Classical integrals beyond the conservative region

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based on arXiv:2005.04236, with J. Parra-Martinez, M. Zeng
and in prepararation with E. Herrmann, J. Parra-Martinez, M. Zeng

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See also Julio Parra-Martinez' talk

Introduction

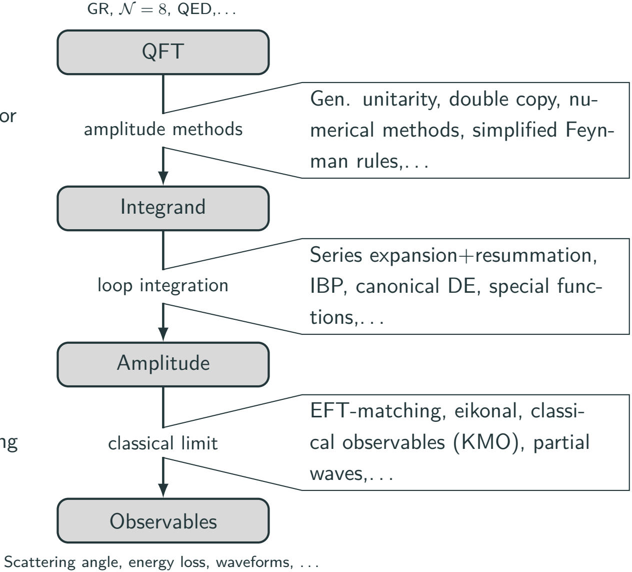
- Next-gen. experiment (ET, CE, ...) with order-of-magnitude improvement in S/N ratio
- In the future, high precision will be key
- Approach to obtain PM-expansion using amplitude methods by now mature
- Multi-loop PM-computations available [Bern, Cheung, Roiban, Shen, Solon, Zeng ('19); Cheung and Solon ('20); Källin, Liu, Porto ('20)]
- Spirit to import as much as possible from the knowledge acquired in perturbative QCD very successful

Amplitude-to-observable pipeline

Hard, but not a problem for the near future

Hard, main bottleneck

By now good understanding



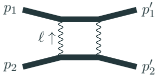
Integrands from numerical unitarity

- Automated C++-framework to compute multiloop amplitudes by numerical unitarity [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, MR, Sotnikov ('20)]
- Use finite field methods and functional reconstruction
- Most powerful when final results simple (e.g. 3PM angle)
- Power shown by computing 2-loop 4-graviton amplitudes [Abreu, Febres Cordero, Ita, Jaquier, Page, MR, Sotnikov ('20)]
- Full quantum, much more than we need for classical physics!
- Geared towards automation and high orders

Caravel

Integrals for $2 \to 2$ scattering

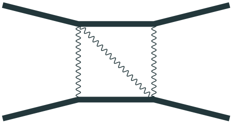
- Integrals common to all approaches (although sometimes one may avoid certain integrals)
- $2 \to 2$ scattering with masses, four scales $(s, t = q^2, m_1, m_2)$
- Typical integral:



$$= \int \frac{\mathrm{d}^D \ell}{\ell^2 (\ell - q)^2 ((\ell + p_1)^2 - m_1^2)((\ell - p_2)^2 - m_2^2)}$$

Integrals for $2 \to 2$ scattering

- Not all integrals for Bhabha scattering at 2 loops known (not even planar!)
- "N integral" is elliptic [Heller, Manteuffel, Schabinger ('19); Broedel, Duhr, Dulat, Penante, Tancredi ('19)]



- We are only interested in the "classical" part of these integrals.

Integrals the BCRSSZ way

- BCRSSZ introduced a method to compute integrals in the potential region
- Works great at 2 loops. But some drawbacks:
  - IR-divergent part not evaluated, but has to cancel with identical term in the EFT
  - Relies on guessing functions based on series expansion
  - Not manifestly Lorentz-invariant
  - Very challenging a higher loop orders
- Let's do the integrals as we would in perturbative QCD!

