MAXIMAL U(1)-VIOLATING $\mathcal{N} = 4$ SYM n-POINT CORRELATORS AND THEIR HOLOGRAPHIC DUALS

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Interplay of **perturbative** and **non-perturbative** properties of correlation functions / amplitudes

Montonen-Olive $SL(2,\mathbb{Z})$ **duality of** $SU(N)$ $\mathcal{N}=4$ **SUSY Yang-Mills** ⟺ $SL(2,\mathbb{Z})$ **S-duality of type IIB superstring**

$SL(2,\mathbb{Z})$ covariance of:

large-N expansion of SUSY YM correlators ⟺ Low energy expansion of type IIB amplitudes in $AdS_5 \times S^5$

1) Exact properties of 4-point correlators in large-N expansion
with Shai Chester, Silviu Pufu, Yifan Wang, Congkao Wen)

arXiv:1912.13365
arXiv:2008.02713

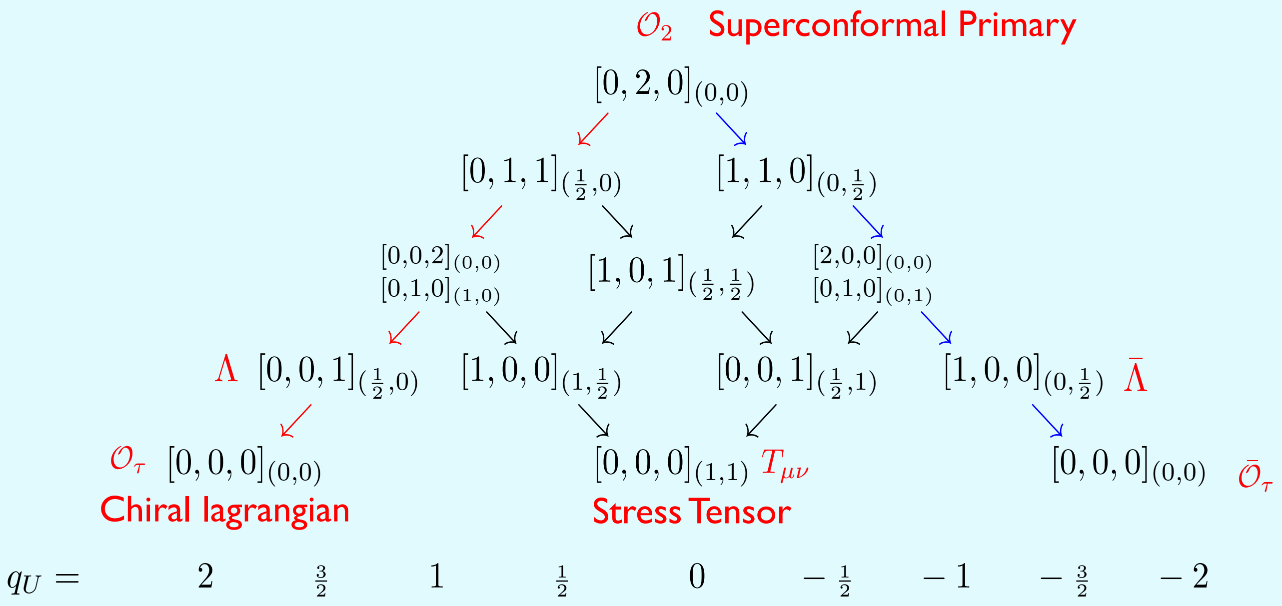
Using SU§SY localisation - Pestun partition function (Extending previous work by Binder, Chester, Pufu, Wang)

2) Modular properties of n-point correlators and closed superstring amplitudes that violate the bonus U(1) symmetry maximally **with Congkao Wen**

arXiv:1904.13394
arXiv:2009.01211

Determines certain precise perturbative and non-perturbative (instantonic) terms.

