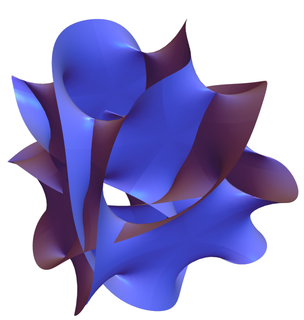
Universität Bonn

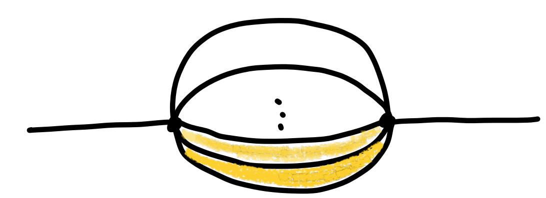
bctp Bethe Center for Theoretical Physics

Calabi-Yau Manifolds and Feynman Integrals

The Family of Banana Graphs



Christoph Nega



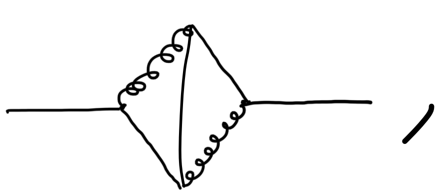
Joint work with:
Kilian Bönisch, Claude Duhr, Fabian Fischbach & Albrecht Klemm

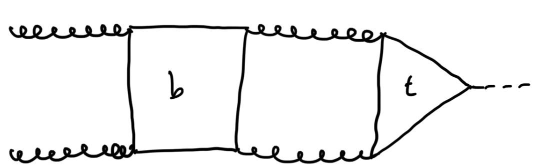
"Feynman Integrals in Dimensional Regularization and Extensions of Calabi-Yau Motives" [1]

Dark Universe Workshop
Hamburg
September 24, 2021

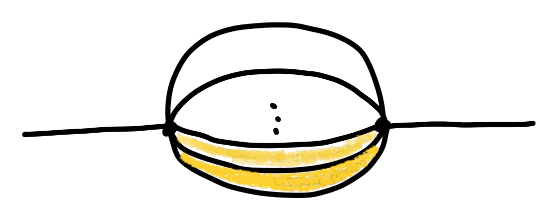
Feynman Integrals

- Perturbative QFT computations are build from Feynman integrals
  - Scattering amplitudes, cross sections, gravitational wave phenomenology, …
- Search for new physics
  - High precision measurements & multi-loop Feynman integral computations
- Feynman integrals themselves have an interesting mathematical structure
  - (New) special functions such as elliptic integrals, polylogarithms, modular forms, CY-periods, …
  - Function space?
  - Algebraic, geometric and number theoretic structures such as Elliptic curves, CYs, motives, …
- Banana graph is subtopology of nearly all multi-loop graphs



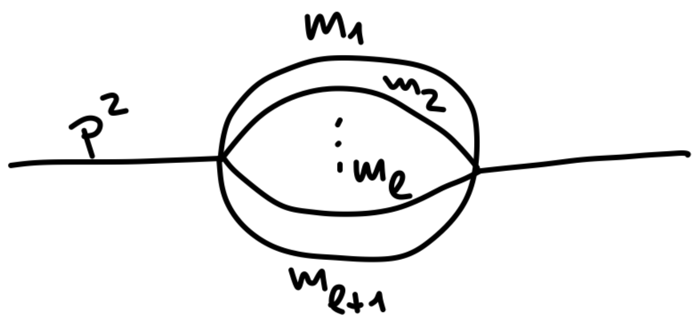


Subtopology



Basics of Feynman Integral Computations

- Momentum space representation:



$\longrightarrow$

$$I_{\underline{\nu}}(\underline{s};D) = \int \left(\prod_{j=1}^{l} \frac{\mathrm{d}^D k_j}{i\pi^{D/2}}\right) \left(\prod_{i=1}^{n} \frac{1}{(q_i^2 - m_i^2 + i0^+)^{\nu_i}}\right)$$

