A Double Copy for Celestial Amplitudes

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QCD meets gravity "@" Northwestern, 30 November 2020

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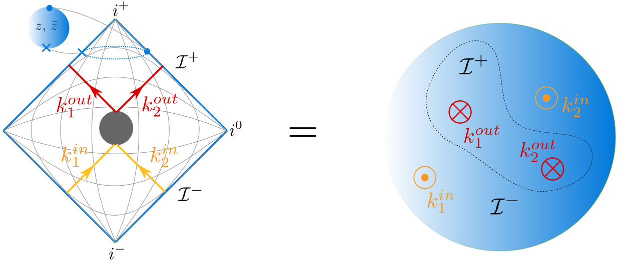
Motivation: Celestial Amplitudes

Conformal Basis for QFT Scattering Amplitudes:

Asymptotic states: plane waves → conformal primary wavefunctions

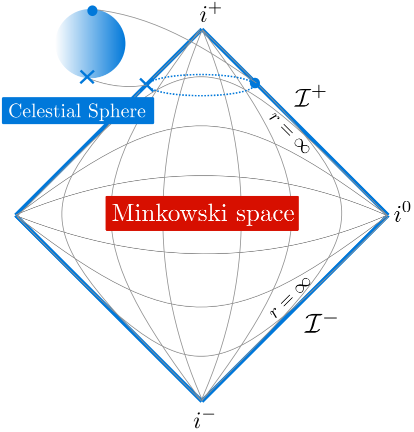
4D Lorentz group $SO(1,3) \simeq SL(2,\mathbb{C})$ acts as global conformal group on celestial sphere: *4D scattering amplitudes recast as 2D correlators!*



- Constraints on Quantum Gravity $\mathcal{S}$-Matrix
- New insights into amplitude structures

Motivation: Flat Space Holography

Is Bulk Physics Encoded on Conformal Boundary ?



What are the symmetries?

- Poincaré
- BMS
- Virasoro/Diff(S^2)
- ...

Dual celestial CFT?

- conformal primaries
- asymptotic symmetry generators
- celestial correlators
- OPEs
- ...

- Non-perturbative def of Quantum Gravity in flat space

Plan of the talk

I. Conformal Basis and Celestial Amplitudes

- From Plane Waves to Conformal Primaries
- Celestial Amplitudes as Conformal Correlators

II. Double Copy for Celestial Amplitudes

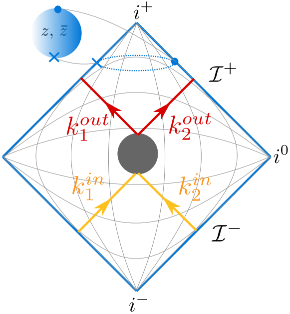
- Momentum-space Double Copy *revisited*
- Celestial Double Copy

III. Discussion

I. Conformal Basis and Celestial Amplitudes

A new basis for QFT scattering amplitudes

4D Spacetime



4D $\mathcal{S}$cattering matrix

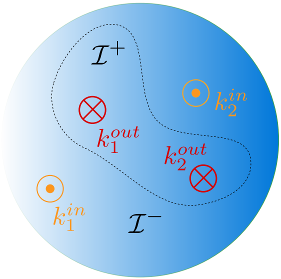
$$\langle out|\mathcal{S}|in\rangle$$

$$k_j^\mu = \pm\omega_j q^\mu(z_j, \bar{z}_j), \ell_j$$

4D Lorentz symmetry

$=$

2D Celestial Sphere



2D correlation function

$$\langle\mathcal{O}_1(z_1, \bar{z}_1)\ldots\mathcal{O}_n(z_n, \bar{z}_n)\rangle$$

$$(z_j, \bar{z}_j),\ \Delta_j,\ J_j$$

2D conformal symmetry

From momentum basis to conformal basis

[de Boer,Solodukhin'03]

Momentum Basis	*Mellin transform* $\longrightarrow$	Conformal Basis
$k^\mu = \pm\omega q^\mu(z,\bar{z})\,,\ \ell$		$z = \dfrac{k^1+ik^2}{k^3+k^0}\,,\ \Delta\,,\ J=\ell$
momentum eigenstates		boost eigenstates
$e^{ik\cdot X}$	$\xrightarrow{\int_0^\infty d\omega\omega^{\Delta-1}}$	$\dfrac{\mathcal{N}(\Delta)}{(-q\cdot X)^\Delta} \equiv \Phi^\Delta$

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$\epsilon_{\mu;\ell}\, e^{ik\cdot X}$		$\epsilon_{\mu;J}\Phi^\Delta \equiv V^\Delta_{\mu;J}$

