Constraining EFTs with causality

1. Dispersion relations:
   gravity vs. other forces

2. Constraints on (non-gravitational) EFTs
   Dimensional analysis scaling:
   rule of thumb or theorem?

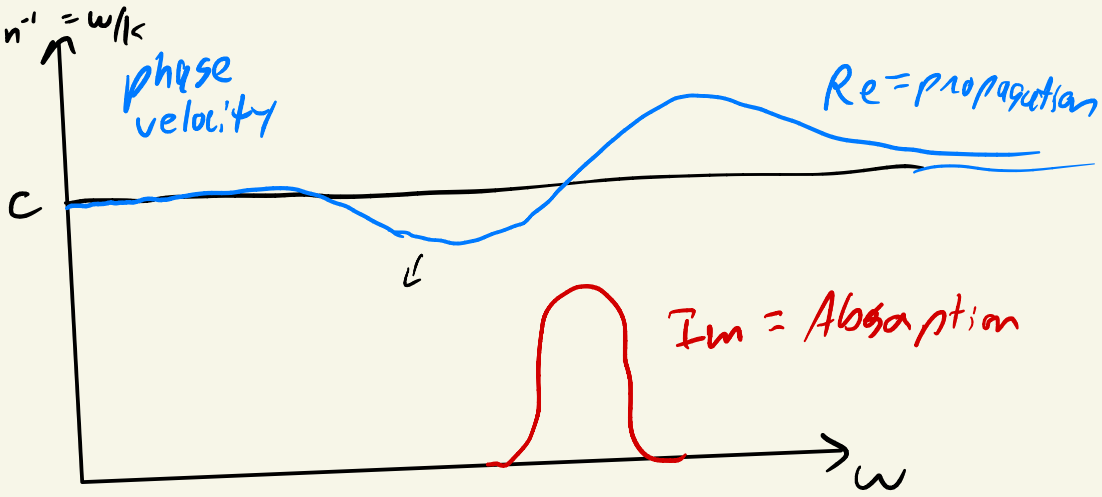
On: SCH + V. Duong 2011.02957
SCH + Mazáč + Rastelli + Simmons-Duffin 2008.04931 + in progress

related: Bellazzini, Miró, Rattazzi, Riembau + Riva,
Tolley, Wang + Zhou

(Caron) + Penedones, Silva, Zhiboedov
Arkani-Hamed + Huang

Kramers-Kronig (20's)

$$\frac{k}{\omega} = n(\omega) = 1 + \frac{1}{\pi}\int \frac{d\omega'\ \mathrm{Im}\, n(\omega')}{\omega'-\omega-i0}$$ ← absorption

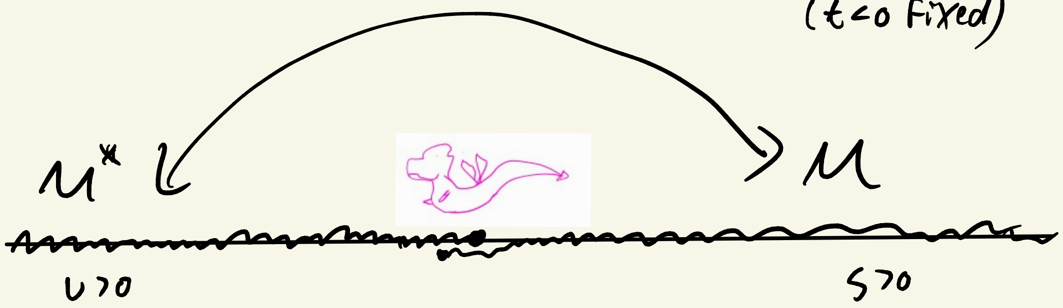


UV can only slow you down.

[level repulsion]

2→2 amplitude $\mathcal{M}(s,t)$

$\underline{\lvert s}$ ($t<0$ fixed)



Causality + Unitarity $\Rightarrow$ $\mathcal{M}(s,t)$ analytic + bounded at large complex s.

$$\boxed{\frac{\mathcal{M}(s,t)}{s^2} \xrightarrow[s\to\infty]{} 0 \qquad (t<0)}$$

$$\oint_\infty \frac{ds\, \mathcal{M}(s,t)}{s^3} = 0$$

Implications

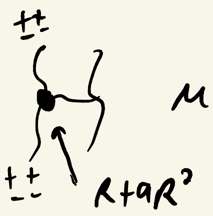
(Adams et al '06)

i) Sign constraints on EFT

$$\mathcal{L} \supset + \frac{g_2}{2}(\partial\phi)^4 \qquad \mathcal{M} \sim + s^2$$

ii) gravity is attractive. $\mathcal{M} \sim + \frac{8\pi G_N s^2}{-t}$.