$\underline{s}$: parameters, dot products

$\underline{\nu} \in \mathbb{Z}^n$

- Integration by parts: $\displaystyle\int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \left(q^\mu \prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}} \right) = 0$ $\longrightarrow$ Relations between integrals

$\longrightarrow$ Master integrals (finite set)

- Dimensional regularization: $D = D_0 - 2\epsilon \; , \quad D_0 \in \mathbb{N}$ $\longrightarrow$ Laurent series $I_{\underline{\nu}} = \displaystyle\sum_{k=-m}^{\infty} I_{\underline{\nu},k}\epsilon^k$

- Dimensional shift relations: $I_{\underline{\nu}}(\underline{s};D) = \displaystyle\sum_{\underline{\alpha}\in\{\text{masters}\}} a_{\underline{\nu},\underline{\alpha}}(\underline{s}) I_{\underline{\alpha}}(\underline{s};D\pm 2)$ $\longrightarrow$ Choose preferred D_0

[Tarasov]

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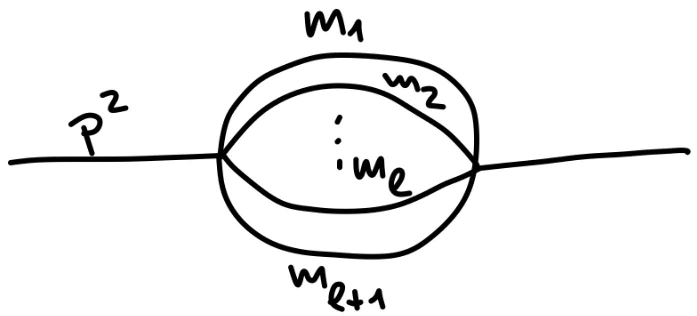

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  [Tarasov]

- Maximal cuts of a Feynman graph: "Residues around propagators" $\longrightarrow$ Similar properties
  Easier to compute

- Periods: (geometric) period integral

  [Kontsevich & Zagier]

$\longleftrightarrow \Pi = \int_\gamma \alpha$, $\alpha \in H^*(M)$, $\gamma \in H_*(M)$

$\longrightarrow$ Satisfy (inhomogeneous) differential equations

Calabi-Yau Manifolds

What is a Calabi-Yau?

"CYs are natural generalizations of elliptic curves"

- Defined by polynomial constraints

Weierstrass form: $y^2 = x^3 + ax + b$

Quintic constraint: $P_5 = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \alpha x_1 x_2 x_3 x_4 x_5 = 0$

- Distinguished differential form

Unique holomorphic differential: $\Omega_1 = \frac{\mathrm{d}x}{y}$

Unique holomorphic $(3,0)$ -form: $\Omega_3 = \int_{S^1} \frac{1}{P_5} \frac{\mu_l}{\prod_i x_i}$

- "Flat"

With "correct" metric torus is flat

CY are defined by Ricci flatness

CYs can be defined for arbitrary dimensions

Periods describe structure of CY, multivalued functions

Point of maximal unipotent monodromy $(\mathbf{M} - \mathbb{1})^{n+1} = 0$

At a MUM point the "periods" have an increasing logarithmic structure

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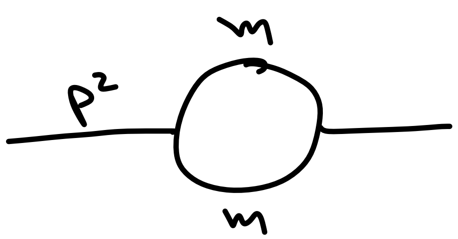
Banana Integrals in D=2

- Start of our analysis $D = 2$ and equal masses $z = \frac{m^2}{p^2}$ $\longrightarrow$ Banana integral is finite
- Associated geometries and functions:

$\underline{\ell = 1}$

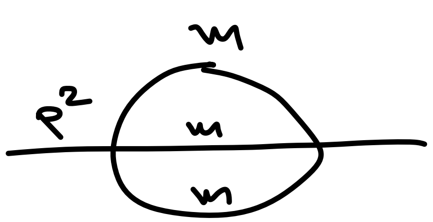


$\longleftrightarrow$ Riemann sphere $\longleftrightarrow$

$$\Pi_1(z) = \frac{z}{\sqrt{1-4z}}$$

$$I_1(z) \sim \Pi_1(z) \int_0^z \frac{dz'}{z'^2} \Pi_1(z')$$

$\underline{\ell = 2}$



$\longleftrightarrow$ elliptic curve $\longleftrightarrow$

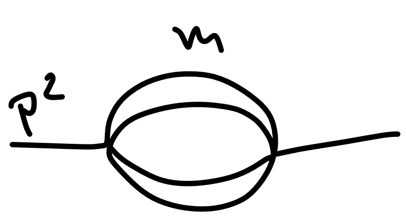
$\Pi_2(z) =$ elliptic functions

$$I_2(z) \sim \Pi_2(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \int_0^z \frac{dz'}{z'^2} \Pi_2(z')$$

$$\sim h_1(\tau) \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} h_3(\tau')\, \tau'$$

[Duhr et al.]

$\underline{\ell = 3}$



$\longleftrightarrow$ K3 surface $\hat{=}$ (elliptic curve)^2 $\longleftrightarrow$ again elliptic functions

[1]

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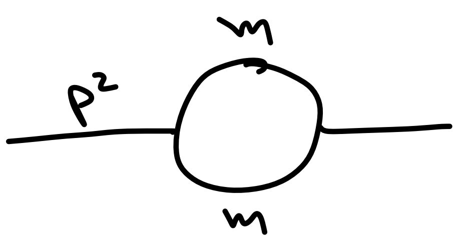
$\ell = 1$ | 

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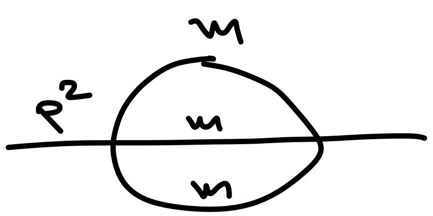
$\ell = 2$ | 

$\longleftrightarrow$ elliptic curve $\longleftrightarrow$ $\underline{\pi}_2(z) =$ elliptic functions

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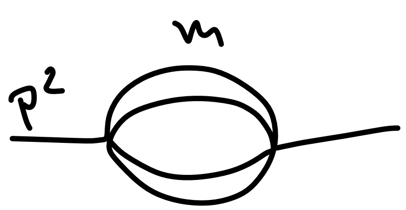
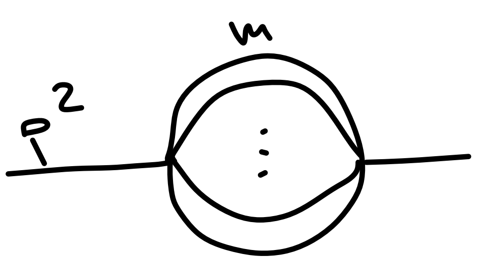
$\ell = 3$ | 

$\longleftrightarrow$ K3 surface $\widehat{=}$ (elliptic curve)^2 $\longleftrightarrow$ again elliptic functions [Duhr et al.]

