The Third Way to Interacting p-form Theories

Matteo Broccoli
Albert-Einstein-Institut, Golm

Phys.Rev.Lett. **127** (2021) · [2103.13243]
with Sadik Deger and Stefan Theisen

DESY Theory Workshop
Hamburg, September 23, 2021

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor
- $T_{\mu\nu}$ is a stress-energy tensor

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor
- $T_{\mu\nu}$ is a stress-energy tensor

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor
- $T_{\mu\nu}$ is a stress-energy tensor

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

There are two standard ways in which $T_{\mu\nu}$ is conserved:

1. as a consequence of matter field equations
2. identically by means of Bianchi-type identity

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor
- $T_{\mu\nu}$ is a stress-energy tensor

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

There are two standard ways in which $T_{\mu\nu}$ is conserved:

1. as a consequence of matter field equations
2. identically by means of Bianchi-type identity
   - $T_{\mu\nu} = \Lambda g_{\mu\nu}$
   - $T_{\mu\nu} = \frac{1}{m} C_{\mu\nu}$ in 3d (Topologically Massive Gravity)

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

- $G_{\mu\nu}$ is the Einstein tensor
- $T_{\mu\nu}$ is a stress-energy tensor

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0$$

There are two standard ways in which $T_{\mu\nu}$ is conserved:

1. as a consequence of matter field equations
2. identically by means of Bianchi-type identity
   - $T_{\mu\nu} = \Lambda g_{\mu\nu}$
   - $T_{\mu\nu} = \frac{1}{m} C_{\mu\nu}$ in 3d (Topologically Massive Gravity)

However, there is also a third way!

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

and $\nabla^\mu T_{\mu\nu} \neq 0$, but $T_{\mu\nu}$ is conserved

3. as a consequence of the gravitational field equation itself

The Third Way

$$G_{\mu\nu} = T_{\mu\nu}$$

and $\nabla^\mu T_{\mu\nu} \neq 0$, but $T_{\mu\nu}$ is conserved

3. as a consequence of the gravitational field equation itself

Examples in $d = 3$

- Minimal Massive Gravity

  [Bergshoeff-Hohm-Merbis-Routh-Townsend 1404.2867]

  $T_{\mu\nu} = -\frac{1}{m} C_{\mu\nu} - \frac{1}{m^2} J_{\mu\nu}$

  $C_{\mu\nu} = \varepsilon_\mu{}^{\rho\sigma} \nabla_\rho S_{\sigma\nu}\,, \quad J_{\mu\nu} = \frac{1}{2} \varepsilon_\mu{}^{\rho\sigma} \varepsilon_\nu{}^{\alpha\beta} S_{\rho\alpha} S_{\sigma\beta}$

- Exotic Massive Gravity [Ozkan-Pang-Townsend 1806.04179]
- ...

The Third Way

Yang-Mills examples in $d = 3$

Consider the equation of motion of a 'Topologically Massive Yang-Mills' gauge field A:

$$\varepsilon^{\mu\nu\rho} D_\nu \tilde{F}_\rho + \mu\, \tilde{F}^\mu = 0\,, \quad \tilde{F}^\mu = \varepsilon^{\mu\nu\rho}\left(\partial_\nu A_\rho + \frac{1}{2}[A_\nu, A_\rho]\right)$$

- add third way consistent sources
  [Arvanitakis-Sevrin-Townsend 1501.07548]

$$\varepsilon^{\mu\nu\rho} D_\nu \tilde{F}_\rho + \mu\, \tilde{F}^\mu = J^\mu + \mathcal{J}^\mu$$

$$J^\mu = -\tfrac{1}{2m}\varepsilon^{\mu\nu\rho}[\tilde{F}_\nu, \tilde{F}_\rho]$$

$$\mathcal{J}^\mu = j^\mu - \tfrac{\varepsilon^{\mu\nu\rho}}{m-\mu}\left(D_\nu j_\rho + \tfrac{1}{m}[\tilde{F}_\nu, j_\rho] + \tfrac{1}{2m(m-\mu)}[j_\nu, j_\rho]\right)$$

The Third Way

- several examples of third way consistent theories
- equations of motion do not follow from an action which contains the dynamical field alone
- not only a mathematical curiosity: gravitational examples have important holographic properties

The Third Way

- several examples of third way consistent theories
- equations of motion do not follow from an action which contains the dynamical field alone
- not only a mathematical curiosity: gravitational examples have important holographic properties

However . . .

- the construction of these models is quite cumbersome
- only 3d theories

The Third Way

Is there any systematic behind third way consistent theories?

Do they exist in $d > 3$?

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

Consider a flat connection A of an arbitrary gauge group G which solves the equation of motion

$$F(A) = \mathrm{d}A + A^2 = 0$$

- $A = A^i_\mu T_i \mathrm{d}x^\mu\,, \quad [T_i, T_j] = f^k{}_{ij} T_k\,, \quad f^i{}_{j\underline{k}} f^j{}_{\underline{mn}} = 0$
- Consistency: $DF(A) = \mathrm{d}F + [A, F] = 0$ (Bianchi id.)