... at all impact parameters:



$$\mathcal{M} \sim \frac{G_N}{b^{d-4}}\begin{pmatrix} 1 & a/b^2 \\ a/b^2 & 1 \end{pmatrix} \succeq 0 \Rightarrow |a| \leq b_{min}^2.$$

[CEMZ '14]

Just for gravity or generic ?

- Causality → dim. analysis scaling?

- IR ⇄ UV (graviton pole predicts ∞ many UV sum rules)

- parametric versus numerical ?
  $\lesssim$ $\qquad$ $<$

↳ EFT for a single real scalar

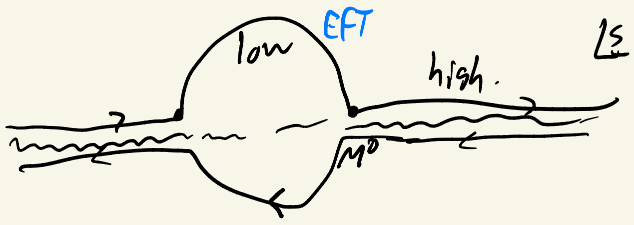
$$\mathcal{M}_{\rm low}(s,t) = -g^2\left[\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right]-\lambda$$
$$+\,g_2(s^2+t^2+u^2)+g_3(stu)+g_4(s^2+t^2+u^2)^2+g_5(s^2+t^2+u^2)(stu)$$
$$+\,g_6(s^2+t^2+u^2)^3+g_6'(stu)^2+g_7(s^2+t^2+u^2)^2(stu)+\cdots$$

Assumptions: - EFT weakly coupled below M^2

- Amplitude causal + unitary above M^2

⇒ sum rules giving g_k ↔ Im M in UV

For $t<0$:



$= 0$

$$B_2^{\rm low}(t) = \oint_{\sim M^2} \frac{ds}{s}\,\frac{M(s,t)}{s(s+t)} = \int_{\rm heavy} (\cdots)\,\mathrm{Im}\,M$$

2-subtracted

$\sum_J \int \frac{dm^2}{m^2}\,\mathrm{Im}\,a \geq 0$ $\Downarrow$

$$B_2:\ 2g_2 - g_3 t + 8g_4 t^2 + \cdots = \left\langle \frac{(2m^2+t)\,P_J\!\left(1+\frac{2t}{m^2}\right)}{m^2(m^2+t)^2}\right\rangle_{\rm heavy}$$

$$B_4:\ 4g_4 + \cdots = \left\langle \frac{(2m^2+t)\,P_J\!\left(1+\frac{2t}{m^2}\right)}{m^4(m^2+t)^3}\right\rangle_{\rm heavy}$$

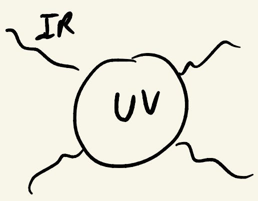
$s^4+t^4+u^4$?

First few cases:

$$g_2 = \left\langle \frac{1}{m^4}\right\rangle,\quad g_3 = \left\langle \frac{3-\frac{4}{d-2}J^2}{m^6}\right\rangle,\quad g_4 = \left\langle \frac{1}{2m^8}\right\rangle$$

$$0 = \left\langle \frac{J^2(2J^2-(5d-4))}{m^8}\right\rangle$$

"Null constraints" on Im M_{UV}



Im $a_J(m)_{\rm heavy}$ knows about IR crossing symmetry!!

$$g_2 = \left\langle \frac{1}{m^4} \right\rangle, \quad g_3 = \left\langle \frac{3 - \frac{4}{d-2} J^2}{m^6} \right\rangle, \quad g_4 = \left\langle \frac{1}{2m^8} \right\rangle$$

$$\left\langle \frac{J^2(2J^2 - (5d-4))}{m^8} \right\rangle = 0 \ (+\text{EFT loops})$$

Easy implications: since $m > M$ cutoff,

$$g_2 > 0, \quad g_3 < \frac{3g_2}{M^2}, \quad g_4 < \frac{g_2}{2M^4} \ \ldots$$

To lower-bound g_3: use Null constraint + Cauchy Schwartz

$$\left\langle \frac{1}{m^4} \frac{\overbrace{J^2}^{=b^2}}{m^2} \right\rangle \leq \frac{1}{M^2} \left\langle \frac{1}{m^4} \right\rangle \cdot \frac{5d-4}{2}$$

As far as sum rules care, all heavy states have size $b \lesssim M$.

(* BH's, and strings, grow faster, but irrelevant due to $\frac{1}{m^4}$)

$\Rightarrow$ two-sided bounds on all coefficients!

