field theories" lecture notes by A. V. Manohar [2].
- "Beyond the Standard Model" lecture notes by Csaba Csaki, and Philip Tanedo (mainly Appendix B for the discussion of CCZW formalism) [3].
- "Quantum Theory of Fields Vol 2" by Steven Weinberg (some parts of Chapter 19).

2 The linear sigma model

Consider a model given by the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \lambda \left(\boldsymbol{\phi} \cdot \boldsymbol{\phi} - v^2\right)^2, \tag{2.1}$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$ is a N-component scalar field. This is the Lagrangian of the *linear sigma model*. It has a global $O(N)$ symmetry where $\boldsymbol{\phi}$ is transformed as an $O(N)$ vector. It is easy to see that the potential for this model is minimized when

$$|\boldsymbol{\phi}| = v. \tag{2.2}$$

The set of field configurations which satisfy this minimization condition is known as the *vacuum manifold* which in this case is given by

$$\text{vacuum manifold} = \left\{ \boldsymbol{\phi} = (\phi_1, \ldots, \phi_N) \ : \ \sum_{i=1}^{N} \phi_i^2 = v^2 \right\}, \tag{2.3}$$

which is nothing but the $(N-1)$-dimensional sphere $\mathcal{S}^{N-1}$ embedded in $\mathbb{R}^N$. We can utilize the $\mathsf{O}(N)$ symmetry of the Lagrangian to fix the *expectation value* of the field $\boldsymbol{\phi}$ by

$$\langle \boldsymbol{\phi} \rangle \equiv (\underbrace{0, \ldots, 0}_{N-1}, v). \tag{2.4}$$

We see that this expectation value is not invariant under the global symmetry $O(N)$ anymore, but it is invariant under a subgroup of it, which is $O(N-1)$. One calls this phenomenon *spontaneous symmetry breaking*, where the vacuum of the Lagrangian has spontaneously broken the $O(N)$ symmetry down to the $O(N-1)$ subgroup.

As we have seen in the first lecture of this Workshop Seminar series[1], a spontaneously broken global symmetry predicts the existence of *massless Goldstone bosons* according to the *Goldstone's theorem* [4]. The number of Goldstone bosons is equal to the number of broken generators. Since the group $O(N)$ has $N(N-1)/2$ generators we have

$$\#\text{ of Goldstone bosons} = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1. \tag{2.5}$$

[1] https://indico.desy.de/indico/event/26025/

These $N-1$ Goldstone bosons correspond to $N-1$ rotations which leave the vacuum state (2.4) invariant. In other words, since these rotations do not change the length of $\boldsymbol{\phi}$, there is no energy cost for doing rotations, thus they correspond to massless degrees of freedom. The remaining degree of freedom is the radial mode, which is the transformation that changes the length of $\boldsymbol{\phi}$, hence it is massive.

To calculate physical observables such as scattering amplitudes from a given Lagrangian, we do one of the following:

1. Include all Feynman diagrams of all orders in perturbation theory.

2. Find some rationale for dropping higher order terms.

The first option is impractical. While the second option is adequate for studying scattering amplitudes in QED or weak interactions since the coupling constant is small, for the Lagrangian given (2.1) there seems to be no reason for justifying such an assumption. However, as we shall see soon, it is possible to recast this Lagrangian into a form that allows us to use Feynman diagrams to generate an expansion for scattering amplitudes in powers of Goldstone boson energies. To accomplish this, we need to make a *local* symmetry transformation such that the fields corresponding to Goldstone degrees of freedom are eliminated from the Lagrangian, but they reappear in the transformed theory as the parameters of the symmetry transformation. This guarantees that only derivatives of the Goldstone degrees of freedom can appear in the new Lagrangian since a constant symmetry transformation should still be a symmetry of the Lagrangian. The fact that the Goldstone fields always arise with their derivatives allows to expand the scattering amplitudes in powers of the Goldstone momenta.