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

Consider a flat connection A of an arbitrary gauge group G which solves the equation of motion

$$F(A) = \mathrm{d}A + A^2 = 0$$

- $A = A^i_\mu T_i \mathrm{d}x^\mu \,, \quad [T_i, T_j] = f^k{}_{ij} T_k \,, \quad f^i{}_{j\underline{k}} f^j{}_{\underline{mn}} = 0$
- Consistency: $DF(A) = \mathrm{d}F + [A, F] = 0$ (Bianchi id.)

Now shift the connection with an arbitrary $C = C^i_\mu T_i \mathrm{d}x^\mu$

$$F(A + C) = F(A) + DC + C^2 = 0$$

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

Consider a flat connection A of an arbitrary gauge group G which solves the equation of motion

$$F(A) = \mathrm{d}A + A^2 = 0$$

- $A = A^i_\mu T_i \mathrm{d}x^\mu \,, \quad [T_i, T_j] = f^k{}_{ij} T_k \,, \quad f^i{}_{j\underline{k}} f^j{}_{\underline{mn}} = 0$
- Consistency: $DF(A) = \mathrm{d}F + [A, F] = 0$ (Bianchi id.)

Now shift the connection with an arbitrary $C = C^i_\mu T_i \mathrm{d}x^\mu$

$$F(A + C) = F(A) + DC + C^2 = 0$$

× $DF(A + C) = D^2 C + DC^2 = [F(A) + DC, C] \neq 0$

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

Consider a flat connection A of an arbitrary gauge group G which solves the equation of motion

$$F(A) = \mathrm{d}A + A^2 = 0$$

- $A = A^i_\mu T_i \mathrm{d}x^\mu \,, \quad [T_i, T_j] = f^k{}_{ij} T_k \,, \quad f^i{}_{j\underline{k}} f^j{}_{\underline{mn}} = 0$
- Consistency: $DF(A) = \mathrm{d}F + [A, F] = 0$ (Bianchi id.)

Now shift the connection with an arbitrary $C = C^i_\mu T_i \mathrm{d}x^\mu$

$$F(A + C) = F(A) + DC + C^2 = 0$$

× $DF(A + C) = D^2 C + DC^2 = [F(A) + DC, C] \neq 0$

✓ on-shell: $DF(A + C) = [F(A) + DC, C] = [C^2, C] = 0$

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

Consider a flat connection A of an arbitrary gauge group G which solves the equation of motion

$$F(A) = \mathrm{d}A + A^2 = 0$$

- $A = A^i_\mu T_i \mathrm{d}x^\mu \,, \quad [T_i, T_j] = f^k{}_{ij} T_k \,, \quad f^i{}_{j\underline{k}} f^j{}_{\underline{mn}} = 0$
- Consistency: $DF(A) = \mathrm{d}F + [A, F] = 0$ (Bianchi id.)

Now shift the connection with an arbitrary $C = C^i_\mu T_i \mathrm{d}x^\mu$

$$F(A + C) = F(A) + DC + C^2 = 0$$

× $DF(A + C) = D^2 C + DC^2 = [F(A) + DC, C] \neq 0$

✓ on-shell: $DF(A + C) = [F(A) + DC, C] = [C^2, C] = 0$

Third way consistency is automatic!

Shifting the Connection in $3d$

MB-Deger-Theisen [2103.13243]

$$F(A + C) = F(A) + DC + C^2 = 0$$

In $d = 3$ the shift in the connection

✓ unveils the systematic behind the construction of third way consistent (Yang-Mills) theories

- choose particular shift C to recover the models in [Arvanitakis-Sevrin-Townsend 1501.07548]

✓ allows to easily construct new models

- choose different (e.g. higher derivatives) shift C

Does this construction make sense in $d > 3$?

New Interacting p-form Theories in $d \geq 3$

MB-Deger-Theisen [2103.13243]

- Consider a 1-form C in $d \geq 3$
- e.g. $C = \kappa \tilde{H}$ with H being a $(d-1)$-form $H = H^i T_i$
- shift the connection $F(A + \kappa \tilde{H}) = 0$

$\Rightarrow$ automatically third way consistent!

New Interacting p-form Theories in $d \geq 3$

MB-Deger-Theisen [2103.13243]

- Consider a 1-form C in $d \geq 3$
- e.g. $C = \kappa \tilde{H}$ with H being a $(d-1)$-form $H = H^i T_i$
- shift the connection $F(A + \kappa \tilde{H}) = 0$

$\Rightarrow$ automatically third way consistent!