Optimal bounds

"Dual extremization problem":

Find positive combinations of sum rules.

Let: $F(J, m^2) = G_3(J, m^2) - a G_2(J, m^2) + \text{null} \geq 0 \ \ \forall_{J}^{m>M}$

$\langle F \rangle \geq 0 \Rightarrow \ g_3 \geq a g_2$.

Results

Box bounds on ratios $\qquad \tilde{g}_k = \frac{g_k}{g_2} \cdot M^{\#} \leftarrow (\partial\phi)^4$

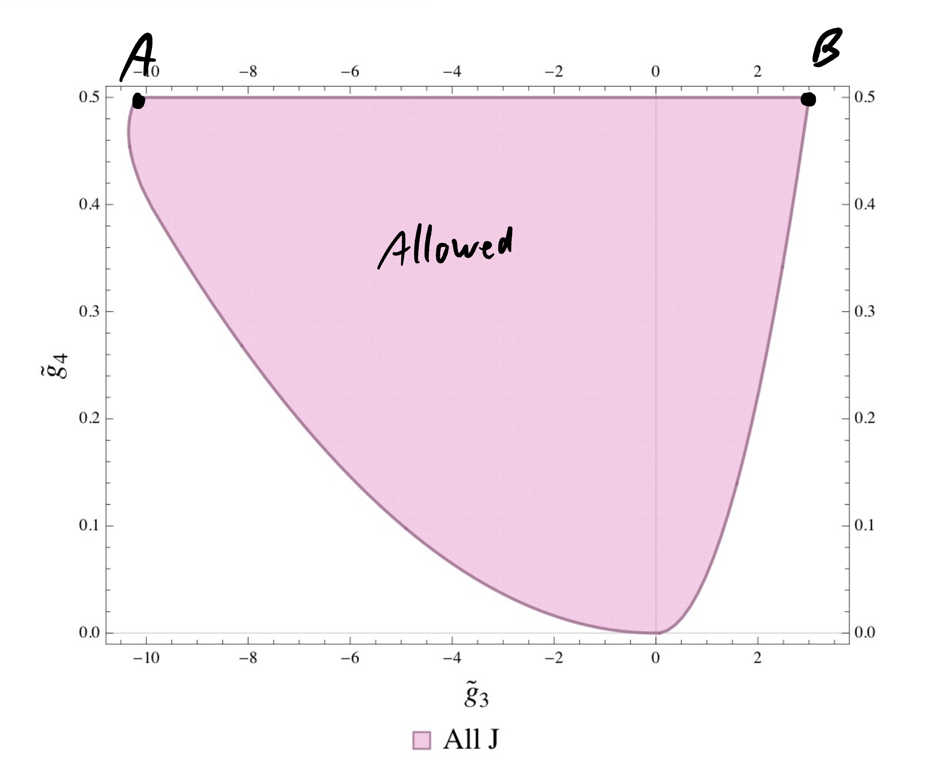
EFT coefficient	Lower bound	Upper bound
$\tilde{g}_3$	-10.346	3
$\tilde{g}_4$	0	0.5
$\tilde{g}_5$	-4.096	2.5
$\tilde{g}_6$	0	0.25
$\tilde{g}'_6$	-12.83	3
$\tilde{g}_7$	-1.548	1.75
$\tilde{g}_8$	0	0.125
$\tilde{g}'_8$	-10.03	4
$\tilde{g}_9$	-0.524	1.125
$\tilde{g}'_9$	-13.60	3
$\tilde{g}_{10}$	0	0.0625
$\tilde{g}'_{10}$	-6.32	3.75

+ strong coupling bound:

$$0 \leq \frac{g_2}{(4\pi)^2} \leq \frac{0.794}{M^4} \qquad (d = 4)\,.$$

= dim. analysis scaling bounds on all couplings!