ℓ | 

$\longleftrightarrow$ CY manifold of dim $\ell - 1$ $\longleftrightarrow$ $\underline{\pi}_\ell(z) =$ CY periods [1]

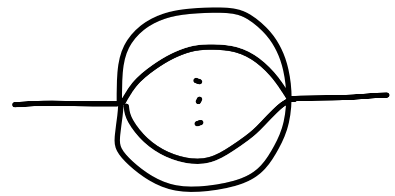
Banana Integrals in D=2

- Associated CY: $\Rightarrow$ Two different descriptions

not equal

$$M^{\mathrm{HS}}_{l-1} = \{\mathcal{F}_l(p^2,\underline{m}^2;\underline{x}) = 0 \,|\, (x_1 : \ldots : x_{l+1}) \in \mathbb{P}^l\}$$

$$\mathcal{F}_l(p^2,\underline{m}^2;\underline{x}) = \left(-p^2 + \left(\sum_{i=1}^{l+1}\frac{1}{x_i}\right)\left(\sum_{i=1}^{l+1} m_i^2 x_i\right)\right)\prod_{i=1}^{l+1} x_i$$



$$M^{\mathrm{CI}}_{l-1} = \left(\left.\begin{array}{c\|cc} \mathbb{P}^1_1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 & 1 \end{array}\right\} l+1\right) \subset \left(\left.\begin{array}{c\|c} \mathbb{P}^1_1 & 1 \\ \vdots & \vdots \\ \mathbb{P}^1_{l+1} & 1 \end{array}\right\} l+1\right) = F_l \subset \bigotimes_{i=1}^{l+1} \mathbb{P}^1_{(i)}$$

Banana Integrals in D=2

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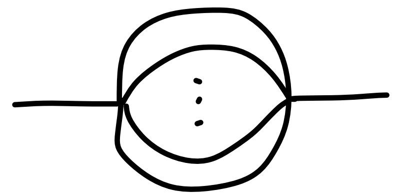

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same periods
cohom groups

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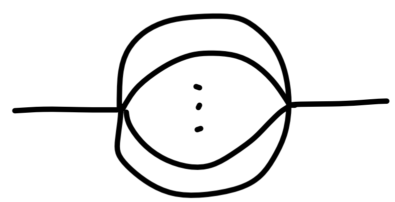

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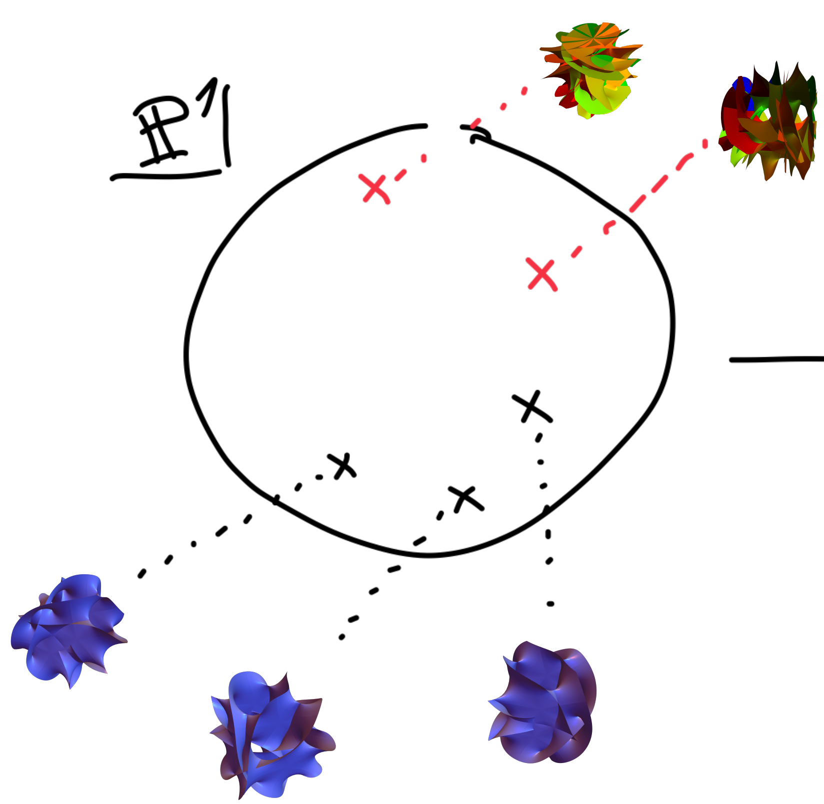
- Parameter space for $z = \dfrac{m^2}{p^2}$:



$$\Pi_i = \int_{\mathcal{C}_i} \Omega(z)$$

Picard-Fuchs differential equation:

$$\mathcal{L}_n \underline{\Pi} = 0$$

$$\begin{aligned}\mathcal{L}_{l=4} = & 1 - 5z - (4-28z)\theta + \left(6 - 63z + 26z^2 - 225z^3\right)\theta^2 \\ & - \left(4 - 70z + 450z^3\right)\theta^3 + (1-z)(1-9z)(1-25z)\theta^4\end{aligned}$$

$$\theta = z\partial_z$$

$$\Pi : H_n(M_n,\mathbb{Z}) \times H^n(M_n,\mathbb{C}) \longrightarrow \mathbb{C}$$

Basis of Cycles
max cuts

Basis of forms
master integrals

Banana Integrals in D=2

- Frobenius method:

  @ MUM-pt. $\varpi_k(z) = z^\alpha \sum_{j=0}^{k} \frac{1}{(k-j)!} \log^{k-j}(z) \Sigma_j(z),$

  $\text{for } 0 \leq k \leq l-1$

  → Fast and efficient way to get periods

→ Basis of solutions $\underline{\varpi}$, $\underline{\Pi} = \mathbf{T}\underline{\varpi}$

$\underline{l=4}$

$$\varpi_0 = \Sigma_0 = z + 5z^2 + 45z^3 + 545z^4 + \ldots$$

$$\Sigma_1 = 8z^2 + 100z^3 + \frac{4148}{3}z^4 + \ldots$$

$$\Sigma_2 = z^2 + \frac{197}{4}z^3 + \frac{33\,637}{36}z^4 + \ldots$$

$$\Sigma_3 = -2z^2 + \frac{89}{4}z^3 - \frac{19\,295}{108}z^4 + \ldots$$

[1]

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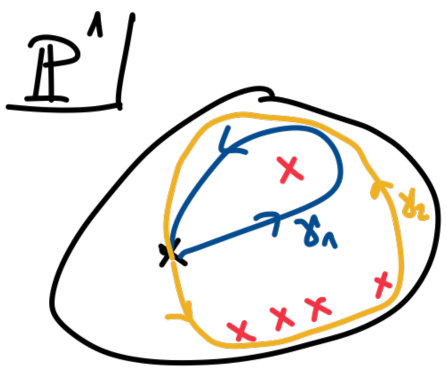
- Singularity structure:

$$\{0,\infty\} \cup \bigcup_{j=0}^{\lceil \frac{l-1}{2} \rceil} \left\{ \frac{1}{(l+1-2j)^2} \right\}$$

$\underline{l=4}$

$$\left\{0, \frac{1}{25}, \frac{1}{9}, 1, \infty\right\}$$

$\longrightarrow$ Monodromies



$$\underline{\Pi} \xrightarrow{\gamma_1} M\,\underline{\Pi}$$
$$\longrightarrow \mathbb{L}\,\underline{\Pi}$$

[1]

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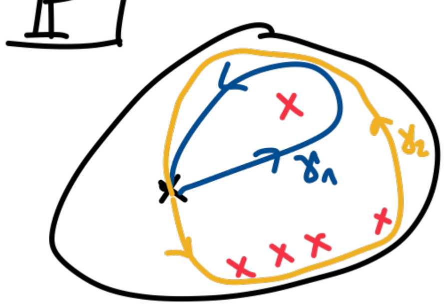
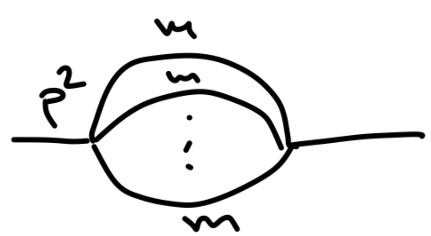
$\underline{\mathbb{P}^1}$



$$\underline{\Pi} \xrightarrow{\gamma_1} M\,\underline{\Pi}$$
$$\longrightarrow \mathbb{1}\,\underline{\Pi}$$