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basis of $\omega \geqslant 0$ wavefunctions		basis of $\Delta \in 1+i\mathbb{R}$ wavefunctions *principal continuous series* of $SL(2,\mathbb{C})$ [Pasterski,Shao'17]
$\vert\omega, z, \bar{z}, \ell\rangle$		$\mathcal{O}^\Delta_J(z,\bar{z}) = \int_0^\infty d\omega\omega^{\Delta-1}\vert\omega, z, \bar{z}, \ell\rangle$

Celestial Amplitudes

Change of basis of asymptotic states in amplitudes achieved by a Mellin transform on each external particle:

[Pasterski,Shao,Strominger'17][Cheung,de la Fuente,Sundrum'17]

4D AMPLITUDES $\xrightarrow{\textit{Mellin transform}}$ 2D CORRELATORS

$$\mathcal{A}_{\ell_1\ldots\ell_n}(\omega_1, z_1, \bar{z}_1, \ldots, \omega_n, z_n, \bar{z}_n) \xrightarrow{\prod_{j=1}^n \int_0^\infty d\omega_j \omega_j^{\Delta_j - 1}} \widetilde{\mathcal{A}}_{J_1\ldots J_n}(\Delta_1, z_1, \bar{z}_1, \ldots, \Delta_n, z_n, \bar{z}_n)$$

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$$= A_{\ell_1\ldots\ell_n}\delta^{(4)}\Big(\sum_{j=1}^{n} k_j^\mu\Big)$$

manifest translation symmetry

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manifest translation symmetry — *manifest conformal symmetry*

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$= A_{\ell_1\ldots\ell_n}\delta^{(4)}\Big(\sum_{j=1}^n k_j^\mu\Big)$		$= \langle \mathcal{O}_{J_1}^{\Delta_1}(z_1, \bar{z}_1)\ldots\mathcal{O}_{J_n}^{\Delta_n}(z_n, \bar{z}_n)\rangle$
manifest translation symmetry		*manifest conformal symmetry*

Under $SL(2,\mathbb{C})$ Lorentz transformations $z \to \frac{az+b}{cz+d}$ with $ad - bc = 1$ celestial amplitudes transform as 2D conformal correlation functions

[Pasterski,Shao,Strominger'17]

$$\widetilde{\mathcal{A}}_n\Big(\Big\{\Delta_j, J_j; \frac{az_j+b}{cz_j+d}, \frac{\bar{a}\bar{z}_j+\bar{b}}{\bar{c}\bar{z}_j+\bar{d}}\Big\}\Big) = \prod_{j=1}^n \Big((cz_j+d)^{\Delta_j+J_j}(\bar{c}\bar{z}_j+\bar{d})^{\Delta_j-J_j}\Big)\widetilde{\mathcal{A}}_n(\{\Delta_j, J_j; z_j, \bar{z}_j\})$$

where $\Delta_j = 1 + i\lambda_j$ are the conformal dimensions and $J_j \equiv \ell_j$ the spins of operators $\mathcal{O}_{J_j}^{\Delta_j}$ inserted at points $(z_j, \bar{z}_j) \in S^2$.

Celestial Amplitudes

Change of basis of asymptotic states in amplitudes achieved by a Mellin transform on each external particle:

[Pasterski,Shao,Strominger'17][Cheung,de la Fuente,Sundrum'17]

4D AMPLITUDES $\xrightarrow{\textit{Mellin transform}}$ 2D CORRELATORS

$$\mathcal{A}_{\ell_1\ldots\ell_4}(\omega_1, z_1, \bar{z}_1, \ldots, \omega_4, z_4, \bar{z}_4) \xrightarrow{\prod_{j=1}^4 \int_0^\infty d\omega_j \omega_j^{\Delta_j - 1}} \widetilde{\mathcal{A}}_{J_1\ldots J_4}(\Delta_1, z_1, \bar{z}_1, \ldots, \Delta_4, z_4, \bar{z}_4)$$

$$= A_{\ell_1\ldots\ell_4}\,\delta^{(4)}\Big(\sum_{j=1}^n k_j^\mu\Big) \qquad = f(z,\bar{z}) \prod_{i<j}^4 z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}$$

$z_{ij} = z_i - z_j,\ z = \frac{z_{12}z_{34}}{z_{13}z_{24}},\ (h_j, \bar{h}_j) = \frac{1}{2}(\Delta_j, J_j),\ h = \sum_{j=1}^4 h_j,\ \bar{h} = \sum_{j=1}^4 \bar{h}_j$

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- MHV 4 gluon amplitude

$$A^{\rm YM}_{--++} = \frac{\omega_1\omega_2}{\omega_3\omega_4}\frac{z_{12}^3}{z_{23}z_{34}z_{41}} \quad\longrightarrow\quad f_{\rm YM}(z,\bar{z}) = z^{\frac{5}{3}}(1-z)^{-\frac{1}{3}}\delta(z-\bar{z})\Theta(z-1)\int_0^\infty d\omega_4\,\omega_4^{i\sum_{j=1}^4 \lambda_j - 1}$$

