Scattering Amplitudes and Navier-Stokes

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Key Features of YM

What makes YM so good for "Amplitudes"?

- Trivalent (fund. 3pt)
- Color
- Transverse (massless/gauge)
- Lorentz

Gives rise to rich structure: BCJ, soft thms, spinor helicity, ...

Look for other theories.

Navier-Stokes

2 out of 4

- Trivalent
- ~~Color~~
- Transverse (solenoidal)
- ~~Lorentz~~

Non-relativistic fluid $\partial_\mu T^{\mu\nu} = 0$, velocity u_i, energy density ρ, pressure p_i, viscosity ν

$$(\partial_0 - \nu\nabla^2)u_i + u_j\partial_j u_i + \partial_i\left(\frac{p_i}{\rho}\right) = J_i$$

$$\partial_i u_i = 0 \text{ (incompressible)}$$

Non-Abelian Navier-Stokes

3 out of 4 (not bad!)

- Trivalent
- Color
- Transverse
- ~~Lorentz~~

Endow fluid with color u_i^a (c.f. gluon A_μ^a).
Conserved $T_{\mu\nu}^a$.

$$(\partial_0 - \nu\nabla^2)u_i^a + f^{abc}u_j^b\partial_j u_i^c + \partial_i\left(\frac{p^a}{\rho}\right) = J_i^a$$

$$\partial_i u_i^a = 0$$

Eliminate pressure

Integrate out pressure

$$(\partial_0 - \nu\nabla^2)u_i^a + f^{abc}u_j^b\partial_j u_i^c = J_i^a$$

Mostly focus on non-Abelian case from now on.

No action, no problem

NS doesn't have a Lagrangian (containing *only* fluid DOF). What to do? Try on-shell methods (amplitudes)!

Goal: study amplitudes of non-Abelian fluid

If it worked for YM, will it work for fluid?

Towards amplitudes

Momentum space

$$u_i^a = \varepsilon_i T^a e^{i(-\omega t + px)}$$

Free Navier-Stokes (kinetic term) gives on-shell conditions

$$i\omega - \nu p^2 = 0 \qquad \text{(disperses)}$$
$$p\varepsilon = 0 \qquad \text{(transverse)}$$

Classical solution

Classical = tree solution obtained recursively. Familiar from Born series (QM) or Berends-Giele recursion (YM). Solve for u

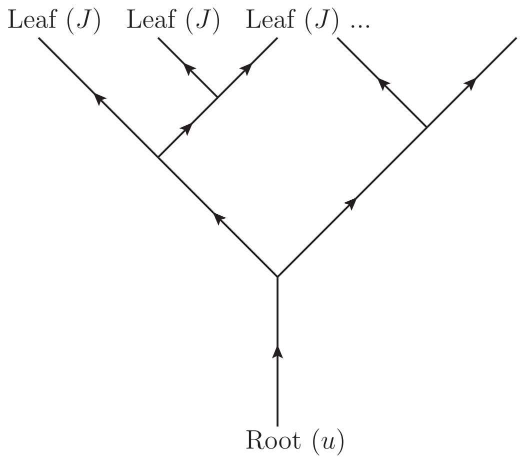
$$
\begin{aligned}
u &\sim \frac{1}{\Box}(J + u\partial u) \\
&\sim \frac{1}{\Box}J + \frac{1}{\Box}(J + u\partial u)\partial\frac{1}{\Box}(J + u\partial u) \\
&\sim \frac{1}{\Box}J + \frac{1}{\Box^2}J\partial J + ...
\end{aligned}
$$

Iterate to get series solution for u.

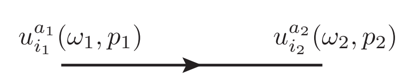
(Abelian case done by Wyld '61.)

Semi-on-shell tree amplitude

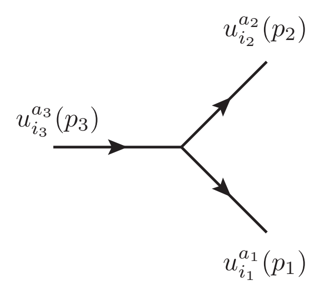
Amputate propagators, leaf legs (J) on-shell, root leg (u, call n'th leg) off-shell



Feynman rules



$$= \frac{\delta^{a_1 a_2}\delta_{i_1 i_2}}{i\omega_1 - \nu p_1^2}$$



$$= f^{a_1 a_2 a_3}(p_1^{i_2}\delta_{i_1 i_3} - p_2^{i_1}\delta_{i_2 i_3})$$

$$\to f^{a_1 a_2 a_3}(p_3^{i_1}\delta_{i_2 i_3} - p_3^{i_2}\delta_{i_1 i_3})$$

$$(\text{used } p_A^{i_A} \to 0)$$

Labels: particle (A, B, C), color (a, b, c), spatial (i, j, k).