$\mathcal{N}=4$ SYM Stress Tensor Supermultiplet



(on-shell) lagrangian

$$\mathcal{L} = -\frac{i}{2\hat{\tau}_2}\left(\hat{\tau}\mathcal{O}_\tau - \bar{\hat{\tau}}\bar{\mathcal{O}}_{\bar{\tau}}\right)$$

$$\hat{\tau} = \frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g^2_{YM}}$$

Complex coupling dual to complex dilaton in IIB

- "Bonus U(1)" symmetry - outer automorphism of $PSU(2,2|4)$ (Intrilligator).

  Broken to $\mathbb{Z}_4$ (as in the dual type IIB superstring). Leads to the following pattern of charge violation:

  All **4-point correlators conserve U(1)** e.g. $\langle\mathcal{O}_2(x_1)\ldots\mathcal{O}_2(x_4)\rangle$, $\langle\Lambda(x_1)\Lambda(x_2)\bar{\Lambda}(x_3)\bar{\Lambda}(x_4)\rangle$ $q_U = 0$

  n-point correlators violate U(1) by $q_U \leq 2(n-4)$ **units**

  Maximal U(1) violating (MUV) correlators violate U(1) by $q_U = 2(n-4)$ units

- **Holographic dual MUV n-point amplitudes have no massless poles**

  – their low energy expansions are contact interactions

Montonen-Olive (SL(2,Z)) Duality

S-Duality group $SL(2,\mathbb{Z})$ $\quad \hat{\tau} \to \dfrac{a\hat{\tau}+b}{c\hat{\tau}+d}$ $\quad a,b,c,d \in \mathbb{Z}$, $\; ad-bc=1$

Modular form $f^{(w,-w)}(\hat{\tau},\bar{\hat{\tau}}) \to \dfrac{(c\hat{\tau}+d)^w}{(c\bar{\hat{\tau}}+d)^w} f^{(w,-w)}(\hat{\tau},\bar{\hat{\tau}}) = e^{2iq_U\phi} f^{w,-w)}(\hat{\tau},\bar{\hat{\tau}})$

Holomorphic anti-holomorphic

weights $(w,-w)$

$U(1)$ Transformation: charge $q_U = 2w$

½-BPS operators are modular forms with weights $(w,-w)$ i.e. U(1) charge $q_U = 2w$

Covariant derivative $\mathcal{D}_w = i\hat{\tau}_2 \dfrac{\partial}{\partial\hat{\tau}} + \dfrac{w}{2}$

$\mathcal{D}_w f^{(w,-w)} = f^{(w+1,-w-1)}$ Changes q_U by 1

$\bar{\mathcal{D}}_{-w} f^{(w,-w)} = f^{(w-1,-w+1)}$ Changes q_U by -1

Note:

Non-holomorphic Eisenstein series (modular function)

$$E(s,\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

weight $(0,0)$

Non-holomorphic Eisenstein modular forms

(act with covariant derivatives on (0,0) form)

$$E_w(s,\hat{\tau},\bar{\hat{\tau}}) = \mathcal{D}_{w-1}\ldots\mathcal{D}_0\, E(s,\hat{\tau},\bar{\hat{\tau}})$$

weight $(w,-w)$

Maximal U(1)-Violating (MUV) Correlation Functions

Violate U(1) by $2(n-4)$ units. e.g. $\underbrace{\langle \mathcal{O}_2(x_1)\dots\mathcal{O}_2(x_4)\mathcal{O}_\tau(x_5)\dots\mathcal{O}_\tau(x_{4+m})\rangle}_{n=4+m}$, $\underbrace{\langle \Lambda(x_1)\dots\Lambda(x_{16})\rangle}_{n=16}$

$w = n-4$

$q_U = 2m$ $\qquad q_U = 24$

Consider differential $\mathcal{D}_w = i\hat{\tau}_2\dfrac{\partial}{\partial\hat{\tau}} + \dfrac{w}{2}$ acting on MUV correlator