- MHV 4 graviton amplitude

$$A^{\rm G}_{--++} = \frac{\omega_1^3\omega_2^3}{\omega_3^2\omega_4^2}\frac{z_{12}^7\bar{z}_{12}}{z_{13}z_{14}z_{23}z_{24}z_{34}^2} \quad\longrightarrow\quad f_{\rm G}(z,\bar{z}) = z^{\frac{10}{3}}(1-z)^{-\frac{2}{3}}\delta(z-\bar{z})\Theta(z-1)\int_0^\infty d\omega_4\,\omega_4^{i\sum_{j=1}^4 \lambda_j + 1}$$

$$= s_{14}(A^{\rm YM}_{--++})^2 + (3 \leftrightarrow 4)$$

where $s_{14} = \omega_1\omega_4 z_{14}\bar{z}_{14}$

II. Double Copy for Celestial Amplitudes

Double copy

Gravity = (Yang-Mills)^2: [Bern,Carrasco,Johansson'10] [...]

[Bern,Carrasco,Chiodaroli,Johansson,Roiban'19]

color → c_γ, kinematic → n_γ

Yang-Mills: $$\mathcal{A}_n^{YM} = \delta^{(4)}\Big(\sum_{j=1}^{n} k_j^\mu\Big) \sum_{\gamma\in\Gamma_n} \frac{c_\gamma\, n_\gamma}{\Pi_\gamma}$$

Lie algebra Jacobi identity: $c_s - c_t + c_u = 0$

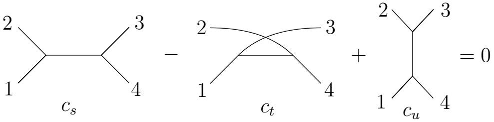

Figure: *Triple of graphs related by a BCJ move.* [Bern,Carrasco,Johansson'08]

Color-kinematics dual (BCJ) numerators: $n_s - n_t + n_u = 0$

Gravity: $$\mathcal{A}_n^{G} = \delta^{(4)}\Big(\sum_{j=1}^{n} k_j^\mu\Big) \sum_{\gamma\in\Gamma_n} \frac{n_\gamma^2}{\Pi_\gamma}$$

Celestial double copy ?

The momentum basis double copy makes heavy use of $\sum_{j=1}^{n} k_j^\mu = 0$.

Puzzle: Conformal basis *obscures translation symmetry*
$\Rightarrow$ Double copy for celestial amplitudes ?

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Puzzle: Conformal basis *obscures translation symmetry*
$\Rightarrow$ Double copy for celestial amplitudes ?

Most natural generalization when computing amplitudes using the usual Feynman rules in a setting that lacks momentum conservation makes use of integral representation

$$\delta^{(4)}\Big(\sum_{j=1}^{n} k_j^\mu\Big) = \int \frac{d^4 X}{(2\pi)^4}\, e^{\sum_{j=1}^{n} i\, k_j \cdot X}\,.$$

Procedure introduced in [Adamo,Casali,Mason,Nekovar'17] in the context of scattering on plane wave backgrounds.

Celestial double copy requires generalization as for curved space double copy but without complications of curved space.

Three-point double copy revisited

Spin-one and spin-two wavefunctions:

$$a_\mu = T^{\mathfrak{a}} \epsilon_\mu e^{ik\cdot X}, \quad h_{\mu\nu} = \epsilon_\mu \epsilon_\nu e^{ik\cdot X}$$

Three-point double copy revisited

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Three-point vertices contributing to the amplitudes:

$$\mathcal{A}_3^{YM} = \frac{1}{2} \mathrm{Tr} \int \frac{d^4 X}{(2\pi)^4} \left(a_1^\mu a_2^\nu \partial_\nu a_{3\mu} - a_2^\mu a_1^\nu \partial_\nu a_{3\mu} + \text{cyclic} \right)$$

$$\mathcal{A}_3^{G} = \frac{1}{2} \int \frac{d^4 X}{(2\pi)^4} \left(h_1^{\mu\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\rho\sigma} - 2 h_1^{\rho\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\mu\sigma} + \text{ permutations} \right)$$

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$$= \frac{1}{4}\int \frac{d^4X}{(2\pi)^4}\left(\epsilon_1\cdot(k_2-k_3)\,\epsilon_2\cdot\epsilon_3 + \text{cyclic}\right)^2\prod_{j=1}^{3} e^{ik_j\cdot X} \quad \text{via integration by parts}$$

Makes use of: $\boxed{\partial_\mu e^{ik\cdot X} = ik_\mu e^{ik\cdot X}}$ where $\partial_\mu k_\nu = 0$

Stripped off $\prod_{j=1}^{3} e^{ik_j\cdot X}$, integrands double copy !

