Scattering Amplitudes & Feynman Integrals from Wilson Loops

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Based on works with Chi Zhang, Zhenjie Li (2019+2020), + Qinglin Yang (to appear)

QCD meets Gravity
December 2020

Motivations

Scattering Amplitudes: QCD @ LHC + Gravity @ LIGO (+ strings, math, cosmology …) [many talks]

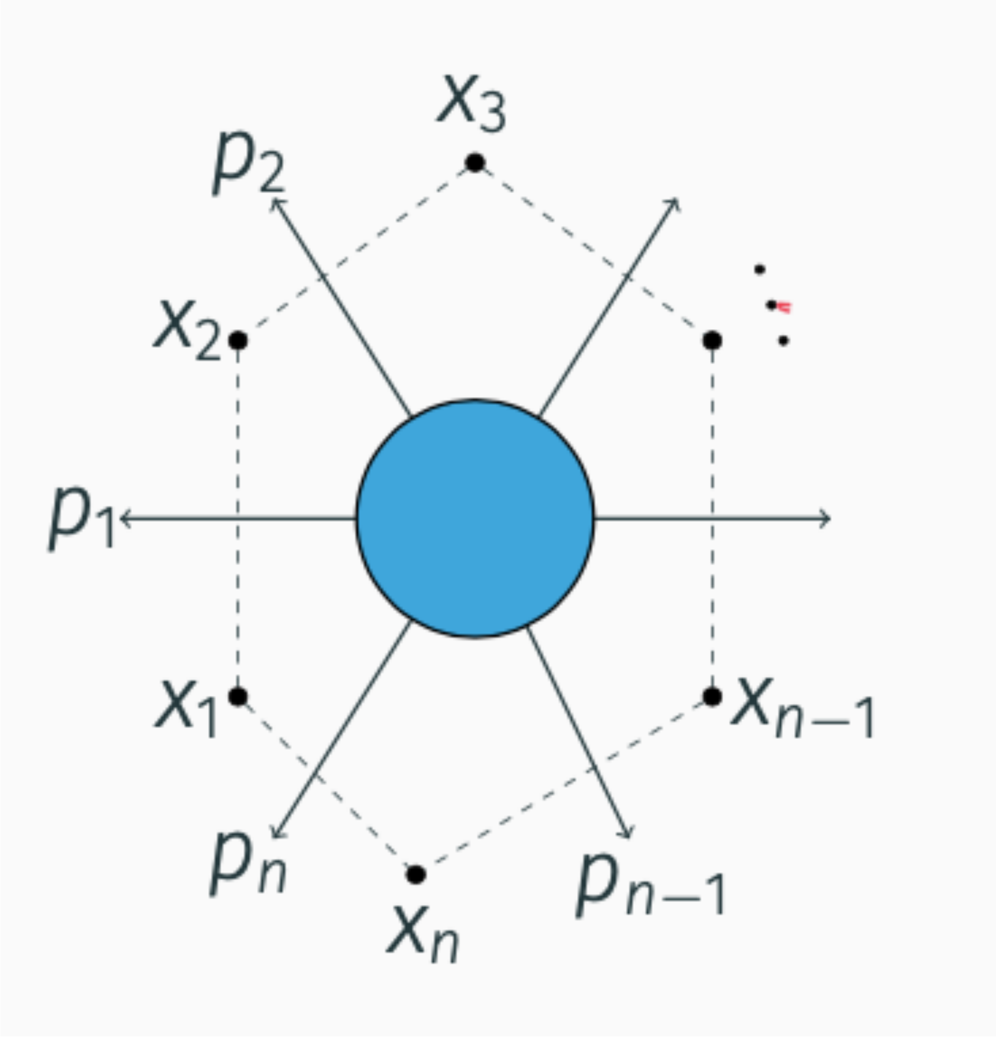
hidden simplicity + new structure in QFT: most remarkable $\mathcal{N}=4$ SYM (planar limit)

All-loop Integrand
gen. unitarity, all-loop recursion, geometric structures
e.g. positive Grassmannian, amplituhedron …

Integrability (any coupling)
AdS/CFT + strong coupling, (pentagon-) OPE,
Yangian symmetry …

(Integrated) Amplitudes + Feynman Integrals
Wilson loops, bootstrap, symbology+ cluster algebra,
differential eqs, iterated integrals (beyond polylogs)…

Amplitudes/Wilson-loop Duality



MHV amplitudes (tree stripped) = null polygonal Wilson loops (strong+ weak coupling)

[Alday, Maldacena][Brandhuber, Heslop, Travaglini] [Drummond, Henn, Korchemski, Sokatchev][…]

$$A_n(p_1, p_2, \cdots, p_n) \leftrightarrow W_n(x_1, x_2, \cdots, x_n) \sim \langle \mathrm{Tr}\mathscr{P} \exp\,(i \oint \mathbf{A} \cdot dx) \rangle$$

generalized to super-amplitudes $\mathscr{A}_{n,k}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$ = super-Wilson loops $\mathscr{W}[\mathbb{A} := \mathbf{A}^{\alpha,\dot{\alpha}} + \bar{\psi}^{\dot{\alpha}}\theta^{\alpha} + \cdots]$