- Full F-integral:

$\ell)$ 

$\longleftrightarrow$ CY mfd $\longleftrightarrow$ $\underline{\Pi}_\ell(z) =$ CY periods

$$I_\ell(z) \sim \underline{\Pi}_\ell^{(z)}\, \Sigma \int_0^z \frac{dz'}{z'^2}\, \underline{\Pi}_\ell(z')$$

[1]

→ Need only initial condition for differential eq. (Gamma class)

Lessons from CY geometry for F-Integrals

Associate a CY geometry to a F-graph $\longrightarrow$ Geometric interpretation/methods [1]

- For one-parameter there are simple conditions to be actually a CY

- Gamma class: $I_l(z(t)) = \int_{F_l} e^{\omega \cdot t} \widehat{\Gamma}_{F_l}(TF_l) + \mathcal{O}(e^{-t})$

$$I_4(z) = (-5\pi^4 + 80i\pi\zeta(3))z + (20i\pi^3 + 80\zeta(3))z\log(z) + 30\pi^2 z^2 \log(z) - 20i\pi z^3 \log(z) - 5z^4 \log(z) + \mathcal{O}(z^2)$$

initial condition as topological integral

- Quadratic relations:

$$0 = \int_{M_{l-1}} \Omega(z) \wedge \Omega(z) = \underline{\Pi}_l^T \mathbf{\Sigma} \underline{\Pi}_l$$
$$0 = \int_{M_{l-1}} \Omega(z) \wedge \partial_z \Omega(z) = \underline{\Pi}_l^T \mathbf{\Sigma} \partial_z \underline{\Pi}_l$$
$$\vdots$$
$$C_l(z) = \frac{1}{z^{l-3} \prod_{k \in \{\text{sing pts}\}} (1 - 1/kz)} \qquad 0 = \int_{M_{l-1}} \Omega(z) \wedge \partial_z^{l-2} \Omega(z) = \underline{\Pi}_l^T \mathbf{\Sigma} \partial_z^{l-2} \underline{\Pi}_l$$
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whole set of quadratic relations between max cuts

$$\mathbf{Z}_l(z) = \mathbf{W}_l(z) \mathbf{\Sigma} \mathbf{W}_l(z)^T$$
$$\longrightarrow \quad \mathbf{W}_l^{-1}(z) = \mathbf{\Sigma} \mathbf{W}_l(z)^T \mathbf{Z}_l(z)^{-1}$$

- Landmann's Theorem: $(\mathbf{M}^k - \mathbb{1})^{n+1}, \quad k \in \mathbb{N}$ $\qquad I_G(\Delta) \sim \log^m(\Delta) \quad \Rightarrow \quad \dim(M) \geq m$

  algebraic n-dim mfd

- Transcendental weight: At MUM-point Frobenius basis has special logarithmic structure

  For CYs not all logarithmic degenerations are possible

- Modular properties: Only for $l \leq 3$

  K3 surface associated to $l = 3$ is a symmetric square

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$$\vdots$$

$$0 = \int_{M_{l-1}} \Omega(z)\wedge\partial_z^{l-2}\Omega(z) = \underline{\Pi}_l^T \mathbf{\Sigma}\partial_z^{l-2}\underline{\Pi}_l$$

$$C_l(z) = \int_{M_{l-1}} \Omega(z)\wedge\partial_z^{l-1}\Omega(z) = \underline{\Pi}_l^T \mathbf{\Sigma}\partial_z^{l-1}\underline{\Pi}_l$$