Naive celestial three-point double copy

Spin-one and spin-two *conformal* wavefunctions:

$$V_\mu(X) = T^{\mathfrak{a}} \epsilon_\mu \phi(X)\,, \quad V_{\mu\nu}(X) = \epsilon_\mu \epsilon_\nu \phi(X)$$

Recall the scalar wavefunctions

$$\phi(X) = \int_0^\infty d\omega \omega^{\Delta-1} e^{ik\cdot X} = \frac{\mathcal{N}(\Delta)}{(-q\cdot X)^\Delta}$$

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Celestial three-point amplitudes:

$$\widetilde{\mathcal{A}}_3^{\text{YM}} = f^{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3} \frac{1}{2} \int \frac{d^4 X}{(2\pi)^4} \left((\epsilon_1 \cdot (K_2 - K_3)\; \epsilon_2 \cdot \epsilon_3 + \text{cyclic} \right) \prod_{j=1}^{3} \phi_j$$

$$\widetilde{\mathcal{A}}_3^{G} = \frac{1}{2} \int \frac{d^4 X}{(2\pi)^4} \left((\epsilon_2 \cdot \epsilon_3)^2\; \epsilon_1 \cdot K_2\; \epsilon_1 \cdot K_3 - 2\, \epsilon_1 \cdot \epsilon_2\; \epsilon_2 \cdot \epsilon_3\; \epsilon_3 \cdot K_2\; \epsilon_1 \cdot K_3 + \text{perm} \right) \prod_{j=1}^{3} \phi_j$$

Naive celestial three-point double copy *fails!*

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$$\neq \frac{1}{4} \int \frac{d^4X}{(2\pi)^4} \left((\epsilon_1 \cdot (K_2 - K_3)\, \epsilon_2\cdot\epsilon_3 + \text{cyclic} \right)^2 \prod_{j=1}^{3} \phi_j \quad \text{via integration by parts}$$

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Celestial three-point amplitudes:

$$\widetilde{\mathcal{A}}_3^{\rm YM} = f^{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3} \frac{1}{2} \int \frac{d^4X}{(2\pi)^4} \left((\epsilon_1 \cdot (K_2 - K_3)\ \epsilon_2 \cdot \epsilon_3 + \text{cyclic}\right) \prod_{j=1}^3 \phi_j$$

$$\widetilde{\mathcal{A}}_3^{G} = \frac{1}{2} \int \frac{d^4X}{(2\pi)^4} \left((\epsilon_2 \cdot \epsilon_3)^2\ \epsilon_1 \cdot K_2\ \epsilon_1 \cdot K_3 - 2\,\epsilon_1 \cdot \epsilon_2\ \epsilon_2 \cdot \epsilon_3\ \epsilon_3 \cdot K_2\ \epsilon_1 \cdot K_3 + \text{perm}\right) \prod_{j=1}^3 \phi_j$$

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Proposal: promote the generalized celestial momenta K_i in the numerators to *differential operators* $\mathcal{K}_i$ defined to act on the conformal wavefunctions ϕ_i as [Casali,AP'20]

$$\boxed{\mathcal{K}_i^\mu \phi_j(X) = K_i^\mu(X)\phi_i(X)\delta_{ij}}$$

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Notice that the $\mathcal{K}_i$ keep track of particle labels and thus are *not equivalent to the usual partial derivatives*!

Celestial Double Copy

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Notice that the $\mathcal{K}_i$ keep track of particle labels and thus are *not equivalent to the usual partial derivatives*!

Instead they admit a representation as

$$\mathcal{K}_i^\mu = q_i^\mu e^{\partial_{\Delta_i}}$$

which is the action of the *translation generator* of the Poincaré algebra acting on the coordiantes of the i-th particle [Stieberger,Taylor'18].

Celestial three-point double copy

Yang-Mills numerator promoted to operator

$$\mathcal{N}_{YM} \prod_{j=1}^{3} \phi_j(X) = \frac{1}{4} \left(\epsilon_1 \cdot (\mathcal{K}_2 - \mathcal{K}_3)\ \epsilon_2 \cdot \epsilon_3 + \text{cyclic} \right) \prod_{j=1}^{3} \phi_j(X)$$

Gravity numerator promoted to operator

$$\mathcal{N}_G \prod_{j=1}^{3} \phi_j(X) = \left(-(\epsilon_2 \cdot \epsilon_3)^2 \epsilon_1 \cdot \mathcal{K}_2\ \epsilon_1 \cdot \mathcal{K}_3 + 2 \epsilon_1 \cdot \epsilon_2\ \epsilon_2 \cdot \epsilon_3\ \epsilon_3 \cdot \mathcal{K}_2\ \epsilon_1 \cdot \mathcal{K}_3 + \text{perm} \right) \prod_{j=1}^{3} \phi_j(X)$$

