"Scattering Amplitude and RG"

1607.06488, 1910.05831

Outline

- S-matrix and RGE
- Example: β-fn of YM theory
- Non-renormalization thm. of anomalous dim. matrix.

[1] S-matrix and RGE

- Basic idea

"−"

RGE: $\oslash \times \ln\left(\frac{-p^2}{\mu^2}\right)$

$= \oslash\left[\ln\left(\left|\frac{p^2}{\mu^2}\right|\right) - i\pi\right]$

$\longrightarrow \quad p^\mu \frac{\partial}{\partial p^\mu} \ln\left(\frac{-p^2}{\mu^2}\right) = -\frac{2}{\pi}\, \mathrm{Im} \ln\left(\frac{-p^2}{\mu^2}\right)$

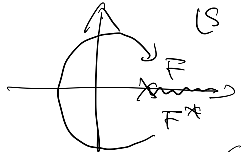
(Im part) = (cutting loops) = (on-shell S-matrix)

- precise formula

$F_{\mathcal{O}}(p_1, \cdots, p_n) \equiv \langle p_1 \cdots p_n | \mathcal{O} | 0 \rangle$

- all ext. particles = out-going, $s_{ij\cdots k} \equiv (p_i + p_j + \cdots + p_k)^2 > 0$

- analyticity $\Rightarrow F^*(s_J - i\epsilon) = F(s_J + i\epsilon)$



$\left(e^{-i\pi D} : \begin{cases} p_i \to e^{-i\pi} p_i \\ s_J \to e^{-2i\pi} s_J \end{cases} \right)$

or $e^{-i\pi D} F^* = F, \quad D = \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu}$

- Furthermore

formally treat $\mathcal{O}$ as a perturbation of S-matrix

$\longrightarrow \quad (S + i\mathcal{O})^\dagger (S + i\mathcal{O}) = 1.$

$\rightsquigarrow \quad \mathcal{O}^\dagger S = S^\dagger \mathcal{O}, \quad \mathcal{O} = S \mathcal{O}^\dagger S$

$\langle p_1 \cdots p_n |$, $|0\rangle$

$\longrightarrow \quad F = S \cdot F^*$

$$e^{i\pi D} F^* = S \cdot F^*$$

High energy (or massless case)

→ F : fn of s_{ij}/μ^2

$$D \simeq -\mu \frac{\partial}{\partial \mu}$$

$$DF = \left[\gamma_{UV} - \gamma_{IR} + \beta(g^2) \frac{\partial}{\partial g^2} \right] F$$

w/ LO (for instance) $S = 1 + iM$ ↓

$$\left[\gamma_0 - \gamma_{IR} + \beta \frac{\partial}{\partial g^2} \right] F = -\frac{1}{\pi} M \cdot F$$

$\gamma \times F \sim$

$F_{\mu\nu}F^{\mu\nu}$ $\langle p_1, p_2 | F_{\mu\nu}F^{\mu\nu} | 0 \rangle = F$

- β → irrelevant for "minimal form factor" that does not vanish for $g=0$
- γ_{IR} → indep of $\mathcal{O}$ w/ subtracted by taking ratio w/ other op.

Example : β-fn of YM @ 1-loop

- spinor-helicity formalism

$$p^{\alpha\dot\alpha} = p^\mu \sigma_\mu^{\alpha\dot\alpha}$$

$$p_\mu p^\mu = 0 \to \det(p^{\alpha\dot\alpha}) = 0$$

$$\to p_j^{\alpha\dot\alpha} = \overset{h}{\lambda_j^\alpha} \overset{h^{-1}}{\tilde\lambda_j^{\dot\alpha}} = |j\rangle [j|$$

※ $p^\mu = p \cdot (1, \sin 2\theta \cos\phi, \sin 2\theta \sin\phi, \cos 2\theta)$

$$\to p^{\alpha\dot\alpha} = p \cdot \begin{pmatrix} 1+\cos 2\theta & \sin 2\theta\, e^{-i\phi} \\ \sin 2\theta\, e^{i\phi} & 1-\cos 2\theta \end{pmatrix}$$

$$\to \lambda^\alpha \propto \begin{pmatrix} \cos\theta \\ \sin\theta\, e^{i\phi} \end{pmatrix}, \quad \tilde\lambda^{\dot\alpha} \propto \begin{pmatrix} \cos\theta \\ \sin\theta\, e^{-i\phi} \end{pmatrix}$$

Mandelstam variables :

$$s_{ij} = 2 p_i \cdot p_j = \langle ij \rangle [ji]$$

↑ $\epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

→ Amplitudes are products of $\langle ij \rangle$, $[ij]$

e.g. YM

· $M^{abcd}_{1^-2^-3^+4^+} \cdot \delta^{cd} = -2g^2 C_A \cdot \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}$

· $\langle 1^a_- 2^b_- | (-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}) | 0\rangle = \frac{1}{2} \delta^{ab}\langle 12\rangle^2$

· $\langle 1^a_- 2^b_+ | T^{\alpha\beta,\dot\alpha\dot\beta} | 0\rangle = 2\delta^{ab}\lambda_1^\alpha\lambda_1^\beta\tilde\lambda_2^{\dot\alpha}\tilde\lambda_2^{\dot\beta}$

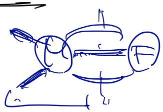
· one-loop β-fn of pure YM

$$F = \langle 1^a_- 2^b_- | (-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}) | 0\rangle \quad \text{"}\mathcal{F}\text{"}$$