To recast the Lagrangian (2.1) into a useful form which is relevant to our goal, we write $\boldsymbol{\phi}$ as

$$\boldsymbol{\phi}(x) = R(x)\boldsymbol{\sigma}(x) \equiv R(x)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sigma(x) \end{pmatrix}, \tag{2.6}$$

where $R(x) \in O(N)$. Note that the vector $\boldsymbol{\sigma}$ has been chosen such that its Goldstone part does vanish. By substituting the transformed field (2.6) into the linear sigma model Lagrangian (2.1) we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sigma^2\left[\left(\partial_\mu R^T\right)\left(\partial^\mu R\right)\right]_{NN} - \lambda(\sigma^2 - v^2)^2 \tag{2.7}$$

where the subscript NN denotes the NN-th element and $\sigma^2 \equiv \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$. In deriving this result, we have used $R^T R = \mathbf{1}$ which comes from the fact that $R \in O(N)$.

Our original degrees of freedom $\{\phi_i\}$ are now replaced by σ and whatever is needed to parameterize the rotation R. One choice is

$$R = \exp\left\{ i \sum_{s=1}^{N-1} \mathbf{X}^s \cdot \boldsymbol{\pi}^s \right\}. \tag{2.8}$$

where $\{\mathbf{X}^2\}_{s=1}^{N-1}$ are the broken generators. The Lagrangian in this basis is given by

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\sigma^2 \left[\partial_\mu \exp\left\{ -i \sum_{s=1}^{N-1} \mathbf{X}^s \cdot \boldsymbol{\pi}^s \right\} \partial^\mu \exp\left\{ i \sum_{s=1}^{N-1} \mathbf{X}^s \cdot \boldsymbol{\pi}^s \right\} \right]_{NN}, \\ - \lambda(\sigma^2 - v^2)^2. \end{aligned} \tag{2.9}$$

At low energies, we can neglect the radial mode and write the Lagrangian

$$\mathcal{L} = \frac{1}{2} v^2 \left[\partial_\mu \exp\left\{ -i \sum_{s=1}^{N-1} \mathbf{X}^s \cdot \boldsymbol{\pi}^s \right\} \partial^\mu \exp\left\{ i \sum_{s=1}^{N-1} \mathbf{X}^s \cdot \boldsymbol{\pi}^s \right\} \right]_{NN}. \tag{2.10}$$

This is an example for the Lagrangian of a *non-linear sigma model*. It is non-linear in the sense that the fields $\{\boldsymbol{\pi}^s\}$ take values on a non-linear manifold $\mathcal{S}^{N-1}$, in contrast to linear sigma model (2.1) where $\boldsymbol{\phi} \in \mathbb{R}^N$. This Lagrangian demonstrates the key properties of the Goldstone bosons:

1. The Goldstone bosons are only derivatively coupled. This can be understood as follows: A constant Goldstone field is a field $\boldsymbol{\phi}$ which is rotated by the same angle everywhere on the spacetime. Therefore by an $\mathcal{O}(N-1)$ rotation, it can be rotated to the standard vacuum $\langle \boldsymbol{\phi} \rangle$. As a result, the Lagrangian can only depend on the derivatives of the Goldstone bosons.

2. The couplings of Goldstone bosons are proportional to their momenta.

3. There is no *global chart* which cover all of the vacuum manifold $\mathcal{S}^{N-1}$, therefore the Goldstone fields $\{\boldsymbol{\pi}^s\}$ are only defined around the vicinity of the vacuum. Such a description is adequate for perturbation theory, however it needs to be modified to study global features or non-perturbative effects.

3 The CCWZ Formalism

In this section, we will describe a general formalism developed by Callan, Coleman, Wess, and Zumino [5, 6] to construct effective Lagrangians for spontaneously broken global symmetries, which we shall apply to QCD in the next section.

Consider a theory with a Lagrangian which is symmetric under the group G, and this symmetry is spontaneously broken to a subgroup $H \in G$ by some order parameter v. This order parameter can be the VEV of some fields $\{\phi_i\}$ which we

shall assume in this section, although this is not always the case. We also assume that $\{\phi_i\}$ transform as a linear representation of G.