Celestial three-point double copy

Yang-Mills numerator^2 promoted to operator

$$(\mathcal{N}_{YM})^2 \prod_{j=1}^{3} \phi_j(X) = \frac{1}{4} \left(\epsilon_1 \cdot (\mathcal{K}_2 - \mathcal{K}_3)\ \epsilon_2 \cdot \epsilon_3 + \text{cyclic}\right)^2 \prod_{j=1}^{3} \phi_j(X)$$

Gravity numerator promoted to operator

$$\mathcal{N}_G \prod_{j=1}^{3} \phi_j(X) = \left(-(\epsilon_2 \cdot \epsilon_3)^2 \epsilon_1 \cdot \mathcal{K}_2\ \epsilon_1 \cdot \mathcal{K}_3 + 2 \epsilon_1 \cdot \epsilon_2\ \epsilon_2 \cdot \epsilon_3\ \epsilon_3 \cdot \mathcal{K}_2\ \epsilon_1 \cdot \mathcal{K}_3 + \text{perm}\right) \prod_{j=1}^{3} \phi_j(X)$$

Celestial three-point double copy

Yang-Mills numerator[2] promoted to operator

$$(\mathcal{N}_{YM})^2 \prod_{j=1}^{3} \phi_j(X) = \frac{1}{4} \left(\epsilon_1 \cdot (\mathcal{K}_2 - \mathcal{K}_3)\, \epsilon_2 \cdot \epsilon_3 + \text{cyclic} \right)^2 \prod_{j=1}^{3} \phi_j(X)$$

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Extra terms from integration by parts:

$$\partial_\mu \partial_\nu \phi_i(X) = \left(1 + \frac{1}{\Delta_i} \right) K_i^\mu(X) K_i^\nu(X) \phi_i(X)$$

Celestial three-point double copy

Yang-Mills numerator^2 promoted to operator

$$(\mathcal{N}_{YM})^2 \prod_{j=1}^{3} \phi_j(X) = \frac{1}{4} \left(\epsilon_1 \cdot (\mathcal{K}_2 - \mathcal{K}_3)\ \epsilon_2 \cdot \epsilon_3 + \text{cyclic}\right)^2 \prod_{j=1}^{3} \phi_j(X)$$

Extra terms from action of celestial momentum operators:

$$\mathcal{K}_i^\mu \mathcal{K}_i^\nu \phi_j(X) = \left(1 + \frac{1}{\Delta_i}\right) K_i^\mu(X) K_i^\nu(X) \phi_i(X) \delta_{ij}$$

Gravity numerator promoted to operator

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Celestial three-point double copy

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$$(\mathcal{N}_{YM})^2\prod_{j=1}^{3}\phi_j(X)=\frac{1}{4}\left(\epsilon_1\cdot(\mathcal{K}_2-\mathcal{K}_3)\,\epsilon_2\cdot\epsilon_3+\text{cyclic}\right)^2\prod_{j=1}^{3}\phi_j(X)$$

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Gravity numerator promoted to operator

$$\mathcal{N}_G\prod_{j=1}^{3}\phi_j(X)=\left(-(\epsilon_2\cdot\epsilon_3)^2\epsilon_1\cdot\mathcal{K}_2\;\epsilon_1\cdot\mathcal{K}_3+2\epsilon_1\cdot\epsilon_2\;\epsilon_2\cdot\epsilon_3\;\epsilon_3\cdot\mathcal{K}_2\;\epsilon_1\cdot\mathcal{K}_3+\text{perm}\right)\prod_{j=1}^{3}\phi_j(X)$$

Extra terms from integration by parts:

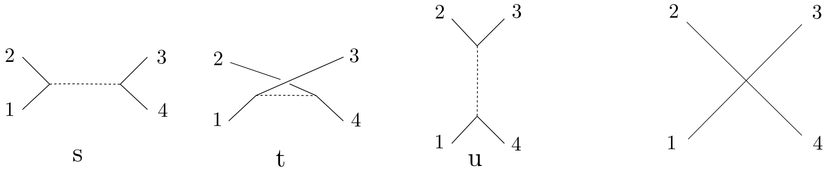
$$\partial_\mu\partial_\nu\phi_i(X)=\left(1+\frac{1}{\Delta_i}\right)K_i^{\mu}(X)K_i^{\nu}(X)\phi_i(X)$$

Extra terms precisely match: $\boxed{\mathcal{N}_G\prod_{i=1}^{3}\phi_i=(\mathcal{N}_{YM})^2\prod_{i=1}^{3}\phi_i}$

The celestial three-gluon amplitude double copies into the celestial three-graviton amplitude!