Dynamics and gauge invariance

- define H to be the Yang-Mills covariant field strength of a $(d-2)$-form B^i, i.e. $H^i = (DB)^i = \mathrm{d}B^i + f^i{}_{jk} A^j B^k$
- from the $(d-2)$-form gauge transformation of B^i and A^i $\rightarrow$ gauge symmetry for $A = 0$

$$F(A + \kappa \tilde{H}) = 0 \quad \Rightarrow \quad \mathrm{d}\tilde{H}^i = -\frac{1}{2}\kappa\, f^i{}_{jk}\, \tilde{H}^j \tilde{H}^k$$

New Interacting p-form Theories in $d \geq 3$

MB-Deger-Theisen [2103.13243]

$$\mathrm{d}\tilde{H}^i = -\frac{1}{2}\kappa\, f^i{}_{jk}\, \tilde{H}^j\, \tilde{H}^k$$

- third way consistent equation of motion of interacting $(d-2)$-forms B^i, $H^i = \mathrm{d}B^i$
- $(d-2)$-form symmetry for the standard $\delta_\xi B^i = \mathrm{d}\xi^i$
- $F(\tilde{H}) = 0$ flatness condition
- $\mathrm{d}H^i = 0$ Bianchi identity

New Interacting p-form Theories in $d \geq 3$

MB-Deger-Theisen [2103.13243]

$$\mathrm{d}\tilde{H}^i = -\frac{1}{2}\kappa\, f^i{}_{jk}\, \tilde{H}^j\, \tilde{H}^k$$

- third way consistent equation of motion of interacting $(d-2)$-forms B^i, $H^i = \mathrm{d}B^i$
- $(d-2)$-form symmetry for the standard $\delta_\xi B^i = \mathrm{d}\xi^i$
- $F(\tilde{H}) = 0$ flatness condition
- $\mathrm{d}H^i = 0$ Bianchi identity
- dual to the principal chiral sigma model
- similarities with Freedman-Townsend model
- several generalisations: coupling to gravity, addition of lower form fields, connection to n-algebras

Shifting the Gravitational Connection

Christoffel symbol $\Gamma^{\nu}_{\mu\rho}$ can also be thought of as the (ν, ρ) component of a Yang-Mills field $\Gamma^{\nu}_{\mu\rho} = (A_\mu)^{\nu}{}_{\rho}$

- shift the connection in Einstein vacuum equation:

$$G_{\mu\nu}(\Gamma + C) = 0$$

Shifting the Gravitational Connection

Christoffel symbol $\Gamma^{\nu}_{\mu\rho}$ can also be thought of as the (ν, ρ) component of a Yang-Mills field $\Gamma^{\nu}_{\mu\rho} = (A_\mu)^{\nu}{}_{\rho}$

- shift the connection in Einstein vacuum equation:

$$G_{\mu\nu}(\Gamma + C) = 0$$

$d = 3$

✓ Minimal Massive Gravity for $C^{\rho}_{\mu\nu} = \kappa\, \varepsilon_{\mu}{}^{\rho\sigma} S_{\sigma\nu}$

✓ Exotic Massive Gravity for $C^{\rho}_{\mu\nu} = \kappa\, (\nabla^{\rho} S_{\mu\nu} - \nabla_{\mu} S^{\rho}{}_{\nu})$

× not every shift leads to third way consistent equations

Shifting the Gravitational Connection

Christoffel symbol $\Gamma^{\nu}_{\mu\rho}$ can also be thought of as the (ν, ρ) component of a Yang-Mills field $\Gamma^{\nu}_{\mu\rho} = (A_\mu)^{\nu}{}_{\rho}$

- shift the connection in Einstein vacuum equation:

$$G_{\mu\nu}(\Gamma + C) = 0$$

$d = 3$

✓ Minimal Massive Gravity for $C^{\rho}_{\mu\nu} = \kappa\, \varepsilon_{\mu}{}^{\rho\sigma} S_{\sigma\nu}$

✓ Exotic Massive Gravity for $C^{\rho}_{\mu\nu} = \kappa\, (\nabla^{\rho} S_{\mu\nu} - \nabla_{\mu} S^{\rho}{}_{\nu})$

× not every shift leads to third way consistent equations

$d > 3$

- algorithmic tool to look for third way consistent gravitational theories

…new theories of gravity?

Conclusion

1. Third way consistent theories in $3d$
2. Underlying mechanism of shifting the connection
3. New interacting $(d-2)$-form theories in $d \geq 3$

Outlook

- supersymmetric version of the new theories
- actions with auxiliary fields to describe the new theories
- (path integral) quantization
- couplings with additional fields (e.g. gravity)
- third way consistent theory of gravity in $d > 3$

Conclusion

1. Third way consistent theories in $3d$
2. Underlying mechanism of shifting the connection
3. New interacting $(d-2)$-form theories in $d \geq 3$

Outlook

- supersymmetric version of the new theories
- actions with auxiliary fields to describe the new theories
- (path integral) quantization
- couplings with additional fields (e.g. gravity)
- third way consistent theory of gravity in $d > 3$

THANK YOU!