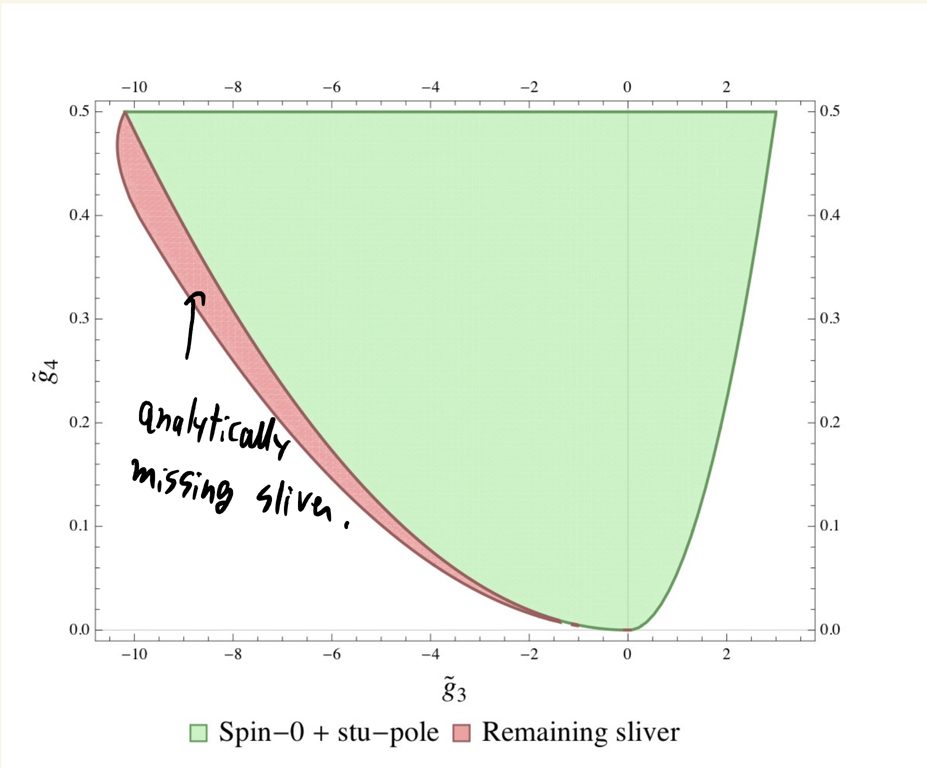
Allowed region for $\frac{g_3}{g_2} \leftarrow \frac{stu}{s^2+t^2+u^2}$, $\frac{g_4}{g_2} \leftarrow (s^2+t^2+u^2)^2$,

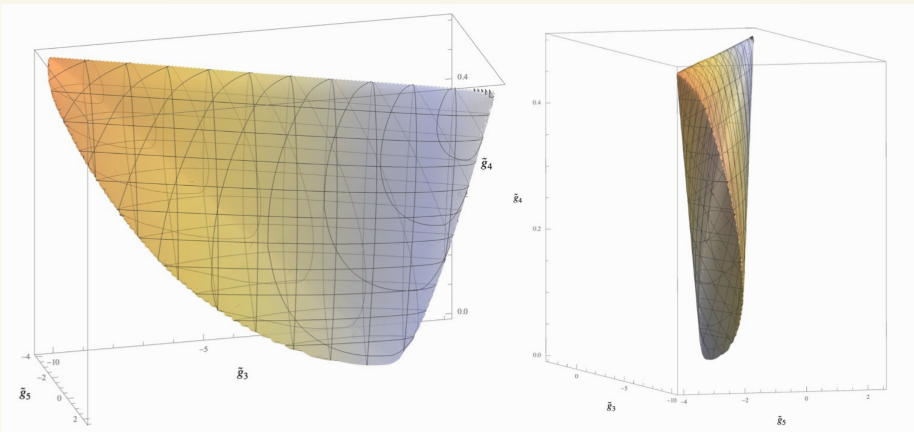


Ruling in: $M_B = \left(\frac{1}{M^2-s} + \frac{1}{M^2-t} + \frac{1}{M^2-u}\right)$

$$M_A = \frac{1}{(M^2-s)(M^2-t)(M^2-u)} - \# M_B$$

satisfy all axioms!





(a) General shape of the volume. (b) View of the narrow ridge along $\tilde{g}_4 = \frac{1}{2}$.

Figure 11: $\tilde{g}_3$ vs. $\tilde{g}_4$ vs. $\tilde{g}_5$ "tortilla chip". Numerics were performed at $n = 10$ Mandelstam order and $J = 0, 2, \ldots, 40$. The volume is surprisingly narrow, showing a strong $\tilde{g}_3$-$\tilde{g}_5$ correlation.

Summary

* Causality ⇒ two-sided bounds on generic EFT coefficients

* Was forward limit important?

No! All sum rules saturated by small impact param's

$$\left\langle \frac{\mathcal{J}^2}{m^6} \right\rangle \leq \frac{b}{M^2} \left\langle \frac{1}{m^4} \right\rangle$$

$(x^+\partial x)^2 \sim s$

Future

↳ Non-identical scalars, fermions etc: dim-6 ops?

↳ $G_N \neq 0$ ⇒
- pole may weaken scalar EFT bounds
- bound → generic corrections to gravity
  → coupling of gravitons to heavy stuff
- WGC?

↳ Embed in AdS: (convergence of 2-sub = thm) ongoing