SMEFT

Philipp Englert

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1 Introduction

Many BSM theories assume New Physics particles to be heavier than some New Physics scale Λ that is significantly larger than the typical energy scales of currently achievable experiments. By virtue of the decoupling theorem by Appelquist and Carazonne [1], these heavy degrees of freedom of the corresponding UV-BSM-theory decouple from its low energy dynamics and hence, can be integrated out. The resulting theory is an effective field theory that is a correction to the renormalizable Standard Model and is called the Standard Model Effective Field Theory (SMEFT). The SMEFT can be represented by the following Lagrangian:

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda}\sum_k C_k^{(5)}\mathcal{Q}_k^{(5)} + \frac{1}{\Lambda^2}\sum_k C_k^{(6)}\mathcal{Q}_k^{(6)} + \mathcal{O}(\frac{1}{\Lambda^3})\,, \tag{1}$$

where $\mathcal{L}_{\text{SM}}$ is the Standard Model Lagrangian, $\mathcal{Q}_k^{(n)}$ are operators with mass dimension n and $C_k^{(n)}$ are the corresponding dimensionless Wilson coefficients. The SMEFT Lagrangian exhibits the same $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry as the Standard Model and it has the same field content.
There are many advantages to this way of describing heavy particle BSM physics but arguably the most important one is the following. First of all, it is a model independent way to describe heavy New Physics, which entails that we can derive bounds on the SMEFT parameters from experiments and subsequently recast them into bounds on the parameters of specific BSM models instead of having to derive them for each of these models from scratch. But even better so, thanks to the SMEFT formalism, we don't even have to come up with a specific BSM model that describes possible New Physics to be able to parameterize deviations of experimental results from the Standard Model.

In the following, will introduce the concept of field redefinitions and their connection to the application of the equation of motion to the Lagrangian, which will be useful for the subsequent section, where we will sketch how to construct a basis for SMEFT and use the equations of motion to eliminate redundancies from this basis. Lastly, we will talk about the renormalization of SMEFT and consequently explore the concepts of operator mixing and the role played by operators vanishing by the equations.

2 Field redefinitions

In this section, we will talk about the concept of field redefinitions and the application of the equations of motion to the Lagrangian, which can be used to remove redundancies from a set of operators. This is necessary if we want to construct a non-redundant basis for the SMEFT.

From the path integral point of view, it is clear that that observables are invariant under redefinitions of the fields[1]. It is possible to show that at the Lagrangian level, redefining a field by applying a small shift always amounts to the addition of an operator that vanishes under the application of the equation of motion of the redefined field. In other words, we have the freedom to apply the equations of motion to the Lagrangian however we want without changing observables.

While this statement is almost trivial if we restrict ourselves to the calculation of tree level diagrams, where all the operators are evaluated between on-shell states and an operator that vanishes by the equations of motion simply amounts to zero, this statement becomes non-trivial if we want to generalize it to general S-matrix elements. We will sketch the proof of this statement in the following.

For simplicity, we will just sketch the proof for the special case of a scalar field ϕ. However, this is just for notational purposes, since the proof is no more complicated for other types of fields. Consider a Lagrangian that can be written as a power series in a small parameter η:

$$\mathcal{L} = \sum_{n=0}^{\infty} \eta^n \mathcal{L}_n \,. \tag{2}$$

In the context of an effective field theory, we can take the small parameter to be e.g. $1/\Lambda^2$. The generating functional for the Green's functions is given as:

$$Z[j_i] = \int \prod_i \mathcal{D}\varphi_i \exp\left(i \int d^4x \left[\mathcal{L}_0 + \eta \mathcal{L}_1 + \sum_i j_i \varphi_i + \mathcal{O}(\eta^2) \right] \right) , \tag{3}$$

where j_i are the sources for each of the fields φ_i. Now let us perform a field redefinition $\phi^\dagger = (\phi')^\dagger + \eta T[\varphi]$ of the scalar field ϕ and express the generating functional in terms of the new field ϕ'. Here, $T[\varphi]$ is any local function of any of the fields φ and their derivatives. We can now express the generating functional as:

$$\begin{aligned} Z[j_i] = &\int \prod_i \mathcal{D}\varphi_i' \left| \frac{\delta \phi^\dagger}{\delta (\phi')^\dagger} \right| \\ &\times \exp\left(i \int d^4x \left[\mathcal{L}_0' + \delta\mathcal{L}_0' + \eta \mathcal{L}_1' + \eta \delta \mathcal{L}_1' + \sum_i j_i \varphi_i + j_{\phi^\dagger} \eta T + \mathcal{O}(\eta^2) \right] \right) , \end{aligned} \tag{4}$$