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle \equiv \int D\Phi\, e^{-\int d^4x\mathcal{L}}\,\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)$$

Soft recursion relation

$$\mathcal{D}_w\,\langle\prod_{r=1}^{n}\mathcal{O}_{j_r}(x_r)\rangle = \frac{1}{2}\int d^4z\langle\mathcal{O}_\tau(z)\prod_{r=1}^{n}\mathcal{O}_{j_r}(x_r)\rangle$$

Uses: (i) Normalisation of ½-BPS operators $\quad i\hat{\tau}_2\dfrac{\partial}{\partial\hat{\tau}}\mathcal{O}_j = -\dfrac{1}{2}\mathcal{O}_j \qquad \mathcal{O}_2 = \dfrac{\hat{\tau}_2}{4\pi}[\mathrm{tr}\varphi^2]_{[0,2,0]}$

(ii) $\quad i\hat{\tau}_2\dfrac{\partial}{\partial\hat{\tau}}e^{-\int d^4z\mathcal{L}} = \dfrac{1}{2}\int d^4z\mathcal{O}_\tau(z)$

(iii) Contact terms in **OPE** $\quad \mathcal{O}_\tau(z)\,\mathcal{O}_j(x_j) = a_j\mathcal{O}_j(x_j)\,\delta^4(z-x_j) + a'_{j'}\dfrac{1}{(z-x)^4}\mathcal{O}_{j'}(x_j) + \dots$

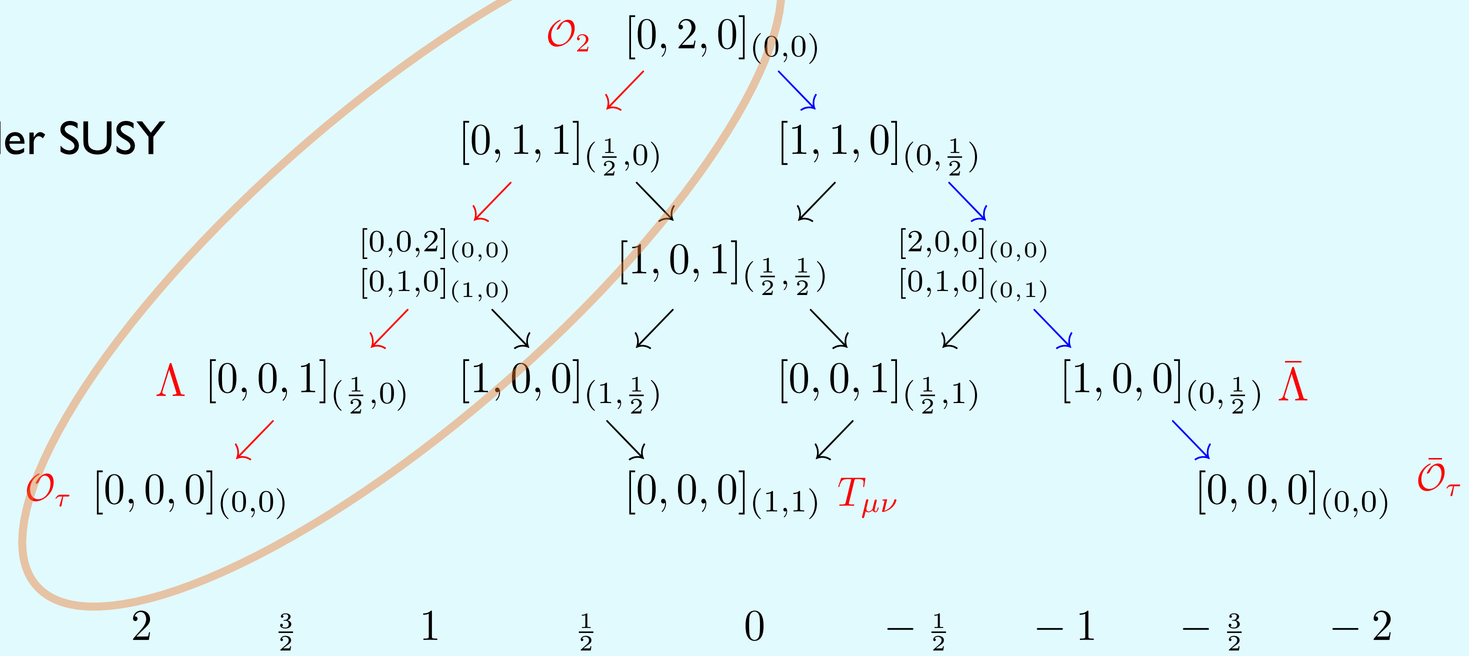
Contact term — Absent for **"chiral" correlators**