Let v denote the vacuum state. Since the symmetry is broken to the subgroup H, an element of H leaves the vacuum invariant:

$$hv = v, \quad \forall h \in H. \tag{3.1}$$

Consider a field configuration ϕ which can be obtained by acting on the vacuum with the group element $g \in G$.

$$g : v \mapsto \phi. \tag{3.2}$$

From (3.1), we can see that the same configuration can be obtained by acting on the vacuum with gh, where $h \in H$. This allows us to define an equivalence relation between two elements of G:

$$g_1 \sim g_2, \quad \text{if} \quad \exists h \in H \ : \ g_1 = g_2 h. \tag{3.3}$$

If G is a group and $H \subseteq G$ is a subgroup of G, then for each element $g \in G$ we can define the set

$$gH = \{gh : h \in H\}, \tag{3.4}$$

which is called the *left coset*. They are the equivalence classes corresponding to the equivalence relation (3.3). The set of all left cosets form the *coset space* G/H which is the vacuum manifold of the theory. The Goldstone bosons will take values on this manifold.

Let $\{T^i\}_{i=1}^{\dim H}$, and $\{X^a\}_{a=1}^{\dim G/H}$ denote the set of unbroken and broken generators respectively[2] so that $\{T^i\} \cup \{X^a\}$ forms all the generators of G. It is possible to choose the generators such that

$$\left[T^i, T^j\right] = i f^{ijk} T^k, \tag{3.5}$$

$$\left[T^i, X^a\right] = i f^{iab} X^b, \tag{3.6}$$

$$\left[X^a, X^b\right] = i f^{abk} T^k + i f^{abc} X^c. \tag{3.7}$$

This is called the *Cartan decomposition*. The first commutation relation is comes from the fact that H is a Lie group and $\{T^i\}$ span the Lie algebra $\mathfrak{h}$ of the Lie group H. To derive the commutator $[T^i, X^a]$, we use the *Cartan inner product* for two generators

$$\langle A|B\rangle \equiv \mathrm{Tr}(AB), \tag{3.8}$$

and the fact that the algebras of H and G/H are orthogonal $\langle T|X\rangle = 0$. Then the cyclic property of the trace implies

$$\left\langle T^i \middle| \left[T^j, X^a\right] \right\rangle = i f^{ijk} \left\langle T^k \middle| X^a \right\rangle = 0, \tag{3.9}$$

[2]Note that $\dim G/H = \dim G - \dim H$

which tells us that $[T^j, X^a]$ does not contain any of the T's. In general, the commutator $\left[X^a, X^b\right]$ will contain both T's and X's, but if there exists a parity transformation P having the properties

- $P^2 = 1$,
- $P\left([g_1, g_2]\right) = [P(g_1), P(g_2)]$,
- $P(X) = -X$ and $P(T) = +T$,

then

$$\left[X^a, X^b\right] = if^{abk}T^k, \tag{3.10}$$

and G/H is called a *symmetric space*.

For any element in G/H, we can pick a representative element of G which we denote by $\hat{g}$, so that any group element can be written as $g \in \hat{g}h$ for some $h \in H$. If the Lie group is connected and compact, then for each $g \in G$, we can use the *exponential map* exp to write

$$g = \exp(i\xi^a X^a)\exp\left(iu^j T^j\right) \equiv \hat{g}(\xi)h(u), \tag{3.11}$$

where $\{\xi^a\}$ and $\{u^j\}$ are real parameters.

Let $\phi(x)$ be a field configuration which transform under a non-trivial representation of the group G. This field contains all the degrees of freedom in the full theory, including the Goldstone degrees of freedom that we want to eliminate. To this end, our goal is to write this field in the form

$$\phi(x) = \gamma(x)\tilde{\phi}(x), \tag{3.12}$$

such that $\gamma(x) \in G$ and $\tilde{\phi}(x)$ does not contain and Goldstone degrees of freedom. If we ignore the massive (radial) excitations of $\phi(x)$, then we can choose $\tilde{\phi}(x)$ to the vacuum configuration v. Then, $\gamma(x)$ becomes a transformation which turns the vacuum configuration into a field configuration:

$$\gamma(x) : v \mapsto \phi. \tag{3.13}$$

But we already know that such a transformation is defined only up to a right multiplication by $h \in H$, so we can choose it to be a function of broken generators only:

$$\gamma(x) \equiv \gamma[\pi] = \exp(i\pi^a(x)X^a). \tag{3.14}$$

This is called the *CCZW construction*. The parameters $\{\pi^a(x)\}$ will be identified with Goldstone bosons up to a multiplicative constant which will restore its mass dimension.