[1]Essentially, those redefinitions just have to leave the one-particle-states invariant.

where

$$\mathcal{L}_i' \equiv \mathcal{L}_i\left((\phi')^\dagger, \partial_\mu(\phi')^\dagger\right) \tag{5}$$

$$\delta\mathcal{L}_i' \equiv \frac{\delta\mathcal{L}_i'}{\delta(\phi')^\dagger}\delta\phi^\dagger - \frac{\delta\mathcal{L}_i'}{\delta\partial_\mu(\phi')^\dagger}\delta\partial_\mu\phi^\dagger\,, \tag{6}$$

with $\delta\phi^\dagger \equiv \phi^\dagger - (\phi')^\dagger = \eta T[\varphi]$ and $\delta\partial_\mu\phi^\dagger \equiv \partial_\mu\phi^\dagger - \partial_\mu(\phi')^\dagger$. What we did here, is Taylor-expanding $\mathcal{L}_i'$ about $(\phi')^\dagger$ but equation (6) should look familiar, since this is essentially the variation of the Lagrangian we know and love from our basic quantum field theory courses. We know that this is exactly the expression we set to zero when applying the principle of least action and after partial integration of this expression and applying the Gaussian theorem, we get

$$\begin{aligned}\delta\mathcal{L}_i' &= \left(\frac{\delta\mathcal{L}_i'}{\delta(\phi')^\dagger} - \partial_\mu\frac{\delta\mathcal{L}_i'}{\delta\partial_\mu(\phi')^\dagger}\right)\delta\phi^\dagger & (7)\\ &= \left(\frac{\delta\mathcal{L}_i'}{\delta(\phi')^\dagger} - \partial_\mu\frac{\delta\mathcal{L}_i'}{\delta\partial_\mu(\phi')^\dagger}\right)\eta T[\varphi]\,. & (8)\end{aligned}$$

The generating functional now reads:

$$\begin{aligned}Z[j_i] = \int\prod_i\mathcal{D}\varphi_i'\left|\frac{\delta\phi^\dagger}{\delta(\phi')^\dagger}\right|\exp\Bigg(i\int d^4x\Bigg[\mathcal{L}_0' + \left(\frac{\delta\mathcal{L}_0'}{\delta(\phi')^\dagger} - \partial_\mu\frac{\delta\mathcal{L}_0'}{\delta\partial_\mu(\phi')^\dagger}\right)\eta T[\varphi]\\ + \eta\mathcal{L}_1' + \sum_i j_i\varphi_i + j_{\phi^\dagger}\eta T + \mathcal{O}(\eta^2)\Bigg]\Bigg)\,.\end{aligned} \tag{9}$$

So we see that a field redefinition of the form $\phi^\dagger = (\phi')^\dagger + \eta T[\varphi]$ amounts to the addition of an "equation of motion"-term derived from the first order Lagrangian $\mathcal{L}_0$ to the total Lagrangian. The "equation of motion"-term derived from the second order Lagrangian $\mathcal{L}_1$ is already $\mathcal{O}(\eta^2)$ so we don't write it explicitly here. Unfortunately, apart from the change of the Lagrangian, there are also a Jacobian and an additional source term that were generated by the field redefinition we performed. It is possible to show that without loss of generality, we can also neglect those order by order in the expansion in η, so the statement that we can apply the equations of motion to the Lagrangian without changing the observables is actually true quite generally. I will not carry out the proof of the fact that the source term and the Jacobian can be neglected, in these notes, but for details, I want to refer you to the original paper of the theorem, which this treatment is based on [2]. In the following section, we will sketch how to systematically construct a basis of SMEFT operators at mass dimension 6 without any redundancies and in the course of this, we will see examples of how to use the equations of motion to reduce redundancies from a set of operators.

3 Construction of a basis for SMEFT

This section is mainly based on [3] and [4]. For further details and a more complete treatment, I would like to refer you to them.