Four-point amplitudes

$$\mathcal{A}_4^{YM} = \mathcal{A}_4^s + \mathcal{A}_4^t + \mathcal{A}_4^u + \mathcal{A}_4^{\text{contact}}$$



The propagator

$$G_{\mu\nu}(X, Y) = \eta_{\mu\nu}\, G(X, Y) = \eta_{\mu\nu} \int d^4k \frac{e^{ik\cdot(X-Y)}}{k^2}$$

satisfies $\Box_X G(X, Y) = (2\pi)^4 \delta^{(4)}(X - Y) = \Box_Y G(X, Y)$.

Proceed by opening up the contact term into s, t, and u contributions.

Four-point double copy revisited

Massaging the four gluon amplitude gives

$$\mathcal{A}_4^{YM} = \int \frac{d^4 X}{(2\pi)^4} \frac{d^4 Y}{(2\pi)^4} \, G(X, Y) \left(c_s n_s \Phi_s + c_t n_t \Phi_t + c_u n_u \Phi_u \right)$$

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where

$\Phi_s = e^{i(k_1+k_2)\cdot X} e^{i(k_3+k_4)\cdot Y}$ $\Phi_t = e^{i(k_1+k_3)\cdot X} e^{i(k_2+k_4)\cdot Y}$ $\Phi_u = e^{i(k_1+k_4)\cdot X} e^{i(k_2+k_3)\cdot Y}$

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and with color factors ($c_s - c_t + c_u = 0$ by Jacobi)

$$c_s = f^{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{b}} f^{\mathfrak{a}_3 \mathfrak{a}_4 \mathfrak{b}} \qquad c_t = f^{\mathfrak{a}_1 \mathfrak{a}_3 \mathfrak{b}} f^{\mathfrak{a}_2 \mathfrak{a}_4 \mathfrak{b}} \qquad c_u = f^{\mathfrak{a}_1 \mathfrak{a}_4 \mathfrak{b}} f^{\mathfrak{a}_2 \mathfrak{a}_3 \mathfrak{b}}$$

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and kinematic numerators ($2 \leftrightarrow 3$ for n_t, $(2,3,4) \to (4,2,3)$ for n_u)

$$\begin{aligned} n_s = & -[\epsilon_1 \cdot \epsilon_2\ (k_1 - k_2)^\mu + 2\ \epsilon_1 \cdot k_2\ \epsilon_2^\mu - 2\ \epsilon_2 \cdot k_1\ \epsilon_1^\mu]\ \eta_{\mu\nu} \\ & \times [\epsilon_3 \cdot \epsilon_4\ (k_4 - k_3)^\nu - 2\ \epsilon_3 \cdot k_4\ \epsilon_4^\nu + 2\epsilon_4 \cdot k_3\ \epsilon_3^\nu] \\ & -(\epsilon_1 \cdot \epsilon_3\ \epsilon_2 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_4\ \epsilon_2 \cdot \epsilon_3)(k_1 \cdot k_2 + k_3 \cdot k_4) \end{aligned}$$

To show color-kinematics $n_s - n_t + n_u = 0$ use IBP and $\partial_X G(X, Y) = -\partial_Y G(X, Y)$.

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The four graviton amplitude is obtained by substituting $\boxed{c \to n}$:

$$\mathcal{A}_4^{G} = \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4}\, G(X,Y)\left((n_s)^2 \Phi_s + (n_t)^2 \Phi_t + (n_u)^2 \Phi_u\right)$$

Four-point double copy revisited

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Stripped off Φ's and propagator, integrands in each channel square.

Naive celestial four-point double copy

From the four gluon amplitude $\mathcal{A}_4^{YM}$ via Mellin transform (effectively $e^{ik_j \cdot X} \to \phi_j$, $k_j^\mu \to K_j^\mu$) we get the celestial four gluon amplitude

$$\widetilde{\mathcal{A}}_4^{YM} = \int \frac{d^4 X}{(2\pi)^4} \frac{d^4 Y}{(2\pi)^4} G(X, Y) \left(c_s \widetilde{n}_s \widetilde{\Phi}_s + c_t \widetilde{n}_t \widetilde{\Phi}_t + c_u \widetilde{n}_u \widetilde{\Phi}_u \right)$$

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where

$$\widetilde{\Phi}_s = \phi_1 \phi_2(X) \phi_3 \phi_4(Y) \quad \widetilde{\Phi}_t = \phi_1 \phi_3(X) \phi_2 \phi_4(Y) \quad \widetilde{\Phi}_u = \phi_1 \phi_4(X) \phi_2 \phi_3(Y)$$