At this point, it is instructive to see how the Goldstone fields $\{\pi^a\}$ transform under the global group G. Using (3.11), (3.12), and (3.14), we get

$$g\phi(x) = g\gamma(x)\tilde{\phi}(x) = g\exp(i\pi^a(x)X^a)\tilde{\phi}(x) \equiv \exp\{i\pi'^a X^a\}\exp\{iu'^i T^i\}\tilde{\phi}(x), \quad (3.15)$$

where both

$$\pi' = \pi'(g,\pi) \quad \text{and} \quad u' = u'(g,\pi) \quad (3.16)$$

are *non-linear* functions. We can express this also by

$$g\gamma[\pi]\tilde{\phi}(x) = \gamma[\pi']h[u']\tilde{\phi}(x). \quad (3.17)$$

This form is particularly useful since it tells us directly how the Goldstone-less field $\tilde{\phi}(x)$ transform under the global group G:

$$\tilde{\phi}(x) \to \tilde{\phi}'(x) = h[u'(g,\pi)]\tilde{\phi}(x), \quad (3.18)$$

with an accompanying implicitly defined transformation of the Goldstones $\pi(x) \to \pi'(x)$. This expression tells us something remarkable. Under the global group G, the field $\tilde{\phi}(x)$ transforms as a representation of H, but a *non-linear* one. In other words, the decomposition (3.12) converts the G-linear representations ϕ, into non-linear realizations of G, π, and $\tilde{\phi}$. (3.18) also tells us that to construct G-invariants out of radial fields $\tilde{\phi}$, it is sufficient to construct H-invariants.

Let us compare these results with the transformations under the unbroken group H. For any $h \in H$ we can write

$$h\tilde{\phi}(x) = h\gamma[\pi]\tilde{\phi}(x) = \left(h\gamma[\pi]h^{-1}\right)h\tilde{\phi}(x). \quad (3.19)$$

Therefore under the group H, the radial and Goldstone field transform respectively as

$$\tilde{\phi}(x) \to h\tilde{\phi(x)} \quad \text{and} \quad \gamma[\pi] \to h\gamma[\pi]h^{-1}. \quad (3.20)$$

In contrast with the transformations under the group G, these are *linear* transformations. To see that the Goldstones transform linearly, we use the Baker-Campbell-Hausdorff formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots, \quad (3.21)$$

and the fact that $[T,X] \sim T$.

We still need to answer the question of how to write effective Lagrangians for the Goldstone fields. Let us assume that the Lagrangian of the full theory is a function of the fields $\phi(x)$ and their derivatives $\partial\phi(x)$. Since the Goldstones can only be derivatively coupled, it is sufficient to concentrate on $\partial\phi(x)$. By using (3.12), it can be written as

$$\partial_\mu\phi(x) = \gamma\left[\partial_\mu + \gamma^{-1}(\partial_\mu\gamma)\right]\tilde{\phi}(x). \quad (3.22)$$

The object $\gamma^{-1}\partial_\mu\gamma$ is called the *Mauer-Cartan form*. Without loss of generality, we can write it as a linear combination of broken and unbroken operators:

$$\gamma^{-1}\partial_\mu\gamma = iD^a_\mu X^a + iE^i_\mu T^i, \tag{3.23}$$

where

$$D^a_\mu = D^{ab}(\pi)\partial_\mu\pi^b \quad \text{and} \quad E^i_\mu = E^{ib}(\pi)\partial_\mu\pi^b. \tag{3.24}$$

Now we differentiate both sides of (3.17) with ∂_μ. Then we find

$$g(\partial_\mu\gamma[\pi])\tilde{\phi} = (\partial_\mu\gamma[\pi'])h\tilde{\phi} + \gamma[\pi'](\partial_\mu h)\tilde{\phi}. \tag{3.25}$$