We make the (sensible) assumptions that the operators of the SMEFT shall be gauge invariant, Lorentz-invariant and all the New Physics is described by heavy particles, hence the new operators only consist of the SM field content. Because we want the SMEFT to be gauge invariant, it is convenient to consider only gauge covariant objects as the building blocks for our construction. The building blocks we are going to use are the fermion fields $\ell^j_{L,p}$, $e_{R,p}$, $q^{\alpha,j}_{L,p}$, $u^{\alpha,j}_{R,p}$, $d^{\alpha}_{R,p}$, the Higgs doublet φ^j, the gauge field tensors $X_{\mu\nu} \in \{G^A_{\mu\nu}, W^I_{\mu\nu}, B_{\mu\nu}\}$ and covariant derivatives thereof. The index convention is as follows $p = 1, 2, 3$ are the generation indices of the fermions, $j = 1, 2$ are indices of the fundamental representation of the $SU(2)_L$ and $\alpha = 1, 2, 3$ are the color indices. The covariant derivative is given as

$$D_\mu = \partial_\mu - ig_s \frac{1}{2}\lambda^A G^A_\mu - ig\frac{1}{2}T^I W^I - ig' Y B \,. \tag{10}$$

The only dimension 5 operator compatible with gauge symmetry is

$$\mathcal{Q}_{\nu\nu} = \varepsilon_{ij} \overline{\ell^c_R}^i \varphi^j \varepsilon_{kl} \ell^k_L \, \varphi^l, \tag{11}$$

where the c indicates charge conjugation. This operator is a Majorana mass term of the neutrinos and it violates the lepton number L, which is why we will not pay too much attention to it in these notes.

We will now proceed with our sketch of how to construct an operator basis for the dimension 6 SMEFT Lagrangian. Considering that in $d = 4$, scalar fields have mass dimension 1, the field strength tensors $X_{\mu\nu}$ have mass dimension 2, fermionic fields have mass dimension 3/2 and derivatives have mass dimension 1, dimensional analysis already constrains the combinations of fields and derivatives for operators at a given mass dimension.

One of the easiest things to notice is that one can only have an even number of fermionic fields in order to construct any operator with integer mass dimension[2]. At dimension 6, we can have four fermionic fields without any bosons or derivatives, which are commonly referred to as four-fermion operators. Another possibility are operators with two fermionic fields and an additional combination of scalar fields, derivatives and a field strength tensor with a total mass dimension of 3. The last possible class of operators at dimension 6 are purely bosonic operators, i.e. a combination of field strength tensors, scalar fields and derivatives.

To give an idea of how to find all the possible operators within these classes, we will focus on operators with purely bosonic field content as an example. Considering the fact that the scalars have half-integer $SU(2)_L$ isospin, while the other gauge bosons have integer isospin, it becomes apparent that there must be an even number of Higgs fields and since the only available objects with an odd number of Lorentz indices are the derivatives, there must also be an even number of those. With these constraints, we can already rule out all the possibilities except X^3, $X^2\varphi^2$, X^2D^2, $X\varphi^4$, XD^4, $X\varphi^2D^2$, φ^6, φ^4D^2 and φ^2D^4.

[2]Of course, this does not only follow from dimensional analysis but also from Lorentz invariance.

Our strategy of classification will be the following. We order the previously listed classes in such a way that classes containing more covariant derivatives are "higher classes". Classes with the same number of covariant derivatives are ordered based on their number of field strength tensors, considering the ones with less field strength tensors to be "higher". Our strategy will be to first use the equations of motion and other relations to move operators to "lower" classes if possible and then list all the independent operators that are left by using representation theoretical arguments.
Since X is antisymmetric and it cannot be contracted with any combination of $\eta_{\mu\nu}$ and $\varepsilon_{\mu\nu\rho\sigma}$ in such a way that it doesn't vanish, the set of operators $X\varphi^4$ is empty. The operators of the class XD^4 can only be contracted in such a way that they contain at least one commutator of the covariant derivatives and since $[D_\mu, D_\nu] \sim X_{\mu\nu}$, all these operators either fall into the category X^2D^2 or X^3.
All the operators in the classes φ^2D^4, $X\varphi^2D^2$ and X^2D^2 can be rewritten in terms of operators of other classes by virtue of the equations of motion. We will show the argument explicitly for the class φ^2D^4 and we can argue analogously for the other two classes.
Let us first write down the equation of motion of the scalar derived from the Standard Model Lagrangian[3]:

$$(D^\mu D_\mu\varphi)^j = m^2\varphi^j - \lambda(\varphi^\dagger\varphi)\varphi^j - \bar{e}\Gamma_e^\dagger\ell^j + \varepsilon_{jk}\bar{q}^k\Gamma_u u - \bar{d}\Gamma_d^\dagger q^j\,, \tag{12}$$

