The Tree Momentum Kernel.

$$M_n = -\lim_{k_n^2 \to 0} \sum_{a,b} \frac{1}{k_n^2} A(1\,\bar{a},n)\, S(a|b)_1\, A(1\,b,n)$$

$$S(a|b)_1 = \prod_{i=2}^{n-1} \Big(s_{1i} + \prod_{\substack{j >_a i \\ j <_b i}} s_{ij} \Big)$$

$$0 = \sum_{b} S(a|b)_1\, A(1\,b,n) \qquad (k_n^2 = 0)$$

Inverse:

$$\sum_b S(a|b)_1 \frac{m(1\,b,n|1\,c,n)}{S_{1,\dots,n-1}} = \delta_{a,c} \qquad k_n \neq 0$$

Actually, $S(a|b)_1$ are matrix elements of nice map that comes from a Lie algebra.

Plan: 1. > What is the Lie algebra?

2. > How do I get KLT matrix from it?

3. > It's related to BG for gauge theories.

Orderings modulo shuffle

Non-commuting vars: $t_1, t_2, \ldots$ $\quad (t_i^2 = 0)$

$1, 2, 3, \ldots$

W = vec space of orderings: $a = t_3 t_7 t_1 \cancel{t_3}$ ...

$W/_{sh}$ = orderings mod shuffles: set ~~st~~ shuffles $= 0$

i.e. $\quad a \sqcup\!\sqcup b = \sum_{c \in OP(a,b)} c = 0$

Example: $\quad pt(12\cdots k) = 1/z_{12} z_{23} \cdots z_{k-1\,k}$.

$$pt(a \sqcup\!\sqcup b) = 0$$

$$pt(\underline{a} : \underline{b}) = (-1)^{|a|}\, pt(i(\bar{a} \sqcup\!\sqcup b))$$

__Other Example__ : Partial tree amplitudes.

$$A(a, n)$$

$$A(a \sqcup\!\sqcup b, n) = 0$$

$$\updownarrow$$

$$A(aib, n) = (-1)^{a} A(i\,\bar{a} \sqcup\!\sqcup b, n).$$

Lie Bracket On W/sh

("S map," "S bracket")

$i \sqcup\!\sqcup j = ij + ji$

Rule: if i and j single vars,

$t_i \; t_j$

$$\begin{cases} \{\underline{i}, \underline{j}\} = \underbrace{S_{ij}}\; ij \\ \{j, i\} = S_{ij}\; ji = - S_{ij}\; ij \end{cases}$$

$$\{iaj, b\} = i\{aj, b\} - j\{ia, b\}$$

$$\{a, ibj\} = \{a, ib\}j - \{a, bj\}i$$

Solve recursion: $$\{i, a\} = \sum_{a = bjc} S_{ij}\; ij\,(\bar{b} \sqcup\!\sqcup c)\; (-1)^{|b|}$$

$$= \sum_{a = bc} (k_i \cdot k_b)\; bic$$

$$\{a, a'\} = \sum_{\substack{a = b i c \\ a' = b' j c'}} s_{ij} \, (b \sqcup\!\sqcup \hat{c}) \, ij \, (\widehat{b'} \sqcup\!\sqcup c)$$

$$\boxed{\text{Fact: It is Lie.}}$$

$$\{1, \{2,3\}\} = \cancel{S_{12} S_{23}\ 123} - S_{13} S_{23}\ 132$$

$$+ S_{23} S_{13}\ 231 - S_{12} S_{13}\ 213$$

+ cyclic

$$+ S_{13} S_{12}\ 312 - \cancel{S_{23} S_{12}\ 321}$$

$$123 - 321 = 0$$ in the space of orders/shuffle.

$$A(a, n) = (-1)^{|a|+1} A(\bar{a}, n)$$

Momentum Kernel Map

Since $\{\ ,\ \}$ is Lie, there is

$$S : \text{Lie} \longrightarrow \text{W/sh}$$

$$t_1, t_2 \cdots \quad : \quad [\ ,\ ] \rightsquigarrow \{\ ,\ \}$$

$$[t_1,[t_2,t_3]] \quad : \quad [1,[2,3]] \longmapsto \{1,\{2,3\}\}$$

e.g. $S(\underbrace{[1,[2,3]] + [[2,3],1]}_{0}) = \{1,\{2,3\}\} + \{\{2,3\},1\}$.
$= 0$.