Next we multiply both sides with $\gamma^{-1}[\pi]g^{-1} = h^{-1}\gamma^{-1}[\pi']$ from the left and obtain

$$\gamma^{-1}[\pi]\partial_\mu\gamma[\pi] = h^{-1}\left(\gamma^{-1}[\pi']\partial_\mu(\gamma[\pi']) + \partial_\mu\right)h. \tag{3.26}$$

Finally, we replace the Mauer-Carton form $\gamma^{-1}\partial_\mu\gamma$ with (3.24):

$$iD^a_\mu[\pi]X^a + iE^i_\mu T^i = ih^{-1}\left(D^a_\mu[\pi']X^a\right)h + ih^{-1}\left(E^i_\mu[\pi']T^i - i\partial_\mu\right)h. \tag{3.27}$$

This result shows that the objects D and E transform under the global group G by

$$D^a_\mu[\pi] \to D^a_\mu[\pi'] = hD^a_\mu[\pi]h^{-1}, \tag{3.28}$$

$$E^i_\mu[\pi] \to E^i_\mu[\pi'] = h\left(E^i_\mu[\pi] - i\partial_\mu\right)h^{-1}. \tag{3.29}$$

We see that both objects transform under G through the representations of H, just like the transformations of $\tilde{\phi}(x)$. The only difference is that while D transforms linearly, E transforms like a *gauge field*. This strengthens our claim that it is sufficient to consider only H-invariant terms when constructing effective Lagrangians. As an example, one can use D to construct the simplest lowest-order effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = \frac{f^2}{4}\,\text{Tr}\left(D^\mu D_\mu\right), \tag{3.30}$$

where f is a parameter of mass dimension 1, which is introduced such that the Lagrangian has correct mass dimension. It is identified with the symmetry breaking scale.

4 The QCD Chiral Lagrangian

In this section, we shall apply the CCZW formalism of the previous section to obtain the *chiral* Lagrangian for the low energy QCD. To begin with, we consider a toy model of QCD with N flavors of *massless* quarks. The Lagrangian is

$$\mathcal{L}_{\text{QCD},N} = -\frac{1}{g_s^2}G^a_{\mu\nu}G^{\mu\nu a} + i\bar{q}\gamma_\mu\mathcal{D}^\mu q, \tag{4.1}$$

where $\{G^a\}$ are the gluon fields, g_s is the strong coupling constant, and q represents all the quark fields. This Lagrangian is invariant under a global symmetry

$$G = SU(N)_L \times SU(N)_R, \tag{4.2}$$

under which left- and right-handed quark fields transform independently:

$$SU(N)_L : q_L(x) \mapsto Lq_L(x) \quad , \quad SU(N)_R : q_R(x) \mapsto Rq_R(x). \tag{4.3}$$

Although we don't know why, at the vacuum of the QCD, the quark bilinearies pick up a non-zero expectation value:

$$\langle \bar{q}q \rangle \neq 0. \tag{4.4}$$

This spontaneously breaks the global symmetry G into the subgroup

$$H = SU(N)_V, \tag{4.5}$$

which denote the set of chiral rotations with $L = R$. The vacuum manifold is the coset space $SU(N)_L \times SU(N)_R / SU(N)_V$. The Lie algebra of $SU(N)$ has dimension $N^2 - 1$, thus it has $N^2 - 1$ generators. Therefore the number of broken generators is

$$2 \times (N^2 - 1) - (N^2 - 1) = N^2 - 1. \tag{4.6}$$

Thus we expect $N^2 - 1$ massless Goldstone bosons in the low energy theory.

So far so good, but in the real QCD quarks are not massless, therefore it is not *exactly* invariant under the global symmetry (4.2). Nevertheless, if the mass terms are small, it is still possible to use the whole machinery of spontaneous symmetry breaking to get an effective Lagrangian at low energies. In this case we say that the Lagrangian is *approximately* invariant under the global symmetry G. The only difference is that due to *explicit* breaking of the symmetry by the mass terms, the Goldstone bosons in the low energy effective theory would not be exactly massless, but they will gain a mass which is proportional to the explicit breaking. If this breaking is small, then the Goldstone bosons will also be light compared with the other degrees of freedom in the theory, such as the radial mode.