where $\Gamma_{e,u,d}$ are the Yukawa matrices and ε_{jk} is the totally antisymmetric tensor.
Let us consider the operators of the class φ^2D^4. To satisfy $SU(2)_L \times U(1)_Y$ invariance, each operator has to contain one φ and one $\varphi^\dagger$. Since the covariant derivative given in equation (10) can be written as $D_\mu = \partial_\mu - iM_\mu$, where M_μ is a hermitian object, we can infer that

$$\begin{aligned}(D_\mu\varphi)^\dagger\varphi + \varphi^\dagger D_\mu\varphi &= (\partial_\mu\varphi^\dagger)\varphi + i(M_\mu\varphi)^\dagger\varphi + \varphi^\dagger\partial_\mu\varphi - i\varphi^\dagger M_\mu\varphi \\ &= (\partial_\mu\varphi^\dagger)\varphi + \varphi^\dagger\partial_\mu\varphi \\ &= \partial_\mu(\varphi^\dagger\varphi)\,,\end{aligned} \tag{13}$$

and therefore, generalizing to multiple applications of the covariant derivative to $\varphi^\dagger$ and φ, we can schematically write

$$(D^n\varphi)^\dagger(D^m\varphi) = -(D^{n+1}\varphi)^\dagger(D^{m-1}\varphi) + \partial\left[(D^n\varphi)^\dagger(D^{m-1}\varphi)\right]\,. \tag{14}$$

Hence, we can reexpress all the operators in the class φ^2D^4 in the form of

$$(D^4\varphi)^\dagger\varphi + \partial(\ldots)\,. \tag{15}$$

Total derivatives have no physical effect by virtue of the Gaussian theorem, so we can safely neglect them, hence the only type of operator we have to consider is $(D^4\varphi)^\dagger\varphi$.

[3]As explained in the previous section, the application of the $\mathcal{O}(\frac{1}{\Lambda})$ terms of the equations of motion to the $\mathcal{O}(\frac{1}{\Lambda^2})$ terms of the Lagrangian are only going to generate $\mathcal{O}(\frac{1}{\Lambda^3})$ terms. Hence, we can savely ignore them at $\mathcal{O}(\frac{1}{\Lambda^2})$.

All possible contractions with $\varepsilon_{\mu\nu\rho\sigma}$ lead to appearances of commutators of the covariant derivative $[D_\mu, D_\nu] \sim X_{\mu\nu}$, which would move the operator to a lower class. This also leads to the observation that we can order the covariant derivatives as we want, since reordering will simply generate an additional operator of a lower class. Because of this, we can rearrange the covariant derivatives of all the operators of this class such that they all contain $D_\mu D^\mu \varphi$, which can be replaced by the right hand side of the equation of motion (12). This moves all the operators of this class to the classes $\varphi^4 D^2$, $\psi^2 \varphi D^2$ and the class of dimension 4 operators $D^2\varphi^2$. In conclusion, by use of the equation of motion of φ and other arguments, we were able to rewrite all the operators of this class in terms of operators of lower classes.

Let us now consider the class $\varphi^4 D^2$. In order for the net hypercharge of the operators to be zero, exactly two φ fields must be complex-conjugated. The two covariant derivatives must be contracted with each other in order to fulfill Lorentz invariance and since the case with two derivatives acting on the same field can be moved to lower classes by using the equations of motion, we only have to consider the case where the derivatives act on different φ fields.

The case where the two derivatives both act on the unconjugated fields or where they both act on the conjugated fields can be rewritten in terms of the case where one derivative acts on an unconjugated field and one on a conjugated field by using equation (14) and partially integrating, e.g.:

$$\begin{aligned}(\varphi^\dagger D^\mu \varphi)(\varphi^\dagger D_\mu \varphi) &\overset{(14)}{=} -(\varphi^\dagger D^\mu \varphi)((D_\mu\varphi)^\dagger \varphi) + \partial_\mu(\varphi^\dagger\varphi)(\varphi^\dagger D^\mu \varphi) \\ &\overset{\text{p.i.}}{=} -(\varphi^\dagger D^\mu \varphi)((D_\mu\varphi)^\dagger \varphi) - (\varphi^\dagger\varphi)\partial_\mu(\varphi^\dagger D^\mu \varphi) \\ &\overset{(14)}{=} -(\varphi^\dagger D^\mu \varphi)((D_\mu\varphi)^\dagger \varphi) - (\varphi^\dagger\varphi)((D_\mu\varphi)^\dagger D^\mu \varphi) - (\varphi^\dagger\varphi)(\varphi^\dagger D_\mu D^\mu \varphi)\,. \end{aligned} \tag{16}$$

We can use the equation of motion on the last term in the last line to move it to a lower class. The other terms each have one covariant derivative acting on an unconjugated field and one acting on a conjugated field. After these reductions, there are only two independent $SU(2)_L$-singlets left we can construct. One possible choice of two independent singlets would be:

$$(\varphi^\dagger \tau^I \varphi)[(D_\mu\varphi)^\dagger \tau^I (D^\mu\varphi)] \qquad\qquad (\varphi^\dagger\varphi)[(D_\mu\varphi)^\dagger (D^\mu\varphi)]\,. \tag{17}$$

The classification of the operators in the other classes can be performed in a similar way.