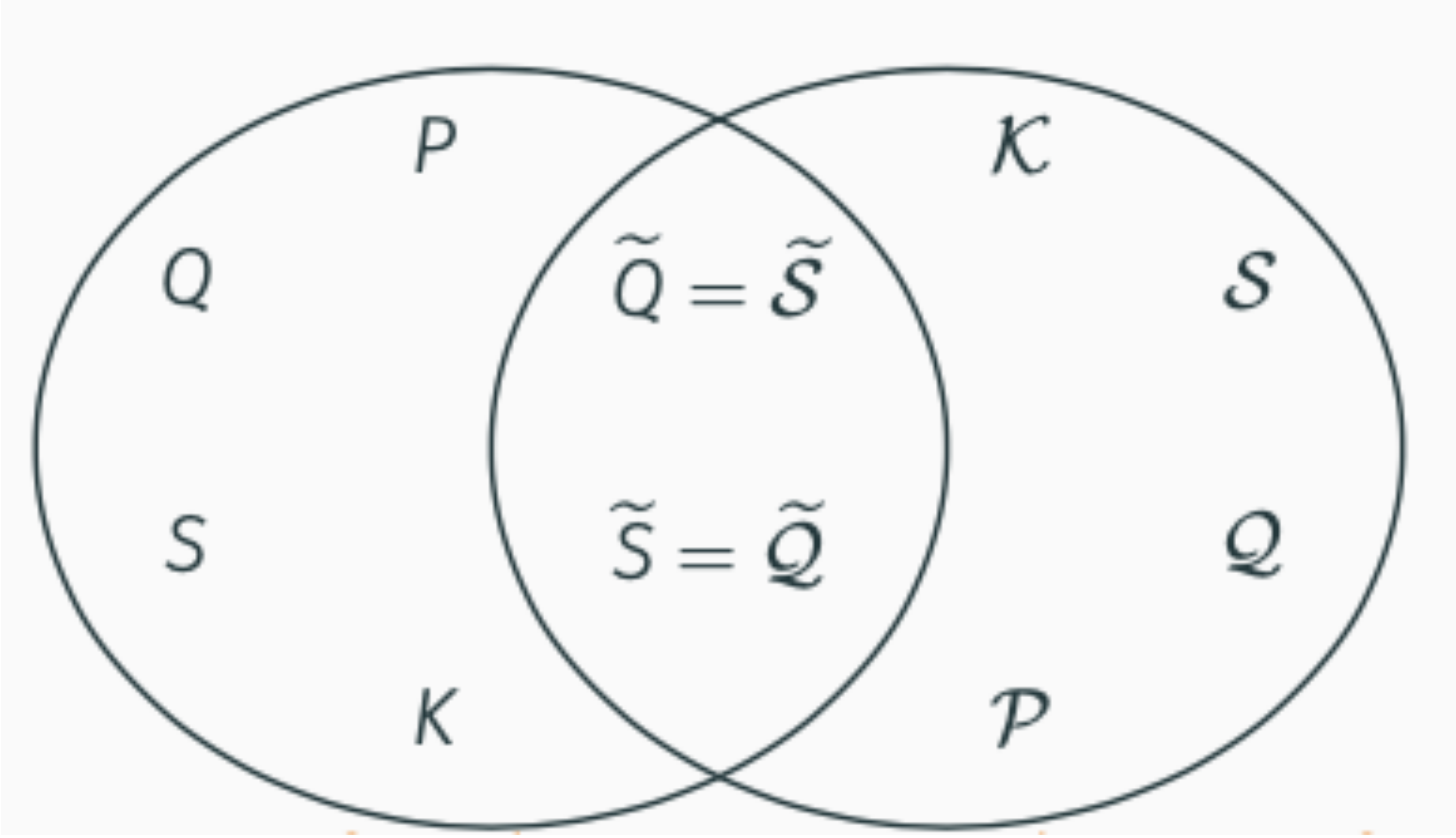
[Mason, Skinner] [Caron-Huot]

defined in dual space with $\mathcal{N} = 4$ SUSY extension: $(x_{i+1} - x_i)^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \quad (\theta_{i+1} - \theta_i)^{\alpha A} = \lambda_i^{\alpha}\eta_i^{A}$

IR divergence of amplitudes <—> UV divergence of Wilson loops (all loops)

Symmetries

PSU(2,2|4)



superconformal (amps) + dual superconformal (WL)

→Yangian symmetry (infinite dim. ↔ integrability)

[Drummond, Korchemski, Sokatchev; + Henn, Smirnov] [Drummond, Henn, Plefka] [...]

Loop-level: symmetry broken by IR/UV divergence!

e.g. $K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde{\lambda}_i^{\dot{\alpha}}}$, $\quad \mathcal{K}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \left[x_i^{\alpha\dot{\beta}} x_i^{\beta\dot{\alpha}} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + \cdots\right]$

Yangian invariants = leading singularities (loop amps @ compact contours)

↔ “on-shell diagrams”/ contour integrals over $G_+(k,n)$ e.g. “1” for MHV

level 0: $\sum_{i=1}^{n} G^{I}_{iJ}$,

level 1: $\sum_{i<j}^{n} (-1)^{|K|}[G^{I}_{iK} G^{K}_{jJ} - (i \leftrightarrow j)]$, $\cdots$

Anomaly eqs for dual conformal sym. $\hat{K}^{\mu}\langle W_n \rangle \propto \int d^D x x^{\mu} \langle \mathscr{L}(x) W_n \rangle$ [Drummond, Henn, Korchemski, Sokatchev]

Loop amplitudes made finite

BDS ansatz [Bern, Dixon, Smirnov]: $A_n^{\rm BDS} \sim \exp(\frac{1}{4}\Gamma_{\rm cusp} F_n^{1-\rm loop}) \implies$ BDS-normalized amps: $R_{n,k} = \mathscr{A}_{n,k}/A_n^{\rm BDS}$

- Dual conformal invariant (DCI) function of $3n-15$ cross-ratios ($n=4,5$ trivial)

invariant under chiral half of dual SUSY, but not the other half!

- $R_{n,k} \sim \underline{(\text{Yangian invariants})} \times \underline{(\text{Transcendental functions})}$

  SUSY, "rational/algebraic" — DCI, "uniform weight"=2 L

MHV & NMHV expected to be simplest: generalized polylogarithms

$$G(\mathbf{a}, t_0) = \int_0^{t_0} \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}$$

[Goncharov] → symbol & letters [Goncharov, Spradlin, Vergu, Volovich]

- For n=6,7, cluster variables of $G_+(4,n)$: only 9 & 42 letters [Golden, Goncharov, Spradlin, Vergu, Volovich] (confirmed up to L=7 for n=6, L=4 for n=7!) [Caron-Huot, Dixon, Dulat, McLeod, von Hippel, Papathanasiou] [Drummond, Foster, Gurdogan, Papathanasiou][…]

Momentum twistors [Hodges]