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where

$$\widetilde{\Phi}_s = \phi_1 \phi_2(X) \phi_3 \phi_4(Y) \quad \widetilde{\Phi}_t = \phi_1 \phi_3(X) \phi_2 \phi_4(Y) \quad \widetilde{\Phi}_u = \phi_1 \phi_4(X) \phi_2 \phi_3(Y)$$

and with color factors

$$c_s = f^{\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{b}} f^{\mathfrak{a}_3 \mathfrak{a}_4 \mathfrak{b}} \quad c_t = f^{\mathfrak{a}_1 \mathfrak{a}_3 \mathfrak{b}} f^{\mathfrak{a}_2 \mathfrak{a}_4 \mathfrak{b}} \quad c_u = f^{\mathfrak{a}_1 \mathfrak{a}_4 \mathfrak{b}} f^{\mathfrak{a}_2 \mathfrak{a}_3 \mathfrak{b}}$$

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where

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and *spacetime dependent* kinematic numerators (similarly for $\widetilde{n}_t$ and $\widetilde{n}_u$)

$$\begin{aligned}\widetilde{n}_s = &\, [\epsilon_1 \cdot \epsilon_2 \, (K_1 - K_2)^\mu + 2\, \epsilon_1 \cdot K_2 \, \epsilon_2^\mu - 2\, \epsilon_2 \cdot K_1 \, \epsilon_1^\mu](X) \, \eta_{\mu\nu} \\ &\times [\epsilon_3 \cdot \epsilon_4 \, (K_4 - K_3)^\nu - 2\, \epsilon_3 \cdot K_4 \, \epsilon_4^\nu + 2\, \epsilon_4 \cdot K_3 \, \epsilon_3^\nu](Y) \\ &+ (\epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3)(K_1 \cdot K_2(X) + K_3 \cdot K_4(Y))\end{aligned}$$

Naive celestial four-point double copy *fails !*

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$$\widetilde{\mathcal{A}}_4^{YM} = \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4} G(X,Y)\left(c_s\widetilde{n}_s\widetilde{\Phi}_s + c_t\widetilde{n}_t\widetilde{\Phi}_t + c_u\widetilde{n}_u\widetilde{\Phi}_u\right)$$

where

$$\widetilde{\Phi}_s = \phi_1\phi_2(X)\phi_3\phi_4(Y) \quad \widetilde{\Phi}_t = \phi_1\phi_3(X)\phi_2\phi_4(Y) \quad \widetilde{\Phi}_u = \phi_1\phi_4(X)\phi_2\phi_3(Y)$$

and with color factors

$$c_s = f^{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{b}}f^{\mathfrak{a}_3\mathfrak{a}_4\mathfrak{b}} \quad c_t = f^{\mathfrak{a}_1\mathfrak{a}_3\mathfrak{b}}f^{\mathfrak{a}_2\mathfrak{a}_4\mathfrak{b}} \quad c_u = f^{\mathfrak{a}_1\mathfrak{a}_4\mathfrak{b}}f^{\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}}$$

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The naive celestial four-point double copy $c \to \widetilde{n}$ *does not work*:

$$\widetilde{\mathcal{A}}_4^{G} \neq \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4} G(X,Y)\left((\widetilde{n}_s)^2\widetilde{\Phi}_s + (\widetilde{n}_t)^2\widetilde{\Phi}_t + (\widetilde{n}_u)^2\widetilde{\Phi}_u\right)$$

Celestial four-point double copy

Celestial double copy prescription $\boxed{K_\mu(X) \to \mathcal{K}_\mu}$ gives:

$$\widetilde{\mathcal{A}}_4^{YM} = \int \frac{d^4X}{(2\pi)^4} \frac{d^4Y}{(2\pi)^4} G(X, Y) \left(c_s \widetilde{n}_s \widetilde{\Phi}_s + c_t \widetilde{n}_t \widetilde{\Phi}_t + c_u \widetilde{n}_u \widetilde{\Phi}_u \right)$$

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$\Downarrow$ promote $\boxed{\widetilde{n}(X,Y) \to \mathcal{N}}$

$$= \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4}\, G(X,Y)\left(c_s\mathcal{N}_s\widetilde{\Phi}_s + c_t\mathcal{N}_t\widetilde{\Phi}_t + c_u\mathcal{N}_u\widetilde{\Phi}_u\right)$$

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Celestial double copy prescription $\boxed{K_\mu(X) \to \mathcal{K}_\mu}$ gives:

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$$\Downarrow \text{ promote } \boxed{\widetilde{n}(X, Y) \to \mathcal{N}}$$

$$= \int \frac{d^4 X}{(2\pi)^4} \frac{d^4 Y}{(2\pi)^4} G(X, Y) \left(c_s \mathcal{N}_s \widetilde{\Phi}_s + c_t \mathcal{N}_t \widetilde{\Phi}_t + c_u \mathcal{N}_u \widetilde{\Phi}_u \right)$$

$$\Downarrow \text{ operator squaring via } \boxed{c \to \mathcal{N}}$$

$$\widetilde{\mathcal{A}}_4^{G} \stackrel{?}{=} \int \frac{d^4 X}{(2\pi)^4} \frac{d^4 Y}{(2\pi)^4} G(X, Y) \left((\mathcal{N}_s)^2 \widetilde{\Phi}_s + (\mathcal{N}_t)^2 \widetilde{\Phi}_t + (\mathcal{N}_u)^2 \widetilde{\Phi}_u \right)$$

To check this need to show that $\widetilde{\mathcal{A}}_4^G$ is the Mellin transform of $\mathcal{A}_4^G$.