$S([1,[2,3]] + \text{cyclic}) = 0$.

$$\boxed{\text{Fact: Matrix elements of } S \text{ are } S(a|b)_1.}$$

i.e. $\tilde{m}(S(T')) = T'$

$\tilde{m}$ & S are <u>inverse maps</u>

$$\mathrm{Comb}(1a) = \tilde{m}(S(\mathrm{Comb}(1a)))$$

$$= \sum_b (S(\mathrm{Comb}(1a)), \mathrm{Comb}(1b))\,\tilde{m}(1b)$$

$S(a|b)_1$

$(R_n^2 \neq 0)$ $\qquad S_{1,\dots,n-1}\,\delta_{a,c} = \sum \underline{S(a|b)_1}\, m(1b,n|1c,n)$

$$S(a,b)_2 = \Big(\underbrace{\{1, \{a_1 \{a_2 \cdots \}\}\}}_{S(\mathrm{Comb}(1a))}, \underbrace{[1, [b_1 [b_2 \cdots]]]}_{\mathrm{Comb}(1b)} \Big)$$

> Can check directly.

> Nicer: properties of $S(a|b)_2$ follow from $\{ , \}$ & bracket.

Off-Shell Fundamental BCJ

(Du Fu Feng)

Biadjoint tree amp is:

$$m_n = \sum_{\Gamma} \frac{\mathrm{tr}(\overbrace{\underbrace{[t_1 \cdots\cdots]}}^{\Gamma}, t_n)\,\mathrm{tr}([t_1' \cdots\cdots], t_n')}{S_{\Gamma}}$$

$$=$$

$$\tilde{m}_n = \sum_{\Gamma} \frac{[t_1 \cdots] \otimes [t_1' \cdots]}{\underbrace{S_{1\cdots n-1}}\, S_{\Gamma}} \qquad (k_n^2 \neq 0)$$

$$\tilde{m}(a) = \sum_{\Gamma} \frac{\Gamma\ (a, \Gamma)}{S_{1\cdots n-1}\, S_{\Gamma}}$$

$$\boxed{\tilde{m}(\{a,b\}) = [\tilde{m}(a), \tilde{m}(b)]}$$

e.g. $\tilde{m}(\{i,a\}) = [t_i, \tilde{m}(a)] \leftarrow$ Du Fu Feng.

$$\Rightarrow \lim \; S_{1\cdots n-1} \; \tilde{m}(\{i,a\}) = 0$$

$$\boxed{\tilde{m}(\{\Gamma\}) = \Gamma}$$

e.g. $\tilde{m}(\underbrace{\{\{1,2\},\{3,4\}\}}) = [[1,2],[3,4]]$

$$\tilde{m}_n = \sum S(a|b)_1 \quad \tilde{m}(1a) \otimes \tilde{m}(1b)$$

Berends-Giele

Perturbinor solve the e.o.m. (e.g. $[D_\mu + A_\mu, F^{\mu\nu}] = 0$) to get series

$$A^\mu = \sum A^\mu_{\Pi}\, \underset{\sim}{\Pi}$$

$$= \sum A_i^\mu\, t_i + \sum A^\mu_{[i,j]}\, \underline{[t_i, t_j]} + \dots .$$

then

$$A^\mu(a) = \sum A_\Pi{}^\mu (a, \Pi)$$

is <u>BG currents</u>.

> This uniquely inverts.

$$A^{\mu}(a) = \sum \frac{N^{\mu}_{\Gamma'}(\Gamma', a)}{s_{1\cdots n-1}\, S_{\Gamma'}}$$

where

$$N^{\mu}_{\Gamma'} = A^{\mu}(\{\Gamma'\}) = \sum_{a,b} (1a, \Gamma')\, S(1a|1b)\, A^{\mu}(1b)$$

> $N^{\mu}_{\Gamma'}$ satisfies Jacobi + antisymmetry.

> Not local for all gauge choices.

$$\begin{cases} N_i^{\mu} = \epsilon_i^{\mu} \\ N_{[i,j]}^{\mu} = (k_i^{\mu} - k_j^{\mu})\, \epsilon_i \cdot \epsilon_j + 2\, \epsilon_i \cdot k_j\, \epsilon_j^{\mu} - (ij). \\ \vdots \end{cases}$$

> What is off-shell BCJ for each gauge?

i.e.

$$A^{\mu}(\{a,b\}) = \{A(a), A(b)\}^{\mu}_{\text{kin}}$$

then $N_{\Gamma}{}^{\mu}$ would satisfy

$$N_{[\Gamma,\Gamma']}{}^{\mu} = \{N_{\Gamma}, N_{\Gamma'}\}^{\mu}_{\text{kin}}.$$

> General structure: relation/pairing between

trees $\longleftrightarrow$ orderings.

quark diagrams $\longleftrightarrow$ marked surfaces.