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$$\mathbf{Z}_l(z) = \mathbf{W}_l(z)\mathbf{\Sigma}\mathbf{W}_l(z)^T$$

$$\longrightarrow \quad \mathbf{W}_l^{-1}(z) = \mathbf{\Sigma}\mathbf{W}_l(z)^T\mathbf{Z}_l(z)^{-1}$$

- Landmann's Theorem: $(\mathbf{M}^k - \mathbb{1})^{n+1}, \quad k\in\mathbb{N}$ $\qquad I_G(\Delta)\sim\log^m(\Delta) \quad\Rightarrow\quad \dim(M)\geq m$

  algebraic n-dim mfd

- Transcendental weight: At MUM-point Frobenius basis has special logarithmic structure

  For CYs not all logarithmic degenerations are possible

- Modular properties: Only for $l\leq 3$

  K3 surface associated to $l=3$ is a symmetric square

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Banana Integrals in Dimensional Regularization

- What can we say for $D = 2 - 2\epsilon$ $\longrightarrow$ Similar story [1]

  - One can construct PF differential equations with ϵ-dependence

$$\begin{aligned}\mathcal{L}_{l=4,\epsilon} =&(1+2\epsilon)(1+3\epsilon)(1+4\epsilon)(1+\epsilon-5z+3z\epsilon)\\&+(-4-30\epsilon+28z+189z\epsilon+26z^2\epsilon-225z^3\epsilon-70\epsilon^2+343z\epsilon^2-225z^3\epsilon^2-50\epsilon^3+84z\epsilon^3+414z^2\epsilon^3)\theta\\&+(6-63z+26z^2-225z^3+30\epsilon-315z\epsilon-675z^3\epsilon+35\epsilon^2-343z\epsilon^2-363z^2\epsilon^2-225z^3\epsilon^2)\theta^2\\&-2\left(2-35z+225z^3+5\epsilon-105z\epsilon+259z^2\epsilon+225z^3\epsilon\right)\theta^3+(1-z)(1-9z)(1-25z)\theta^4\end{aligned}$$

  - Initial condition from hypergeometric series expansion of F-integral (generalized Gamma class)

$$I_l(z;2-2\epsilon) = -\sum_{k=1}^{l+1}\binom{l+1}{k}\frac{\Gamma(-\epsilon)^k\,\Gamma(\epsilon)^{l+1-k}}{\Gamma(-k\epsilon)}\frac{\Gamma(1+(k-1)\epsilon)}{\Gamma(1+l\epsilon)}e^{(k-1)i\pi\epsilon}\,z^{1+(k-1)\epsilon}+\mathcal{O}\left(z^2\right)$$

  - Extended CY geometry (left-extended PF equations, iterated integrals of CY periods)

$$\mathcal{L}_l^{(n,\text{inh})}\mathcal{L}_l^{(n)}\cdots\ \mathcal{L}_l^{(1,\text{inh})}\mathcal{L}_l^{(1)}\ \mathcal{L}_l^{(0,\text{inh})}\mathcal{L}_l^{(0)}\ I_{l,n}=0 \qquad\qquad I_l(z;2-2\epsilon)=\sum_{i=0}^{\infty}I_{l,i}(z)\epsilon^i$$

$\longrightarrow$ Banana integrals can be computed in equal— and generic-mass case also in dim reg!

Conclusions

- Banana integrals can be solved in $\mathrm{D}{=}2$ as well as in $\mathrm{D}{=}2\text{-}2\epsilon$ dimensions

  Equal-mass case ✓ Generic-mass case ✓

- Many interesting implications from CY geometry for F-integral

  Quadratic relations, Gamma class, relation log power and dim mfd, ...

- Not unique geometry but same periods

Further Questions:

- General connection between CY manifolds and F-integrals has to be elaborated
- Which (class of) Feynman graphs correspond to CY manifolds?
- Are there more general objects than CY manifolds or motives?

Can one associate a CY motive to every F-integral?

Thank you for your attention

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