Check of celestial four-point double copy

The four graviton amplitude

$$\mathcal{A}_4^G = \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4}\,G(X,Y)\left((n_s)^2\Phi_s + (n_t)^2\Phi_t + (n_u)^2\Phi_u\right)\,,$$

depends on the energy of each particle k_j^0 in a polynomial fashion.

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When Mellin transforming get numerator linear in the energy

$$\int d\omega_j \omega_j^{\Delta_j - 1}(ik_j^\mu)\,e^{ik_j\cdot X} = K_j^\mu\,\phi_j(X)$$

and numerator quadratic in the same energy

$$\int d\omega_j \omega_j^{\Delta_j - 1}(ik_j^\mu)(ik_j^\nu)\,e^{ik_j\cdot X} = \left(1 + \frac{1}{\Delta_j}\right)K_j^\mu K_j^\nu\,\phi_j(X)\,.$$

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This mirrors the action of the $\mathcal{K}_j$ operators on the ϕ_j and thus

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The four graviton amplitude

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Celestial four-gluon amplitude double copies
into celestial four-graviton amplitude!

III. Discussion

Comments and open questions I

- Interesting presentation for amplitudes:

$\widetilde{\mathcal{A}}^{scalar} \to \widetilde{\mathcal{A}}^{YM} \to \widetilde{\mathcal{A}}^{G}$ via *celestial momentum operator* $\mathcal{K} \ni e^{\partial_{\Delta_j}}$

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Three-point gluon and graviton amplitudes from scalar amplitudes:

$$\widetilde{\mathcal{A}}_3^{\mathrm{YM}} = f^{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3}\mathcal{N}_{YM}\int \frac{d^4X}{(2\pi)^4}\prod_{j=1}^{3}\phi_j^{\Delta_j}(X)$$

$$\widetilde{\mathcal{A}}_3^{\mathrm{G}} = (\mathcal{N}_{YM})^2\int \frac{d^4X}{(2\pi)^4}\prod_{j=1}^{3}\phi_j^{\Delta_j}(X)$$

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At higher points dress each analog of the scalar trivalent graphs with $\mathcal{N}'s$:

$$\widetilde{\mathcal{A}}_4^{YM} = c_s\,\mathcal{N}_s\,\mathfrak{s} + c_t\,\mathcal{N}_t\,\mathfrak{t} + c_u\,\mathcal{N}_u\,\mathfrak{u}$$

$$\widetilde{\mathcal{A}}_4^{G} = (\mathcal{N}_s)^2\mathfrak{s} + (\mathcal{N}_t)^2\mathfrak{t} + (\mathcal{N}_u)^2\mathfrak{u}$$

where

$$\mathfrak{s}(\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \int \frac{d^4X}{(2\pi)^4}\frac{d^4Y}{(2\pi)^4}\,G(X,Y)\prod_{i=1}^{2}\phi_i^{\Delta_i}(X)\prod_{j=3}^{4}\phi_j^{\Delta_j}(Y)\,,$$

with $\mathfrak{t}$ and $\mathfrak{u}$ obtained in an analogous way.

Comments and open questions II

- Celestial double copy generalizes to higher points.

  First rewrite the amplitude in terms of $n-2$ spacetime integrals by opening up 4 point contact terms, and then promote $K(X) \to \mathcal{K}$. [Casali,Sharma'20]

  Also: compact n-point formulas using celestial scattering equations and color-kinematical dual numerators from ambitwistor string numerators.

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- Celestial double copy beyond tree-level and for other theories?
  Celestial MHV 4-gluon amplitude in planar $\mathcal{N}=4$ SYM too all loops in dimensional regularization (internal momenta in $D=4-2\varepsilon$) [González,AP,Rojas'20]
  $\tilde{\mathcal{A}}_{\text{all loops}} = \mathcal{M}_\varepsilon \tilde{\mathcal{A}}_{\text{tree}}$ via *celestial loop operator* $\mathcal{M}_\varepsilon \ni \prod_{j=1}^{4} e^{\frac{\varepsilon}{2}\partial_{\Delta_j}}$

Thank you!