- Unconstrained variables for any massless kinematics, + DCI $\to 3(n-5)$ (bosonic) d.o.f.
- "Light-rays" of dual space, linearly realize dual symmetry SL(4|4): $\mathcal{Z}_i = (Z_i^a \mid \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha,\dot\alpha}\lambda_{i,\alpha} \,|\, \theta_i^{\alpha,A}\lambda_i^\alpha)$
- Basic SL(4) invariant: 4-bracket $\langle i\,j\,k\,l\rangle := \varepsilon_{abcd}Z_i^a Z_j^b Z_k^c Z_l^d$ e.g. $\langle i-1\,i\,j-1\,j\rangle \propto (x_i - x_j)^2$
- Dual symmetries: $K_b^a = \sum_i Z_i^a \frac{\partial}{\partial Z_i^b}$, $R_B^A = \sum_i \chi_i^A \frac{\partial}{\partial \chi_i^B}$, $Q_A^a = (Q_a^\alpha, \bar{S}_A^{\dot\alpha}) = \sum_i Z_i^a \frac{\partial}{\partial \chi_i^A}$ annihilate $R_{n,k}$,
  but not $\bar{Q}_a^A = (\bar{Q}_\alpha^A, S_{\dot\alpha}^A) = \sum_i \chi_i^A \frac{\partial}{\partial Z_i^a}$ (chiral nature of super WL!)
- Usual symmetry generators become level-1, just need one: $s_A^\alpha = \sum_i \frac{\partial}{\partial \lambda_{i,\alpha}} \frac{\partial}{\partial \eta_i^A}$ (parity of $S_{\dot\alpha}^A$)

Yangian anomaly equations [Caron-Huot, SH]

$$\bar{Q}_a^A R_{n,k} = a\ \mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3}\mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\mathrm{tree}}\right] + \text{cyclic},$$

loop parameter $\quad a := \frac{1}{4}\Gamma_{\mathrm{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots,$

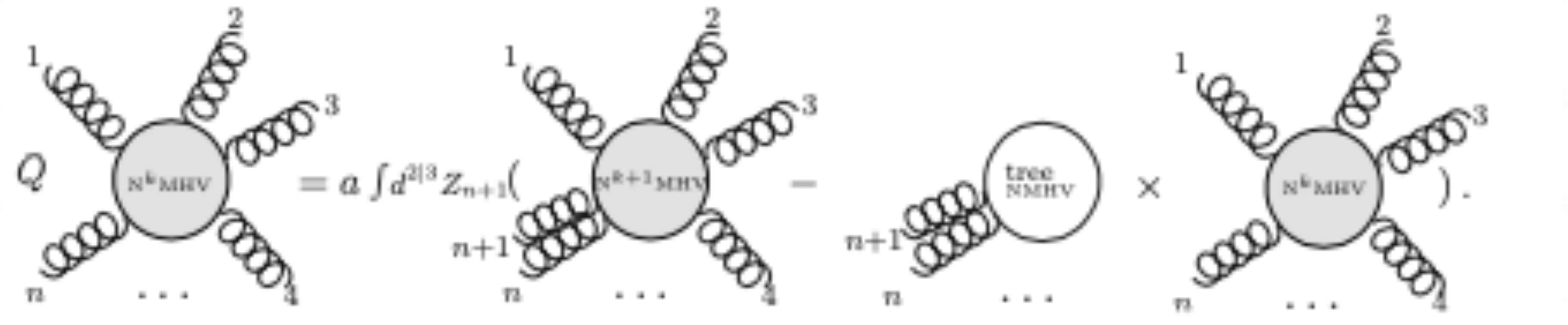


Figure 1. All-loop equation for planar $\mathcal{N}=4$ S-matrix.

- Regulator independent, expect to hold @ any coupling
- Insert fermion on edges of WL -> collinear integral

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon(\mathcal{Z}_{n-1} - \tau \mathcal{Z}_1) + \mathcal{O}(\epsilon^2)$$

- 2nd term on RHS: $\bar{Q}$ on BDS-ansatz (fixes Γ_{cusp})

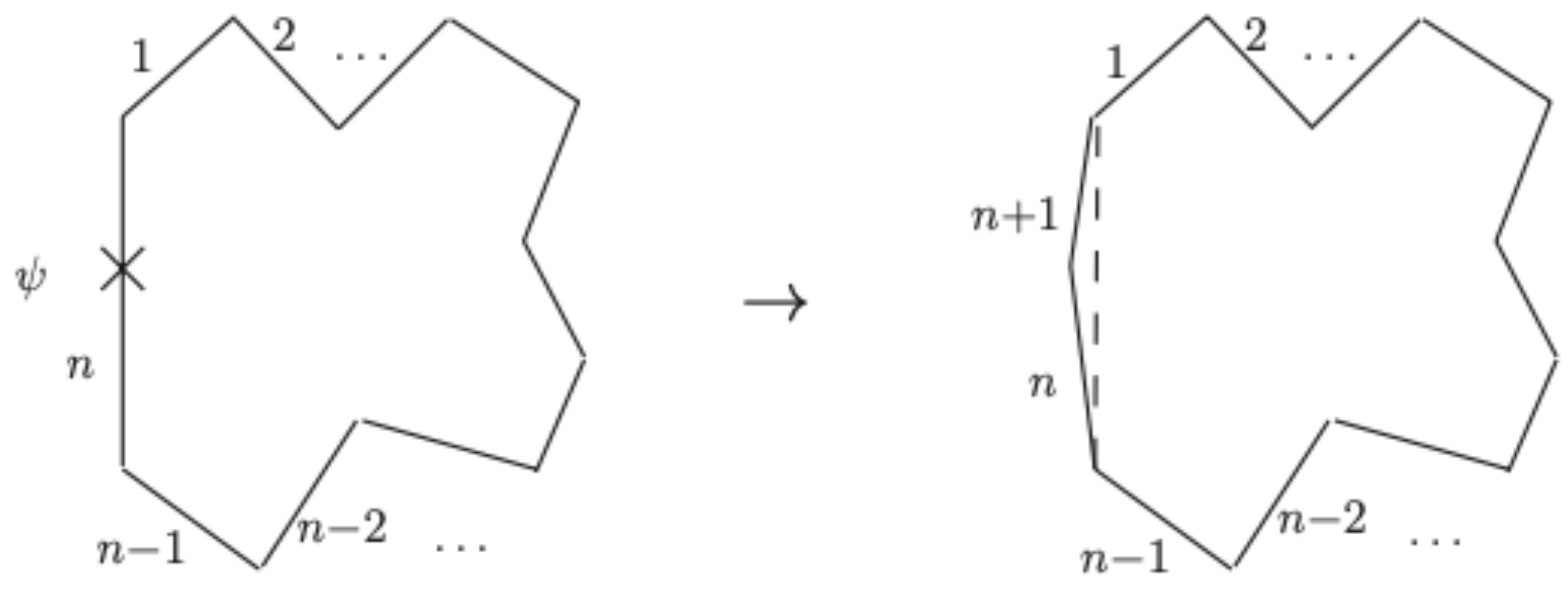


Figure 2. Fermion insertion on the Wilson loop versus kink insertion

space-time parity $\implies$ $\bar{Q}^{(1)}$ equations; determine "anomalies" of all Yangian generators!