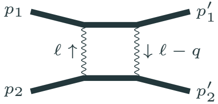
Strategy: use differential equations + method of regions

Expansion-by-region [Beneke, Smirnov ('98)]

- Hierarchy of scales in classical limit:

$$1 \ll J^2 \sim \frac{s}{q^2} \sim \frac{m_i^2}{q^2} \quad \rightarrow \quad q^2 \ll m_i^2 \sim s$$

- Relativistic regions:
  hard: $\ell \sim m$ $\leftarrow$ short range, UV
  soft: $\ell \sim q$ $\leftarrow$ classical physics

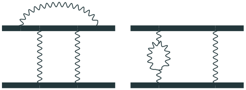


- Soft region further splits $|\boldsymbol{v}| = q^0/|\boldsymbol{q}|$
  potential: $(\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|)$ $\leftarrow$ conservative dynamics
  radiation: $(\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}||\boldsymbol{v}|)$

Let's not split the soft region !

Soft vs. potential region

- 👍 Captures dissipate effects
- 👍 Manifestly relativistic (no resummation)
- 👍 Straightforward to use dim-reg
- 👍 Avoids artifacts from splitting regions (tail effects)
- 👎 Additional contributions (must upgrade BCRSSZ integrand)

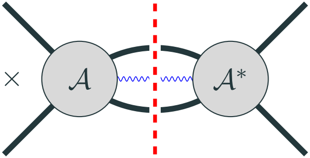


- 👎 Need more general way to extract classical physics (beyond potential in EFT)

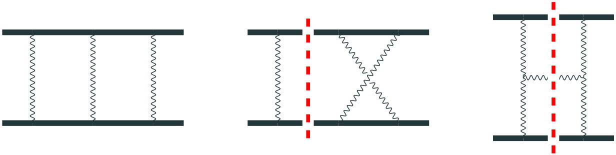
  Use KMO (here) or eikonal (talks by Heissenberg, Veneziano)

- Appropriate observables have smooth classical limit
- Example: LO (3PM) radiated momentum

$$R^\mu = \int \mathrm{d}^D q \delta(2p_1 \cdot q)\delta(2p_2 \cdot q) e^{ib\cdot q} k^\mu \times$$

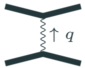


- In general we need virtual integrals and integrals with cuts



Soft expansion

- Sudakov-parametrization manifest q-scaling



$$u_i = \frac{\overline{p}_i}{|\overline{p}_i|}\,,\; u_i\!\cdot\! q = 0\,,\; u_1\!\cdot\! u_2 = y = \sigma + \mathcal{O}(q^2)\,.$$

- Matter propagators eikonalize

$$(\ell - p_1)^2 - m_1^2 = 2m_1(u_1 \cdot \ell) + \mathcal{O}(q^2)$$

- Mass scale factors, q-dependence fixed by dimensional analysis

$$\mathcal{I}(q^2, y) = (-q^2)^\alpha \tilde{\mathcal{I}}(y)$$

Only a single variable to all orders in the PM-expansion!

Soft expansion

- Box integral after expansion

$$\mathcal{I}_{\Box}^{\text{soft}} \simeq \frac{(-q^2)^{D/2-3}}{4m_1m_2} \underbrace{\left[(-q^2)^{-D/2+3} \int \frac{\mathrm{d}^D\ell}{\ell^2(\ell-q)^2(u_1\cdot\ell)(-u_2\cdot\ell)}\right]}_{\tilde{\mathcal{I}}(y)}$$

- Integrals reduced to a finite set of *master* integrals using IBP-identities

$$\int \mathrm{d}^D\ell \frac{\partial}{\partial\ell^\mu}\left[\frac{v^\mu}{\ell^2(\ell-q)^2\ldots}\right] = 0$$

- Various public implementations (KIRA, FIRE, Reduze,. . .)
- Compute single-scale master integrals by differential equations

Canonical differential equations

[Kotikov, Remiddi, Gehrmann ('91),('98),('99); Henn ('13)]