Rewrite propagator

$$\frac{1}{i\sum_A \omega_A - \nu\Big(\sum_A p_A\Big)^2} = -\frac{1}{\nu}\frac{1}{\sum_{A\neq B} p_A p_B}$$

- Energy independence (dot products p, ε)
- Eff. coupling const $1/\nu$ (dissipation dominated (mud)/laminar flow/low Reynolds #)

Ex: 3pt & 4pt

$$A(123) = f^{a_1 a_2 a_3}\left[(p_1\varepsilon_2)(\varepsilon_1\varepsilon_3) - (1 \leftrightarrow 2)\right]$$

$$A(1234) = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

$$s = p_1 p_2$$

$$c_s = f^{a_1 a_2 \sigma} f^{\sigma a_3 a_4}$$

$$n_s = (p_1\varepsilon_2)(p_3\varepsilon_1)(\varepsilon_3\varepsilon_4) + (p_1\varepsilon_2)(p_3\varepsilon_3)(\varepsilon_1\varepsilon_4) - (1 \leftrightarrow 2)$$

Energy independent, as advertised.

$$A_n \sim \sum \frac{(p\varepsilon)^{n-2}(\varepsilon\varepsilon_n)}{(pp)^{n-3}}$$

Spinor helicity

- Introduce $p_0 \sim \sqrt{\omega}$ so that on-shell becomes $p_\mu p^\mu = 0$ (Maldacena & Pimentel '11).
- Non-relativistic object $\varepsilon_{a\dot{b}}$

Spinor helicity 3pt

$$A(1^-2^-3^-) = f^{a_1 a_2 a_3} \frac{\langle 12\rangle\langle 23\rangle\langle 31\rangle}{\langle 11]\langle 22]}$$

$$A(1^+2^-3^-) = f^{a_1 a_2 a_3} \frac{[12\rangle\langle 23\rangle\langle 31]}{\langle 11]\langle 22]}$$

$$A(1^-2^-3^+) = f^{a_1 a_2 a_3} \frac{[31\rangle\langle 12\rangle\langle 23]}{\langle 11]\langle 22]}$$

$$= f^{a_1 a_2 a_3} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \left(1 - \frac{\langle 11]}{\langle 22]} - \frac{\langle 22]}{\langle 11]}\right)$$

- 4pt and MHV $(1/p^2 \neq 1/\langle ij\rangle[ij])$

Soft thm root leg

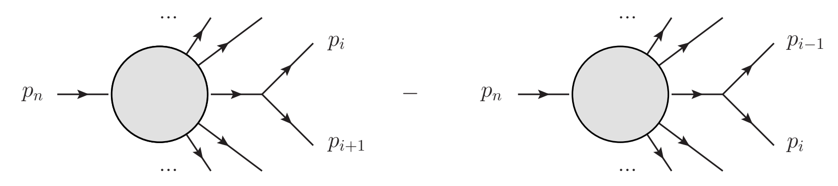
Vertex is proportional to root leg momentum

$$f^{a_1 a_2 a_n}(p_n^{i_1}\varepsilon_n^{i_2} - p_n^{i_2}\varepsilon_n^{i_1}) \propto p_n$$

Adler zero in root leg $p_n \to 0$

$$A(p_1, \ldots p_n) = \mathcal{O}(p_n)$$

Weinberg-type soft thm on leaf legs



$$\lim_{p_i \to 0} A_n[p_1, ...p_i, ...p_n]$$

$$= \left(\frac{e_i p_{i+1}}{p_i p_{i+1}} - \frac{e_i p_{i-1}}{p_i p_{i-1}} \right) A_{n-1}[p_1, ...p_{i-1}, p_{i+1}, ...p_n]$$

Changes slightly for p_i adjacent to p_n

On-shell constructibility

- Don't have $\mathcal{L}$ so try on-shell construction
- Given limited on-shell data (3pt, 4pt) can you obtain everything?
- Answer: Recursion

Momentum only recursion

- Shift leaf legs

$$p_A \to p_A + z\tau_A \varepsilon_A$$

- On-shell for circular polarization $\varepsilon_A^2 = 0$
- Scales as z^{4-n} (mass dim)

$$A_n \sim \sum \frac{(p\varepsilon)^{n-2}(\varepsilon\varepsilon_n)}{(pp)^{n-3}}$$