Move to LHS as quantum corrections → exact symmetries for S-matrix: $\hat{\bar{Q}}\mathcal{M} = \hat{Q}^{(1)}\mathcal{M} = \hat{K}\mathcal{M} = Q\mathcal{M} = 0$

Jumpstarting all-loop amplitudes [Caron-Huot, SH]

Diff. equations (1st order) determine S-matrix up to Yangian inv. (“algebraic”), fixed by collinear limits

In practice, $\bar{Q}$ alone determine MHV & NMHV amps uniquely, once lower-loop amps are known!

Key Q: kernel in replacing $\bar{Q} = \sum_i \chi_i \frac{\partial}{\partial Z_i} \to d = \sum_i Z_i^a \frac{\partial}{\partial Z_i^a}$? Trivial for MHV (no χ)

General amplitudes: $\bar{Q} R_{n,k} \sim Y_{n,k}\, \bar{Q} I_{n,k}$, no non-trivial (DCI) kernel for NMHV $(k=1)$!

$$\to dR^{(L)}_{n,k=0,1} = \begin{cases} \sum d\log s_0 \int d\log f(\tau)\, I^{(L-1)}_{k=1}(\tau) & \text{MHV} \\ \sum Y_{n,1}\, d\log s_1 \int d\log f(\tau)\, I^{(L-1)}_{k=2}(\tau) & \text{NMHV} \end{cases}$$

“Last entry” for all loops (important for bootstrap); full results for L=2,3, … from lower loops

Wilson-loop τ-integrals

Collinear limit: $\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon(\mathcal{Z}_{n-1} - C\tau\mathcal{Z}_1) + C'\epsilon^2\mathcal{Z}_2$, with $C = \dfrac{\langle n-1n23\rangle}{\langle n123\rangle}$ etc.

Measure: $\displaystyle\int d^{2|3}\mathcal{Z}_{n+1} := C(n-1n1)\ \mathrm{Res}_{\epsilon=0}\int \epsilon d\epsilon \int_0^\infty d\tau d^{0|3}\,\chi_{n+1}$

- Fermionic integral e.g. "R-inv." to MHV last entry: $Y_{n,1} \to Y_{n,1} \to \bar{Q}\log\dfrac{\langle\bar{n}i\rangle}{\langle\bar{n}j\rangle} \to d\log\dfrac{\langle\bar{n}i\rangle}{\langle\bar{n}j\rangle}$
- Residue: extract coefficients of $\dfrac{d\epsilon}{\epsilon}$ (non-trivial it exists, $\log\epsilon$ divergence cancel)
- τ-integral (of lower loops in collinear limit) $\displaystyle\int_0^\infty d\log\langle X(\tau)\,n\,i\,j\rangle\ I^{(L-1)}(\epsilon=0)\,, \quad X := Z_{n-1} - C\tau\,Z_1$

Totally straightforward for 2-loop n-point MHV (1-d integral of 1-loop NMHV x last entry)

Multi-loop amps/WL

Beyond 2-loop MHV, 2-loop NMHV heptagon from (integral of) 1-loop NNMHV octagon

→ symbol of 3-loop hexagon & all-n for 2d kinematics (nice $\bar{Q}$ eqs in 2d) [Caron-Huot, SH]

All-loop predictions (for bootstrap): $Y_{n+1,1} \sim [i,j,k,l,m] \implies dR_{n,0} = \sum_{i,j} C_{i,j}\, d\log\langle \bar{i}\, j\rangle$

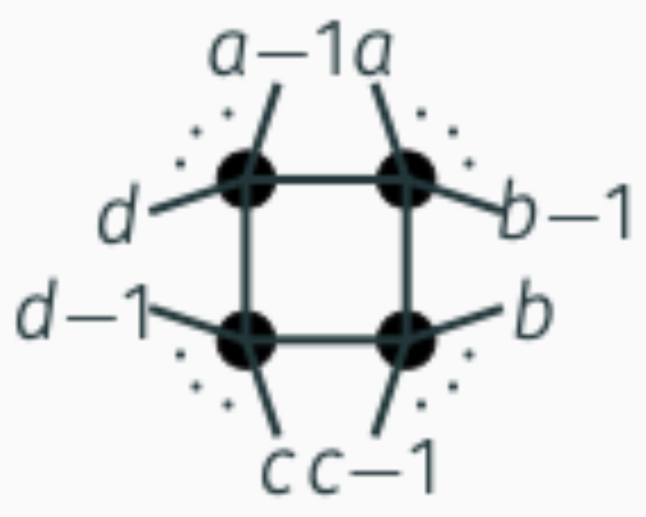
with $C_{i,j}$ 1-d τ-integrals of NMHV functions

For NMHV, $Y_{n+1,2} \implies$ R-inv. $\times\, d\log s_1$: $[i,j,k,l,m] \times d\log \frac{\langle \bar{n}(ij) \cap (klm)\rangle}{\langle \bar{n} i\rangle\langle jklm\rangle}$ etc. [SH, Z. Li, C. Zhang]

Total #=$42\binom{n}{6} - \binom{n-1}{5}$ (41 for n=6, 288 for n=7, further simplified for BDS-like)

Starting n=8: algebraic letters (square roots), what is the alphabet (cluster becomes infinity)?