½-BPS OPE: $q_U^{j'} = 2 + q_U^{j}$

Double trace and long operators

“Chiral” n-Point MUV Correlators

The subset of chiral operators close under SUSY



Universal features of MUV chiral n-point correlators $\quad q_U = 2n - 8$

$$\widehat{G}_n(j_1, j_2, \cdots, j_n) = \mathcal{I}_n(\{x_i, y_i\})\, \mathcal{G}_n(x_1, \cdots, x_n; \hat{\tau}_2)$$

- $\mathcal{I}_n(\{x_i, y_i\})$ encodes $SU(4)$ and spin quantum numbers labelled by j_i.
- $\mathcal{G}_n(x_1, \ldots, x_n; \hat{\tau}_2)$ depends on x_i and is independent of the species labelled by j_i
- No short single trace operators contribute to OPE $\mathcal{O}_\tau(z)\mathcal{O}_j(x_j) \sim \mathcal{O}_{j'}(x_j)$,
- $\mathcal{G}_n(x_1, \ldots, x_n; \hat{\tau}_2)$ is dual to $AdS_5 \times S^5$ **contact interactions** (no massless poles).

"Chiral" n-Point MUV Correlators

Reduced recurrence relation $\mathcal{D}_w \, \mathcal{G}_{n-1}(x_1, \cdots, x_{n-1}; \hat{\tau}) = \frac{1}{2} \int d^4 x_n \, \mathcal{G}_n(x_1, \cdots, x_n; \hat{\tau})$

Relates n-point correlator to (n-1)-point correlator

Large-N expansion

$$\mathcal{G}_n(x_1, \cdots, x_n; \hat{\tau}) = \sum_{\alpha=0,2,3} c^{\frac{1-\alpha}{4}} \, \mathcal{G}_n^{(\alpha)}(x_1, \cdots, x_n; \hat{\tau}) + O(c^{-\frac{3}{2}}) \, ,$$

$$c = \frac{N^2 - 1}{4}$$

General structure

$$\mathcal{G}_n^{(\alpha)}(x_i; \hat{\tau}) = \sum_{m=0}^{\alpha} \mathcal{F}_{n,m}^{(\alpha)}(\hat{\tau}) \, A_n^{(m)}(x_i)$$

Modular form of weight (n-4,4-n)

kinematic basis – contact interactions with m derivatives

Mellin transform of correlator

~ amplitude in $AdS_5 \times S^5$

$$\mathcal{M}_n^{(\alpha)}(\gamma_{ij}; \hat{\tau}) = \sum_{m=0}^{\alpha} \mathcal{F}_{n,m}^{(\alpha)}(\hat{\tau}) \, M_n^{(m)}(\gamma_{ij}) \, ,$$