Method is divided into steps

1. Computing the differential equations (DE)
2. Finding a good basis; desirable canonical form
3. Computing boundary conditions (BND)
4. Integrating the system

Canonical differential equations

[Kotikov, Remiddi, Gehrmann ('91),('98),('99); Henn ('13)]

- Most powerful in canonical form [Henn ('13)]

$$\frac{\partial}{\partial y}\vec{\mathcal{I}}(y,\epsilon) = \epsilon \left[\sum_k A_k \, \frac{\partial \log(w_k(y))}{\partial y} \right] \vec{\mathcal{I}}(y,\epsilon)$$

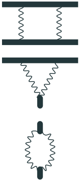
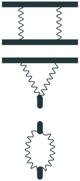
matrix of rational numbers

symbol letters (singularities)

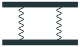
- Single-variable problem automatized [Lee ('14)] (We used the program `Epsilon` [Prausa ('17)])
- At most logarithmic singularities
- Solved iteratively by multiple polylogartihms

Example DE at 1 loop

- At 1 loop system of 3 master integrals

$$\frac{\partial}{\partial y}$$

$$= \epsilon \frac{\partial}{\partial y} \underbrace{\log\left(\frac{y+\sqrt{y^2+1}}{y-\sqrt{y^2+1}}\right)}_{=2\,\mathrm{arccosh}\,y} \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$


- Can be integrated to all orders in ϵ:


$$= \epsilon\ \mathrm{arccosh}\,y \ \times$$

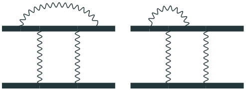
$$+\ \mathrm{BND}$$

- BND have to be provided. Fixes the region
- E.g. potential BND conditions $\Longrightarrow$ potential integral

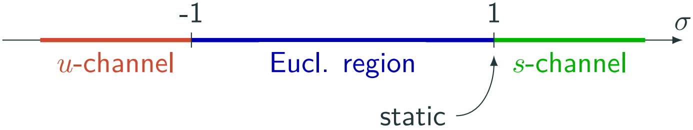
Soft Integrals

[Di Vecchia, Heissenberg, Russo, Veneziano; Hermann, Parra-Martinez, MR, Zeng]

- Canonical basis known [Parra-Martinez, MR, Zeng ('20)]
- Additional master integrals (top-level mushroom integrals)



- s and u-channel related through analytic continuation



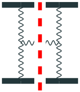
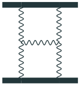
- Regularity at $\sigma = 0$ (trivial at 2 loops; constraints at 3 loops)
- More checks (numerically with e.g. PySecDec; Integrals are real in Euclidean region)

Reverse unitarity [Anastasiou, Melnikov ('02)]

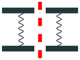
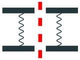
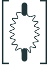
- Reverse unitarity: cut integrals satisfy same IBP and DE

$$2\pi \mathrm{i}\delta(2u\cdot\ell) = \frac{1}{2u\cdot\ell - \mathrm{i}\varepsilon} - \frac{1}{2u\cdot\ell + \mathrm{i}\varepsilon}$$

- Sufficient set of BND conditions from unitarity and static limit

 $= 2\,\mathrm{Im}\Bigg($  $\Bigg)$

- Trivial example:

 $= \epsilon\,\mathrm{arccosh}\,y \times 0 + \Big[$  $\Big]_{y=1} \sim \Big[$  $\Big]_{D=3-2\epsilon}$

Conclusions

- Integrals are a main bottleneck moving forward in the PM-expansion (common to all approaches)
- Powerful approach: method of regions + canonical DE
- Computed all integrals relevant at the 2 loop order (virtual/cut, potential/soft)
- Can compute generic "inclusive enough" observable.
- All integrals from *same* DE; different BND conditions
- Methods are scalable, ready for 3 loops.
- New result for radiated energy at $\mathcal{O}(G^3)$ (see Parra-Martinez' talk)