Rationalizing square roots [SH, Z. Li, C. Zhang]



$$\begin{cases} u_{abcd} = \frac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2}, \; v_{abcd} = \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2}, \; \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv} \\ z_{abcd} = \frac{1}{2}(1+u-v+\Delta), \; \bar{z}_{a,b,c,d} = \frac{1}{2}(1+u-v+\Delta), \end{cases}$$

$$f_{a,b,c,d} = \frac{1-u-v\pm\Delta}{2\Delta}[\alpha_\pm, b{-}1, b, c-1, c][\delta_\pm, d{-}1, d, a{-}1, a]$$

$$\mathcal{I}_{a,b,c,d} = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

where $\alpha_\pm$ and $\delta_\pm$ are two solutions of Schubert problem
$\alpha = (a{-}1\, a) \cap (d\, d{-}1\, \gamma)$, $\gamma = (c{-}1\, c) \cap (b\, b{-}1\, \alpha)$

- 2-loop NMHV ($n \geq 8$), need τ-integral of 4-mass box: naively LS polluted by (square-root) prefactor
- Need "rationalize" square roots: rational points of quadratic curve $y^2 = x^2 + ax + b$
- change of variable: e.g. $\tau \to z(\tau)$, $\bar{z} = \frac{az+b}{cz+d}$ constant $SL(2)$, integral becomes manifestly pure:

$$\int_{z^{-1}(0)}^{z^{-1}(\infty)} d\log\frac{z-w}{z-\bar{w}}\,\mathrm{dilog}(z,\bar{z}) + (z \leftrightarrow \bar{z}) \quad \text{with } \bar{w} = \frac{aw+b}{cw+d}$$

- beautiful weight-3 symbol: "4-mass box ⊗ algebraic letters", $\left(u \otimes \frac{1-z}{1-\bar{z}} + v \otimes \frac{\bar{z}}{z}\right) \otimes \frac{(z-w)(\bar{z}-\bar{w})}{(\bar{z}-w)(z-\bar{w})}$

Algebraic letters & words [SH, Z. Li, C. Zhang]

$$
\begin{aligned}
&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c-1}_{a,b,c,d} \otimes x^{c-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c-1]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{c}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c]\\
+\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b-1}_{a,b,c,d} \otimes x^{b-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,c-1\,c]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{b}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b\,c{-}1\,c]\\
+\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a-1}_{a,b,c,d} \otimes x^{a-1}_{a,b,c,d}\,[a{-}1\,b{-}1\,b\,c-1\,c]\\
-\,&\mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}^{a}_{a,b,c,d} \otimes x^{a}_{a,b,c,d}\,[a\,b{-}1\,b\,c{-}1\,c]\,,
\end{aligned}
$$

$$
\mathcal{X}^{*}_{a,b,c,d} := \frac{(x^{*}_{a,b,c,d}+1)^{-1}-\bar{z}_{d,a,b,c}}{(x^{*}_{a,b,c,d}+1)^{-1}-z_{d,a,b,c}}\,, \qquad \tilde{\mathcal{X}}^{*}_{a,b,c,d} := \frac{(x^{*}_{a,b,c,d-1}+1)^{-1}-z_{d,a,b,c}}{(x^{*}_{a,b,c,d-1}+1)^{-1}-\bar{z}_{d,a,b,c}}
$$

with 6 choices $a{-}1, a, b{-}1, b, c{-}1, c$ of the superscript, where

$$
x^{a}_{a,b,c,d} = \frac{\langle \bar{d}(c{-}1\,c)\cap(a\,b{-}1\,b)\rangle}{\langle \bar{d}\,a\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, \qquad x^{a-1}_{a,b,c,d} = x^{a}_{a,b,c,d}|_{a\leftrightarrow a-1}
$$

$$
x^{b}_{a,b,c,d} = \frac{\langle \bar{d}(c{-}1\,c)\cap(a{-}1\,a\,b)\rangle}{\langle \bar{d}(a{-}1\,a)\cap(b\,c{-}1\,c)\rangle}\,, \qquad x^{b-1}_{a,b,c,d} = x^{b}_{a,b,c,d}|_{b\leftrightarrow b-1}
$$

$$
x^{c}_{a,b,c,d} = \frac{\langle \bar{d}\,c\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle \bar{d}(a{-}1\,a)\cap(b{-}1\,b\,c)\rangle}\,, \qquad x^{c-1}_{a,b,c,d} = x^{c}_{a,b,c,d}|_{c\leftrightarrow c-1}
$$

Remarkably constrained & compact "algebraic part": 4-mass $\otimes\, \text{algebraic}^i \otimes \text{final}^i \times R_i$ (all correlated!)

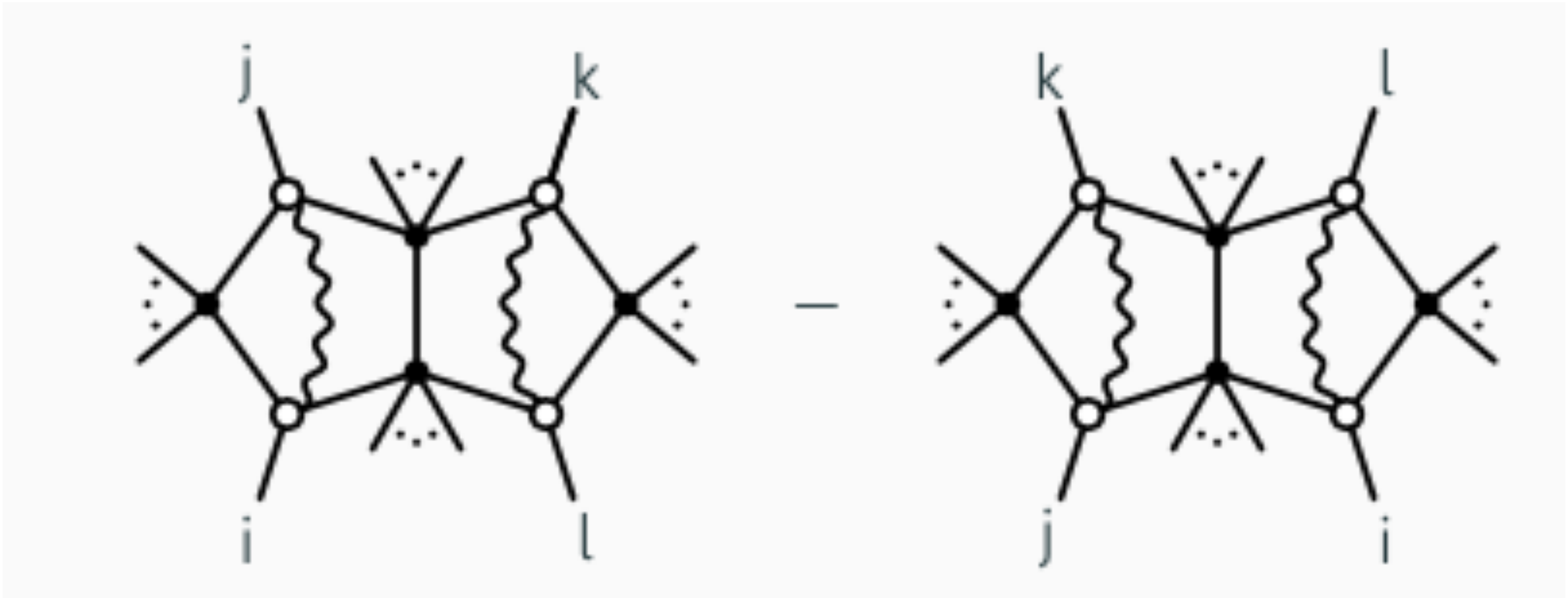
For each $\Delta_{a,b,c,d}$: at most 17 (multiplicative) independent algebraic letters $\dfrac{a^i-\bar{z}}{a^i-z}$