4 Operator Mixing and EOM-vanishing operators

In order to be able to use the SMEFT quantitatively, it's Wilson coefficients $C_k^(n)$ need to be renormalized order by order in the $\frac{1}{\Lambda}$ expansion in order to account for UV divergences. The EFT operators can mix into each other under renormalization, i.e. in general

$$C_j^{(n),\text{bare}} \mathcal{Q}_j^{(n)} = \sum_i C_i \mathcal{Z}_{ij} \mathcal{Q}_j^{(n)}\,, \tag{18}$$

where $\mathcal{Z}_{ij}$ is the matrix of renormalization constants that relates the bare couplings to the renormalized ones:

$$C_j^{(n),\text{bare}} = \sum_i C_i \mathcal{Z}_{ij} \, . \tag{19}$$

Here, $\mathcal{Q}_j^{(n)}$ denotes the SMEFT operators of mass dimension n consisting of renormalized Standard Model fields.
Observables do not depend on the renormalization scale μ, hence the bare Wilson don't either. From this, we can derive that

$$0 = \frac{\mathrm{d}C_j^{(n),\text{bare}}}{\mathrm{d}\mu} = \sum_i \left[\frac{\mathrm{d}C_i^{(n)}}{\mathrm{d}\mu} \mathcal{Z}_{ij} + C_i^{(n)} \frac{\mathrm{d}\mathcal{Z}_{ij}}{\mathrm{d}\mu} \right] , \tag{20}$$

which is commonly rewritten as

$$\frac{\mathrm{d}C_j^{(n)}}{\mathrm{d}\log\mu} = C_i^{(n)} \gamma_{ij} \, , \tag{21}$$

where we defined the anomalous dimension matrix (ADM)

$$\gamma_{ij} = \sum_k -(\mathcal{Z}^{-1})_{kj} \frac{\mathrm{d}\mathcal{Z}_{ij}}{\mathrm{d}\log\mu} \tag{22}$$

that describes mixing of the Wilson coefficients under the renormalization group evolution.
There is a caveat in the renormalization procedure, though: Some operators can generate divergencies proportional to operators that are not part of our basis but appear in the equations of motion. For example in the renormalization of the effective Lagrangian for weak decays, one can construct a basis that only consists of four-quark operators like the operator

$$\mathcal{O}_q = \bar{u}\gamma^\mu P_L s \bar{d} \gamma_\mu P_L u \, . \tag{23}$$

This operator induces the divergent penguin diagram displayed in figure 1, which has a divergence proportional to

$$\mathcal{O}_P = \bar{d} T^A \gamma^\mu P_L s g_s [D^\nu, G_{\nu\mu}]^A \, , \tag{24}$$

which is no linear combination of the operators in the basis of four-quark operators, even though the basis is complete. However, when calculating physical observables, we can use the equations of motion to rewrite this operator in terms of the four-quark operators and this enables us to study the ADM in the four-quark operator basis.
It is important to note that while redundant operators like $\mathcal{O}_q$ may appear in the calculations, they should not be included in the basis because otherwise the ADM is not uniquely determined, since one can arbitrarily add operators that vanish by the equations of motion to the renormalization group equation by virtue of field redefinitions. The arbitrary parameters we would introduce in the ADM could be renormalization scheme and gauge dependent, which is obviously not physical.
For further details, please see [5] and the corresponding follow-up papers.

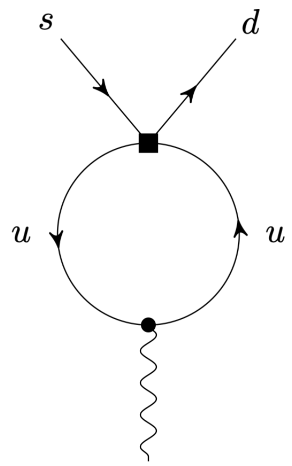

Figure 1: Penguin diagram for $s \to d$ transition

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