Mellin variables ~ Mandelstam variables for $AdS_5 \times S^5$ amplitude

THE n=4 CASE - THE FOUR-$\mathcal{O}_2$ CORRELATOR

DUAL TO FOUR-GRAVITON AMPLITUDE IN $AdS_5 \times S^5$

Binder, Chester, Pufu, Wang arXiv:1902.06263

Chester, Pufu, arXiv::2003.08412

Chester, MBG, Pufu, Wang Wen arXiv:1912.13365; arXiv:2008.02713

- **Start with the PESTUN PARTITION FUNCTION OF $\mathcal{N}=2^*$ SYM on S^4** (Supersymmetric Localisation)

  mass-deformed $\mathcal{N}=4$ SYM

- **Consider** $\partial_m^2 \partial_\tau \partial_{\bar\tau} \log Z|_{m=0}$ - Four-$\mathcal{O}_2$ **integrated correlation function**
- **Large-N expansion, fixed** g_{YM} (expansion in α' - keeps instantons)

Up to order N^{-1} i.e. up to terms in dual theory $\sim d^6 R^4$

Contact Witten diagrams ("D-functions")

$N^{\frac{1}{2}}$ $\quad \mathcal{G}_4^{(0)}(x_i;\hat\tau) = \dfrac{15E(\frac{3}{2},\hat\tau)}{4\sqrt{2\pi^3}} A_4^{(0)}(x_i)$ $\qquad A_4^{(0)}(x_i) = D_{4444}(x_i)$ $\qquad \sim R^4$

$N^{-\frac{1}{2}}$ $\quad \mathcal{G}_4^{(2)}(x_i;\hat\tau) = \dfrac{315\,E(\frac{5}{2},\hat\tau)}{32\sqrt{2\pi^5}} \left[A_4^{(2)}(x_i) - \dfrac{19}{4} A_4^{(0)}(x_i)\right]$ $\qquad A_4^{(2)}(x_i) = (x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)\, D_{5555}(x_i)$ $\qquad \sim d^4R^4$

N^{-1} $\quad \mathcal{G}_4^{(3)}(x_i;\hat\tau) = -\dfrac{945\,\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\hat\tau)}{32\pi^3} \left[A_4^{(3)}(x_i) + \dfrac{9}{2} A_4^{(2)}(x_i) - 32 A_4^{(0)}(x_i)\right]$

$$A_4^{(3)}(x_i) = x_{14}^2 x_{24}^2 x_{34}^2 D_{5557}(x_i) + x_{13}^2 x_{23}^2 x_{34}^2 D_{5575}(x_i) + x_{12}^2 x_{23}^2 x_{24}^2 D_{5755}(x_i) + x_{12}^2 x_{13}^2 x_{14}^2 D_{7555}(x_i) \qquad \sim d^6R^4$$

(a "generalized" Eisenstein series)

- **Coefficients are modular functions** $E(\frac{3}{2},\hat\tau,\bar{\hat\tau})$, $E(\frac{5}{2},\hat\tau,\bar{\hat\tau})$ and $\mathcal{E}(3,\frac{3}{2},\frac{3}{2},\hat\tau,\bar{\hat\tau})$

  which give precisely defined expressions for **perturbative terms** and **multiple instanton/anti-instanton** contributions to low-energy expansion of $AdS_5 \times S^5$ string amplitude.

ALL-ORDERS FEATURES OF $SL(2,\mathbb{Z})$-INVARIANT INTEGRATED CORRELATOR

- **CONJECTURED ALL ORDERS EXPANSION OF INTEGRATED CORRELATION FUNCTION** $\partial_m^2 \partial_\tau \partial_{\bar\tau} \log Z|_{m=0}$