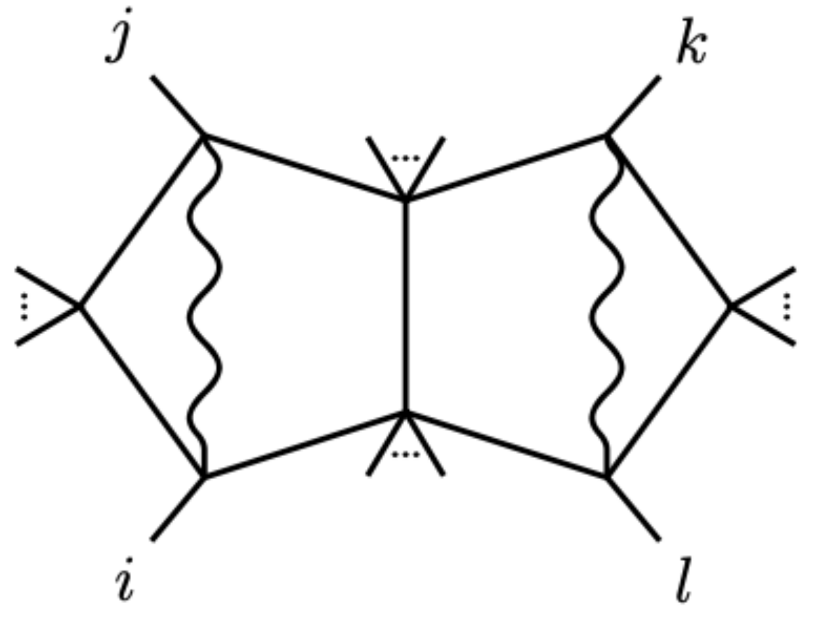
Most degenerate: $\Delta_{1,3,5,7}$ & $\Delta_{2,4,6,8}$ for $n=8$, 9+9 independent algebraic letters (+ 180 rational letters)

Origin of alphabet? tropical $G_+(4,8)$ etc. [Drummond, Foster, Gurdogan, Kalousios] [Henke, Papathanasiou][Arkani-Hamed, Lam, Spradlin]
poles/"letters" of Yangian invariants [Mago, Schreiber, Spradlin, Volovich] [SH, Z. Li]

Simplest NMHV components

Component $\chi_i^1\chi_j^2\chi_k^3\chi_l^4$ for non-adjacent i, j, k, l (vanishes for L=0,1) given by 2 double-pentagon integrals





9 propagators, numerators $\langle \ell_1 \bar{i} \cap \bar{j} \rangle$, $\langle \ell_2 \bar{k} \cap \bar{l} \rangle$

Surprise: free of roots in the difference (such components vanish for algebraic words)!
For n=8: confirms observation of [Bourjaily et al] by evaluating $I_{\rm dp}(1,3,5,7)$ at a numeric point

Longstanding problem: compute generic double-pentagon analytically (12 legs, lots of square roots)?

All we need for MHV: $A^{2-\text{loop}}_{n,\text{MHV}} = \sum_{i<j<k<l} I_{\rm dp}(i,j,k,l)$ [Arkani-Hamed et al]; how to see cancellation of square roots?

Feynman integrals from WL

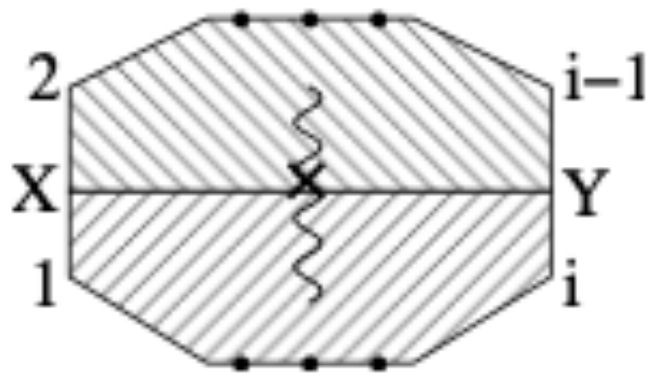
WL powerful for not only (full) amps, also a large class of Feynman integrals (=WL diagrams)

How Simon originally computed 2-loop MHV: $dR_{n,0} = \sum_{i<j} C_{i,j} d\log\langle \bar{i} \bar{j}\rangle$ w. $C_{i,j}$ (super-) WL diagrams

L=1:

$$C_{2,i} = \int_0^\infty d\tau_X d\tau_Y \frac{\langle \bar{2} i\rangle\langle \bar{i} 2\rangle}{\langle XY\rangle^2} = \log u_{2,i-1,i,1}$$

L=2: 1-d τ-integral of box integrals



Simplest NMHV: difference of two WL diagrams=double pentagon!