- Any choice of τ_A is ok but some are better (probe Adler)
- Quadratic poles (like soft recursion)

Risager-type recursion

- Leaf legs

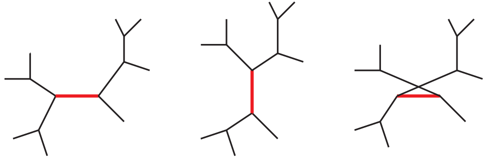
$$p_A \to p_A + z(p_A\eta)\eta$$
$$\varepsilon_A \to \varepsilon_A - z(\varepsilon_A\eta)\eta$$

- $\eta^2 = 0$ and $\eta\varepsilon_n = 0$
- Scales as z^{3-n} (# props)

$$A_n \sim \sum \frac{(p\varepsilon)^{n-2}(\varepsilon\varepsilon_n)}{(pp)^{n-3}}$$

- Fluid: Risager $\to$ Feynman
  YM: Risager $\to$ MHV

Off-shell color-kinematics duality



$$\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \text{ with } c_s + c_t + c_u = 0$$

$$c_s = f^{a_1 a_2 \sigma} f^{\sigma a_3 a_4}$$

$$n_s = p_1^{i_2}(p_1^{i_3} + p_2^{i_3})\delta_{i_1 i_4} + p_2^{i_1} p_3^{i_2}\delta_{i_3 i_4} - (1 \leftrightarrow 2)$$

Discover $n_s + n_t + n_u = 0$

No $p_i \cdot p_j$ in numerator helps

Double copy at EOM level for free!

$$(\partial_0 - \nabla^2) u_i^a + \frac{1}{2} f^{abc} f_{ijk} u_j^b u_k^c = J_i^a$$

$f_{ijk} v_j w_k = v_j \partial_j w_i - w_j \partial_j v_i$ (see SD YM and NLSM)

Diffeo algebra! $[v_j \partial_j, w_k \partial_k] = f_{ijk} v_j w_k \partial_i$ (gen trans)

Double copy: $f^{abc} \to f_{\bar{i}\bar{j}\bar{k}}$

$$(\partial_0 - \nabla^2) u_{i\bar{i}} + \frac{1}{2}(u_{j\bar{j}} \partial_j \partial_{\bar{j}} u_{i\bar{i}} - \partial_j u_{i\bar{j}} \partial_{\bar{j}} u_{j\bar{i}}) = J_{i\bar{i}}$$

Tensor fluid!

Classical double copy

’t Hooft-Polyakov-type monopole

$$u_i^a = f(r)\epsilon_{aij}x_j$$

$$f''(r) + \frac{4f'(r)}{r} - \frac{f^2(r)}{\nu} = 0$$

Double copy

$$u_{i\bar{i}} = g(r)\delta_{i\bar{i}} + h(r)x_i x_{\bar{i}}$$

Reminiscent of Kerr-Schild

$$u_i^a = -\frac{2\nu\epsilon_{aij}x_j}{r^2} \qquad u_{i\bar{i}} = 2\nu\delta_{i\bar{i}} + \frac{Cx_i x_{\bar{i}}}{r^4}$$

Color-kinematics: sym and currents

$$\text{color: } u_i^a \to u_i^a + f^{abc}\theta^b u_i^c$$
$$\text{kinematic: } u_i^a \to u_i^a + f_{ijk}\theta_j u_k^a \text{ (translation)}$$

Noether without $\mathcal{L}$?

$$J_{li} = f_{ijk} u_j^a \overleftrightarrow{\partial}_l u_k^a$$
$$\partial_l J_{li} = \frac{1}{\nu} f_{ijk} u_j^a \overleftrightarrow{\partial}_0 u_k^a$$
$$\partial_0 Q = \int d^3x \partial_l J_{li} = 0$$

Summary

- Punchline 1: fluids & YM common ground
- Punchline 2: $\mathcal{A}$ at EOM level (without $\mathcal{L}$)

Mandelstams $\mathcal{A}(\{p, \varepsilon\})$		✔			
Spinor helicity		✔	Bi-adjoint scalar ϕ^3		✔
MHV/CSW/PT	?		KLT kernel	?	
Soft thms		✔	Classical sol	?	
Sub-leading	?		Currents w/o Noether		✔
Recursion		✔	SUSY		?
BCFW, 3-line	?		Web		?
BCJ double copy		✔	CHY & Stringy		?
Off-shell	✔		Turbulence & LSS		?
Classical double copy		✔			