$\rightarrow \beta \frac{\partial}{\partial g^2} F = 0$ @ this order

$(\gamma_{UV} - \gamma_{IR}) F \sim -\frac{1}{\pi} M \otimes F.$

$\langle 1^a_- 2^b_- | M \otimes \mathcal{F} | 0\rangle$



$$= \frac{1}{16\pi}\int \frac{d\Omega}{4\pi} \sum_{h_1', h_2', a', b'} \langle 1^a_- 2^b_- | M | 1'^{a'}_{h_1'} 2'^{b'}_{h_2'}\rangle \langle 1'^{a'}_{-h_1'} 2'^{b'}_{-h_2'} | \mathcal{F} | 0\rangle$$

$$= -\frac{g^2 C_A}{64\pi^2}\delta^{ab}\int d\Omega \frac{\langle 12\rangle^4 \langle 1'2'\rangle^2}{\langle 11'\rangle\langle 1'2\rangle\langle 22'\rangle\langle 2'1\rangle}$$

$p_1' \propto \hat{z}$. $\lambda_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\lambda_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$\rightarrow \begin{pmatrix}\lambda_1'\\ \lambda_2'\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\, e^{i\varphi}\\ \sin\theta\, e^{-i\varphi} & \cos\theta\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$$

$$\rightarrow \langle 1^a_- 2^b_- | M \otimes \mathcal{F} | 0\rangle = \frac{g^2 C_A}{32\pi}\cdot\delta^{ab}\langle 12\rangle^2 \int_{-1}^{1} d\cos 2\theta \cdot \frac{1}{\cos^2\theta\cdot\sin^2\theta}$$

$$\gamma_{UV} - \gamma_{IR} = -\frac{1}{\pi}\cdot\frac{\langle 1^a_- 2^b_- | M\otimes\mathcal{F}|0\rangle}{\langle 1^a_- 2^b_- | \mathcal{F} | 0\rangle} = -\frac{g^2 C_A}{16\pi^2}\int_{-1}^{1} d\cos 2\theta \cdot \frac{1}{\cos^2\theta\cdot\sin^2\theta}$$

to extract γ_{UV} consider energy-stress tensor

$\dot{x}^1 T^{\mu\nu}$ (or conserved quantity in general)

$\gamma^{(1)}_{UV} = \qquad \gamma_{UV} = 0$

$$-\frac{1}{\pi}\left[\frac{\langle 1^a_- 2^b_- | M\otimes\mathcal{F}|0\rangle}{\langle 1^a_- 2^b_- | \mathcal{F}|0\rangle} - \frac{\langle 1^a_- 2^b_+ | M\otimes T|0\rangle}{\langle 1^a_- 2^b_+ | T|0\rangle}\right]$$

$$= -\frac{g^2 C_A}{8\pi^2}\int_{-1}^{1} d\cos 2\theta \frac{1}{\sin^2\theta\cos^2\theta}\left[1 - (\cos^8\theta + \sin^8\theta)\right]$$

$$= -\frac{g^2}{8\pi^2}\cdot\frac{11 C_A}{3}$$

this γ and β are related as

$$\gamma = g^2 \cdot \frac{\partial}{\partial g^2}\left(\frac{\beta}{g^2}\right)$$

$$\longrightarrow \beta = -\frac{g^4}{8\pi^2} \cdot \frac{11 C_A}{3}$$

※ Matter contributions also appear correctly from energy stress tensor

In general $M \otimes F$ Melua several different formulation

$\longrightarrow$ Anomalous dimension matrix

$\lambda \Phi^4$, $y \phi \bar{\psi} \psi$.

☑ Non-renormalization thm of anomalous dim matrix

- $\Delta \mathcal{L} = \sum_i C_i \mathcal{O}_i$

$$\frac{\partial C_i}{\partial \log \mu} = \frac{\gamma_{ij}^{UV}}{16\pi^2} \cdot C_j$$

γ_{ij}^{UV} = Anomalous dim matrix.

- fact

Some entries of γ_{ij}^{UV} of SMEFT are "non-trivially" zero

$\longrightarrow$ understand this within the current method.

def

$\ell[\mathcal{O}]$: # of fundamental fields involved in $\mathcal{O}$.

$\ell[\phi^4] = 4$, $\ell[\phi \bar{\psi}\psi] = 3$ . . . .

thm

O_s is renormalized at L-loop by O_ℓ

only when

$$L > \ell[O_\ell] - \ell[O_s]$$

proof

- first non-trivial loop order → $\gamma_{IR} = 0$
- minimal FF → $\frac{\partial}{\partial s_2} F = 0$

$\Rightarrow$ $\gamma^{(L)}_{s\ell} \langle p_1 \cdots p_{\ell[O_s]} | O_s | 0 \rangle$

$$= -\frac{1}{\pi} \langle p_1 \cdots p_{\ell[O_s]} | M \otimes O_\ell | 0 \rangle$$

k-cut (at $\otimes$); n_M-pt. ; n_F-pt.

- $(n_M - k) + (n_F - k) = \ell[O_s]$
- $n_M \geq k+2$
- $n_F \geq \ell[O_\ell] - (L_F - 1) - \delta_{L_F,0}$ (L_F: loops on F-side)
- $L_F + (k-1) \leq L$

→ You can show the above inequality.