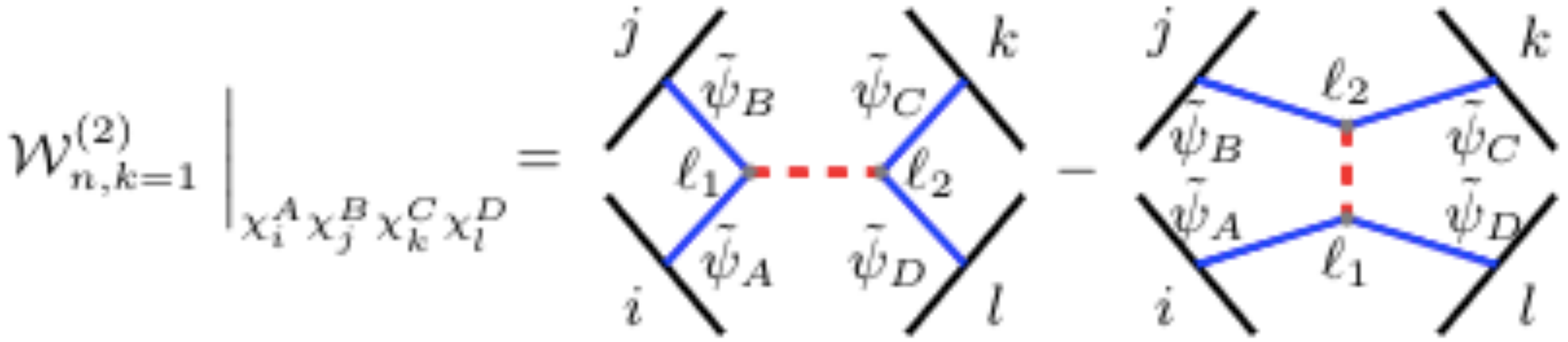


FIG. 2. NMHV component of super-WL as difference of two diagrams, each equals to a double-pentagon integral.

WL $d\log$-integral: chiral pentagon

Why useful? swap order of integrals, left with simple line integrals ("smart parametrization") [SH, Z. Li, Q. Yang, C. Zhang]

chiral pentagon: $\frac{1}{\langle \ell i-1 i\rangle \langle \ell i i+1\rangle} = \int_0^\infty \frac{d\tau}{\langle \ell i X(\tau)\rangle^2}, \quad X(\tau) := Z_{i-1} + \tau_X\, Z_{i+1}$ "fermions" at $x := (iX)$ & $y := (jY)$

$$\Longrightarrow I_{\text{pent.}} = \int d\tau_X d\tau_Y \int \frac{d^4\ell \langle \ell \bar{i} \cap \bar{j}\rangle \langle I i j\rangle}{\langle \ell\, i\, X\rangle^2 \langle \ell\, j\, Y\rangle^2 \langle \ell\, I\rangle} = \int d^2\tau \frac{\langle I\bar{i} \cap \bar{j}\rangle \langle I i j\rangle}{\langle I\, i\, X\rangle \langle I\, j\, Y\rangle \langle i\, X\, j\, Y\rangle}$$ (star-triangle identity)