To determine which of the quark masses are small enough to be ignored, we need an energy scale to compare them. For QCD, this energy scale can be chosen as the $\Lambda_{\text{QCD}} \sim \mathcal{O}(300\,\text{MeV})$, which is roughly the energy scale at which QCD becomes non-perturbative. If we compare the quark masses with this scale we find

$$\frac{m_{u,d}}{\Lambda_{\text{QCD}}} \sim \frac{1}{50} \quad , \quad \frac{m_s}{\Lambda_{\text{QCD}}} \sim \frac{1}{3} \quad , \quad \frac{m_{c,b,t}}{\Lambda_{\text{QCD}}} > 1. \tag{4.7}$$

This comparison tells us that neglecting up and down quark masses is quite a reasonable approximation. Neglecting the strange quark mass may also be fair, if we are not looking for precise results.

Let us see that if our estimate (4.7) agrees with the observed particle spectrum of low energy QCD. If we only neglect the up and down quark masses then the global symmetry group will be $G = SU(2)_L \times SU(2)_R$ and from (4.6) the number of light degrees of freedom will be $2^2 - 1 = 3$. Indeed the lightest particles in the meson spectrum are the pions and they are 3 of them with masses

$$m_{\pi^0} \approx 135\,\text{eV} \quad , \quad m_{\pi^\pm} \approx 140\,\text{MeV}. \tag{4.8}$$

If we include the strange quark, the symmetry group is $G = SU(3)_L \times SU(3)_R$, and we expect $3^2 - 1 = 8$ light degrees of freedom. These are three pions, four kaons $K^\pm, K^0, \bar{K}^0$, and the eta-meson η. The mass of the kaons are $\approx 500\,\text{MeV}$ while mass of the eta-meson is $\approx 550\,\text{MeV}$. The next meson is η' with mass $\approx 950\,\text{MeV}$, which is quite heavy compared to the lighter ones.

In the remaining part of this section, we will construct the low energy effective Lagrangian of QCD, i.e. the *chiral lagrangian*, in a bottom-up approach by using the CCWZ formalism of the previous section, and then study some of its properties.

4.1 Chiral Lagrangian from breaking of $SU(3)_L \times SU(3)_R$

Let us forget about the QCD Lagrangian given in (4.1). Only information we have is that there is global symmetry $G = SU(3)_L \times SU(3)_R$ which is spontaneously broken to $H = SU(3)_V$ by the vacuum.

We start by choosing the set of unbroken generators. Since the unbroken group H has the Lie algebra $\mathfrak{su}(3)$, its generators are the Gell-mann matrices $\{T^a\}$. The set of broken generators are the same since the group G is a product of two $SU(3)$s. To be concrete, we choose $\{T^a_L\}$ to be the broken commutators. Then using the CCZW construction (3.14), we write

$$\gamma[\pi] = \exp\left(i\begin{bmatrix}\pi^a T^a_L & 0\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}\Sigma(x) & 0\\ 0 & 1\end{bmatrix}, \tag{4.9}$$

where

$$\Sigma(x) = \exp(i\pi^a T^a_L). \tag{4.10}$$

We will restore the mass dimensions of Goldstones later. From the relation (3.17), we can see that $\gamma[\pi]$ transforms under the group G by

$$g : \gamma[\pi] \mapsto \gamma[\pi'] = g\gamma[\pi]h^{-1}. \tag{4.11}$$

Recall that for the transformations under the unbroken group H, we need $L = R$, so we can express an element of H by

$$h = \begin{pmatrix}U & 0\\ 0 & U\end{pmatrix}. \tag{4.12}$$

Then the transformation (4.11) can be written by

$$\begin{pmatrix} \Sigma(x) & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Sigma(x) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix}. \tag{4.13}$$