AdS_5 interactions – scale L

$$\begin{aligned}
\tau_2^2 \partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\Big|_{m=0} =& \frac{N^2}{4} - \frac{3N^{\frac{1}{2}}}{2^4\,\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) && R^4 \\
&+ \frac{45}{2^8 N^{\frac{1}{2}} \pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) && d^4 R^4 + \frac{1}{L^4} R^4 \\
&+ \frac{1}{N^{\frac{3}{2}}} \left[-\frac{39}{2^{13}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) + \frac{4725}{2^{15}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau) \right] && d^8 R^4 + \frac{1}{L^2} d^6 R^4 + \frac{1}{L^4} d^4 R^4 + \frac{1}{L^8} R^4 \\
&+ \frac{1}{N^{\frac{5}{2}}} \left[-\frac{1125}{2^{16}\pi^{\frac{5}{2}}} E(\tfrac{5}{2},\tau,\bar\tau) + \frac{99225}{2^{18}\pi^{\frac{9}{2}}} E(\tfrac{9}{2},\tau,\bar\tau) \right] && d^{12} R^4 + \cdots + \frac{1}{L^{12}} R^4 \\
&+ \frac{1}{N^{\frac{7}{2}}} \left[\frac{4599}{2^{22}\pi^{\frac{3}{2}}} E(\tfrac{3}{2},\tau,\bar\tau) - \frac{2811375}{2^{25}\pi^{\frac{7}{2}}} E(\tfrac{7}{2},\tau,\bar\tau) + \frac{245581875}{2^{27}\pi^{\frac{11}{2}}} E(\tfrac{11}{2},\tau,\bar\tau) \right] && d^{16} R^4 + \cdots + \frac{1}{L^{16}} R^4 \\
&+ O(N^{-\frac{9}{2}})\,.
\end{aligned}$$

- The quantity $\partial_m^4 \log \mathcal{Z}\Big|_{m=0}$ is an independent integrated 4-point correlator and has an expansion that includes both **½-integer** and **integer** powers of $\frac{1}{N}$. i.e. With coefficients $E(s,\tau,\hat\tau)$ and $\mathcal{E}(r,s,s',\tau,\hat\tau)$

  Eisenstein and generalized Eisenstein

- The terms up to order $d^6 R^4$ - i.e. **BPS terms** - reproduce precisely the results of the flat-space type IIB superstring.

MUV n-Point Correlators (very brief)

MBG, Congkao Wen arXiv: arXiv:2008.02713

- The soft recursion relation uniquely determines MUV (n>4)-point correlators in terms of the 4-point correlator.

$N^{\frac{1}{2}}$ $\qquad \mathcal{G}_n^{(0)}(x_i;\hat\tau) = \mathcal{F}_n^{(0)}(\hat\tau) A_n^{(0)}(x_i) \qquad \mathcal{F}_n^{(0)}(\hat\tau) = \frac{\Gamma(2n-2)\Gamma(n-\frac{5}{2})}{16\sqrt{2}\,\pi^{2n-6}} E_{n-4}(\tfrac{3}{2},\hat\tau) \qquad A_n^{(0)}(x_i) = D_{\underbrace{44\ldots4}_{n}}(x_i)$

Eisenstein modular form weight (n-4,4-n)

- Similarly the $N^{-\frac{1}{2}}$ contribution involves $E_{n-4}(\frac{5}{2},\hat\tau)$ and combinations of $A_n^{(0)}(x_i)$ and $A_n^{(2)}(x_i)$

$$A_n^{(2)}(x_i) = (x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)\, D_{\underbrace{55554\ldots4}_{n}}(x_i)$$

(two derivatives of D-function)

- At order N^{-1} and $n \geq 6$ there are two independent kinematic basis elements

$$x_{14}^2 x_{24}^2 x_{34}^2 D_{555744}(x_i)\,, \quad x_{12}^2 x_{34}^2 x_{56}^2 D_{555555}(x_i)$$

- Also two independent (n-4, 4-n) modular forms for each value of $n \geq 6$.

  One of these is determined from the n=5 case by the recursion relation.

  The other can be fixed by the flat-space limit.

FINALLY

- These considerations have determined a number of exact terms in the low energy expansion of type IIB string amplitudes in the large radius limit of $AdS_5 \times S^5$.

- This extends our knowledge of scattering amplitudes expanded around the large-radius (flat-space) and low energy limits.

- They have also led to tantalizing modular-covariant expressions for the all-orders expansion of integrated correlators in $SU(N) \mathcal{N} = 4$ super Yang – Mills theory, iwhich include precisely defined non-perturbative contributions.