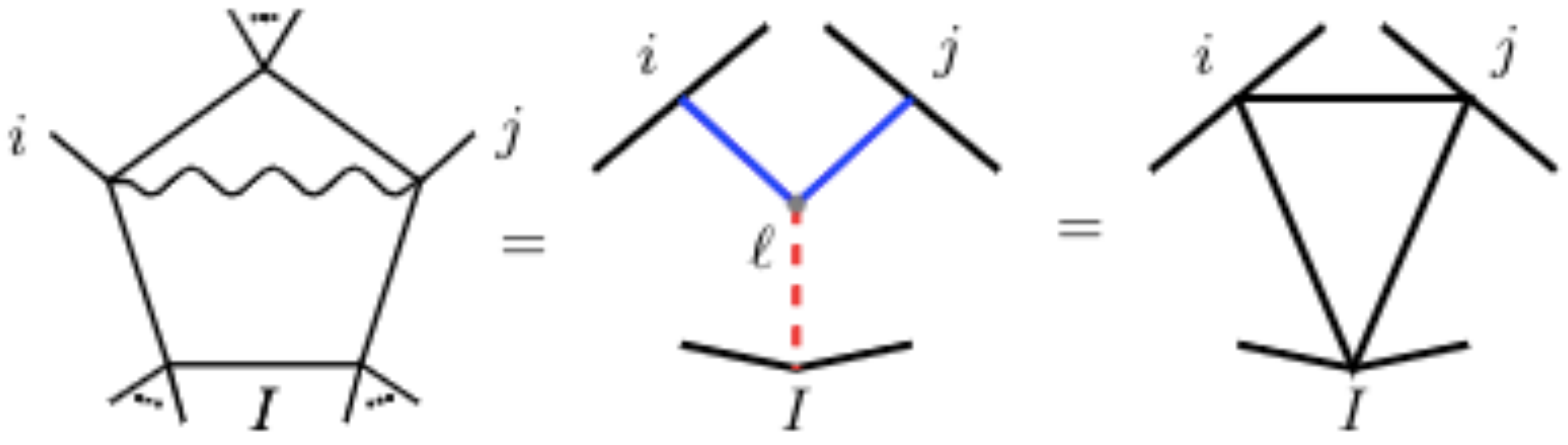


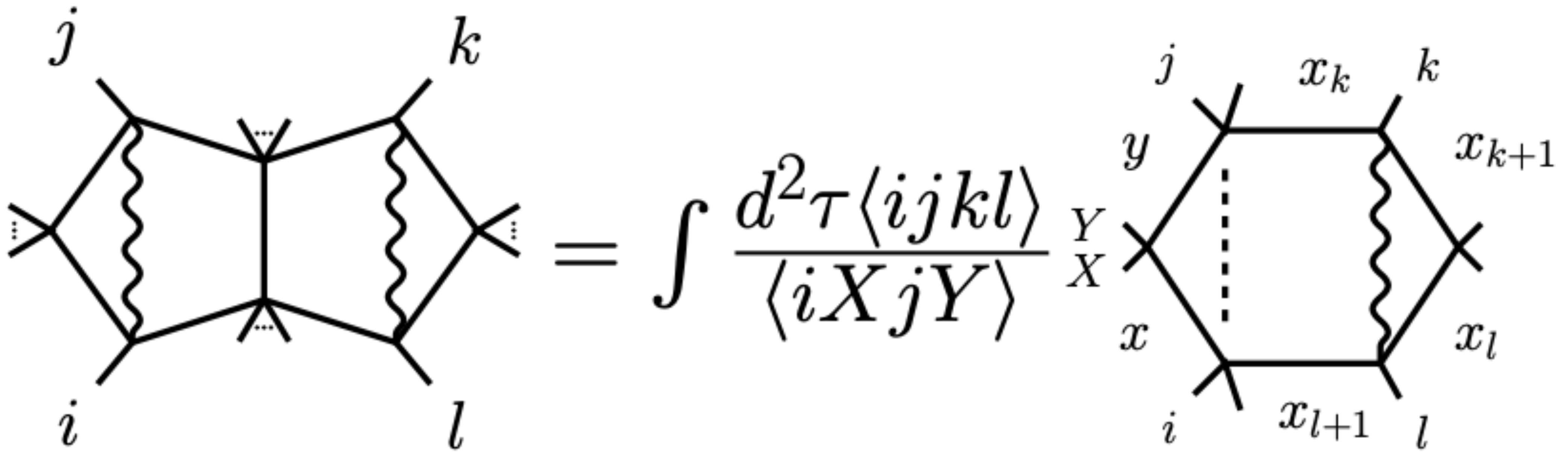
FIG. 3. The chiral pentagon written as a WL diagram, and loop integral performed using "star-triangle" identity.

Nice $d\log$ 2-form: $\int_{(i,j)} d\log \frac{\langle IjY\rangle}{\langle \bar{i}(jY) \cap (iI)\rangle} d\log \frac{\langle iXjY\rangle}{\langle IiX\rangle}$

Trivially give well-known dilog (manifest DCI + weight-2)

Geometry: integrating $\Omega(\Delta')$ in Δ (similar to Aomoto)

Double pentagon from WL [SH, Z. Li, Q. Yang, C. Zhang]



double-pentagon from WL: could perform 2 loop integrals → 4-fold line integrals

mixed method: apply to 1-loop → 2-d integral of hexagon ($k = j+1,\ l = i-1$: chiral hexagon)

- hexagon= sum of 15 boxes w. "2-form LS"; straightforward to integrate, but prefactor for 4-mass boxes!
- rationalizing precisely as τ-integral in $\bar{Q}$ of NMHV: $\int d\log\frac{z-w}{z-\bar{w}}\,\mathrm{dilog}(z,\bar{z}) + (z \leftrightarrow \bar{z})$ (manifest pure functions!)
- same weight-3 algebraic symbol (τ'-integrand): $\left(u \otimes \frac{1-z}{1-\bar{z}} + v \otimes \frac{\bar{z}}{z}\right) \otimes \frac{(z-w)(\bar{z}-\bar{w})}{(\bar{z}-w)(z-\bar{w})}\Bigg|_{\tau=0}^{\tau=\infty}$
- already this level: (algebraic symbol)$|_{(i,j,k,l)-(j,k,l,i)}$ =0 ! nicely explain cancellation of square roots

Algebraic words

Weight 4 → final-entry free of square roots (similar to 2-loop NMHV to 3-loop MHV!)

Remarkably compact algebraic words : sum of 16 blocks with square roots $\Delta(a,b,c,d)$

$$\sum_{\sigma_a\in\{0,1\}} (-)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4}\, S^{4-\mathrm{m}}(i+\sigma_1, j+\sigma_2, k+\sigma_3, l+\sigma_4) \otimes W^{i,j,k,l}_{\sigma_1,\cdots,\sigma_4}$$

each with "4-mass box" times

$$W^{i,j,k,l}_{a-i,\cdots,d-l} = \chi^{j,k}_{a,b,c,d} \otimes \frac{\langle x_a\, j\, l\rangle\langle x_b\, i\, k\rangle}{\langle x_a\, j\, k\rangle\langle x_b\, i\, l\rangle} + \text{cyclic. } (i,j,k,l)$$
$$+\frac{1}{2}\left(\frac{\bar{z}(1-z)}{z(1-\bar{z})}\prod \chi\right) \otimes \frac{\langle x_a\, k\, l\rangle\langle x_b\, i\, l\rangle\langle x_c\, i\, j\rangle\langle x_d\, j\, k\rangle}{\langle x_a\, j\, l\rangle\langle x_b\, i\, k\rangle\langle x_c\, j\, l\rangle\langle x_d\, i\, k\rangle}$$

with algebraic letters

$$\chi^{j,k}_{a,b,c,d} := \left(\frac{\frac{\langle x_a x_b\rangle\langle x_d\, jk\rangle}{\langle x_d x_b\rangle\langle x_a\, jk\rangle} - z_{a,b,c,d}}{\frac{\langle x_a x_b\rangle\langle x_d\, jk\rangle}{\langle x_d x_b\rangle\langle x_a\, jk\rangle} - \bar{z}_{a,b,c,d}}\right)$$

in addition to $W^{i,j,k,l} - W^{j,k,l,i} = 0$, trivial to see 16 W's with the same $\Delta(a,b,c,d)$ cancel:

$$\sum_{\sigma_a\in\{0,1\}} W^{a-\sigma_1,\cdots,d-\sigma_4}_{\sigma_1,\cdots,\sigma_4}(x_a, x_b, x_c, x_d) = 0 \implies$$ 2-loop n-point MHV from Feynman integrals!

Discussions

- Wilson loops powerful for perturbative computations: amplitudes, Feynman integrals, …

- Yangian anomaly eqs ($\bar{Q}$ + parity): all-loop data & explicit computations

  3-loop MHV octagon (& higher) [WIP w. Z. Li, C. Zhang] eventually need level-1 (or parity)

- Structures of amplitudes & integrals: algebraic letters, first/last entries, etc. bootstrap?

  connections to amplituhedron? cluster algebra & non-perturbative S-matrix?

- WL rep for integrals: systematic “smart parametrization”? geometries of the dlog forms?

Thank You!