This implies $U = R$, thus

$$g : \Sigma(x) \mapsto L\Sigma(x)R^\dagger. \tag{4.14}$$

Now we can construct the most general effective Lagrangian consistent with spontaneously broken $SU(3) \times SU(3)$ symmetry. Since all the Goldstone fields are contained inside the object $\Sigma(x)$, the goal is to construct the most general possible Lagrangian invariant under the symmetry transformation given in (4.14). Most general term involving no derivatives is

$$\text{Tr } \Sigma\Sigma^\dagger \ldots \Sigma\Sigma^\dagger, \tag{4.15}$$

where Σ and $\Sigma^\dagger$ alternate. But, $\Sigma\Sigma^\dagger = \mathbb{1}$, so all these terms are constant. The lowest order term is with two derivatives and it is given by

$$\mathcal{L}_2 = \frac{f^2}{4} \text{Tr } \partial_\mu \Sigma \partial^\mu \Sigma^\dagger. \tag{4.16}$$

The full chiral Lagrangian can be obtained by adding higher derivative terms:

$$\mathcal{L} = \frac{f^2}{4} \text{Tr } \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \mathcal{L}_4 + \mathcal{L}_6 + \ldots, \tag{4.17}$$

where $\mathcal{L}_4$ contains terms with four derivatives, $\mathcal{L}_6$ contains six and so on.

We still didn't write down the physical Goldstone fields (pions, kaons, ...) explicitly. They should be defined such that the Goldstone fields have the correct mass dimension and they have the correct canonical normalization for the kinetic term inside the chiral Lagrangian. The standard choice is to redefine the Goldstone fields $\{\pi^a\}$ such that

$$\Sigma(x) = \exp\left(\frac{2i}{f}\pi^a T^a\right). \tag{4.18}$$

The fields $\{\pi^a\}$ in terms of meson fields is given by

$$\pi^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{4.19}$$

Then by expanding (4.16) in terms pion fields, we get

$$\mathcal{L}_2 = \text{Tr } \partial_\mu \pi \partial^\mu + \frac{1}{3f^2} \text{Tr } [\pi, \partial_\mu \pi]^2 + \ldots. \tag{4.20}$$

With this Lagrangian we can calculate all the pion scattering amplitudes up to order p^2.

4.2 The Chiral Currents

We can use the Noether's theorem to compute the currents for $SU(3)_L$ and $SU(3)_R$. Recall that the definition of a symmetry current is

$$\delta\mathcal{L} = \partial_\mu \epsilon(x) j^\mu(x), \tag{4.21}$$

where $\epsilon(x)$ is the function parametrizing the infinitesimal symmetry transformation at $\delta\mathcal{L}$ is the change in the Lagrangian by this transformation. The infinitesimal form of the $SU(3)_L$ transformation $\Sigma \to L\Sigma$ is

$$\Sigma \to \Sigma + i\epsilon_L^a T^a \Sigma. \tag{4.22}$$

From this transformation, we can calculate the $SU(3)_L$ current as

$$j_L^{\mu a} = \frac{i}{2} f^2 \,\mathrm{Tr}\; T^a \Sigma \partial^\mu \Sigma^\dagger. \tag{4.23}$$

Similarly, we can obtain the right-handed current as

$$j_R^{\mu a} = \frac{i}{2} f^2 \,\mathrm{Tr}\; T^a \Sigma^\dagger \partial^\mu \Sigma. \tag{4.24}$$

Using these two expressions, the axial current becomes

$$j_A^{\mu a} = j_R^{\mu a} - j_L^{\mu a} = -f\partial^\mu \pi^a. \tag{4.25}$$

This expressions is important, since the matrix element $\langle j_A^{\mu a}|0|j_A^{\mu a}\rangle\, \pi^b = ifp^\mu \delta^{ab}$ is defined so that f is the *pion decay constant*. By measuring the decay rate $\pi \to \mu\bar{\nu}$ we can learn that

$$f \approx 93\,\mathrm{MeV}. \tag{4.26}$$

4.3 Weinberg's Power Counting Argument

We have seen in (4.17) that the chiral Lagrangian is an expansion in derivatives, which implies that it is an expansion in p/Λ. There are two questions we can ask immediately:

1. What terms in the effective Lagrangian are required to compute to a given order in p/Λ?

2. What is the value of Λ?

Let us write the chiral Lagrangian as

$$\mathcal{L} = \sum_{k\geq 2} \mathcal{L}_k, \tag{4.27}$$

where $\mathcal{L}_k$ contains terms with k derivatives. The general form of an arbitrary loop graph is

$$\mathcal{A} \sim \int \left(\mathrm{d}^4 p\right)^L \frac{1}{(p^2)^I} \prod_k \left(p^k\right)^{m_k}, \tag{4.28}$$

where L is the number of loops, I is the number of internal lines, and m_k is the number of interaction vertices. Notice that each $\mathcal{L}_k$ brings m_k vertices. In a mass-independent subtraction scheme, the only dimensionful quantity is the momentum p, so we need $\mathcal{A} \sim p^D$. This implies

$$D = 4L - 2I + \sum_k k m_k. \tag{4.29}$$

Furthermore, for any Feynman graph the following identity holds:

$$V - I + L = 1. \tag{4.30}$$

By using these relations together with $V = \sum_k m_k$, we can solve for D and find

$$D = 2 + 2L + \sum_k (k-2) m_k. \tag{4.31}$$

The lowest order term in the Lagrangian is $k = 2$ so all terms in the above expression are positive.

Let us say that we want to calculate a scattering amplitude up to p^4 order. There are solutions to the equation above for $D = 4$. These are

$$L = 0 \Rightarrow m_4 = 1, m_{k>4} = 0 \quad \text{and} \quad L = 1 \to m_{k>2}. \tag{4.32}$$

Therefore, in order to calculate a scattering amplitude up to p^4 order, we need all the tree-level graphs with one insertion of $\mathcal{L}^4$, and all one-loop graphs with the lowest order Lagrangian $\mathcal{L}_2$. Hereby we answered the first question.

To answer the second question, consider the $\pi - \pi$ scattering up to order p^4. As we have seen above, there are two contributions to this. One comes from the loop graph using $\mathcal{L}_2$, the other one is from the tree level graph with one insertion of $\mathcal{L}_4$. Let us try to estimate the loop graph:

$$I \sim \underbrace{\int \frac{\mathrm{d}^4 k}{(2\pi)^4}}_{\text{loop}} \underbrace{\left(\frac{p^2}{f^2}\right)^2}_{\text{vertices}} \underbrace{\left(\frac{1}{k^2}\right)^2}_{\text{internal lines}} \sim \text{log divergent} \tag{4.33}$$

$$\sim \frac{p^4}{(4\pi)^2} \frac{1}{f^4} \log(\mu), \tag{4.34}$$

where μ is the $\overline{MS}$ renormalization scale. Now consider the contribution from the tree graph, which is produced by a four-derivative term in the Lagrangian of the form

$$a \operatorname{Tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\mu \Sigma. \tag{4.35}$$

This pion interaction will be on the order of ap^4/f^4 which can be seen by expanding the Σ field in a power series in π/f. The total pion scattering amplitude should be invariant under the renormalization scale μ. Thus any change in μ should be compensated by a change in a. Since a change in μ of order one produces a shift in a of order $\delta a \sim (4\pi)^2$, generically we expect

$$|a| \gtrsim |\delta a| \sim \frac{1}{(4\pi)^2}. \tag{4.36}$$

If we write the chiral Lagrangian as

$$\mathcal{L} = \frac{f^2}{4}\left[\mathrm{Tr}\ \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \frac{1}{\Lambda^2}\mathcal{L}_4 + \frac{1}{\Lambda^4}\mathcal{L}_6 + \ldots\right]. \tag{4.37}$$

Then the estimate (4.36) implies that

$$\Lambda \lesssim 4\pi f. \tag{4.38}$$

It has been shown by many calculations that the bound is actually saturated. Therefore we can assume that the cutoff of our effective field theory prescription is around

$$\Lambda \sim 4\pi f \sim 1\,\mathrm{GeV}. \tag{4.39}$$

5 Final remarks

In these lecture notes, I tried to give an introduction to the theory of chiral Lagrangians by focusing on their use in the study of low-energy QCD. Of course this introduction barely scratches the surface of this very deep topic. I couldn't fit many interesting topics due to the time constraint as well as my lack of preparation. An incomplete list of these topics are

- Use of the spurion analysis to get meson masses in terms of the quark masses.
- Detailed discussion of the $\pi - \pi$ scattering.
- Interaction of Goldstone bosons with other matter fields such as baryons or heavy mesons.

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