QCD meets gravity 2020

Northwestern University

Two-loop five-point amplitudes in string and field theory

Oliver Schlotterer (Uppsala University)

based on 2006.05270 and 2008.08687

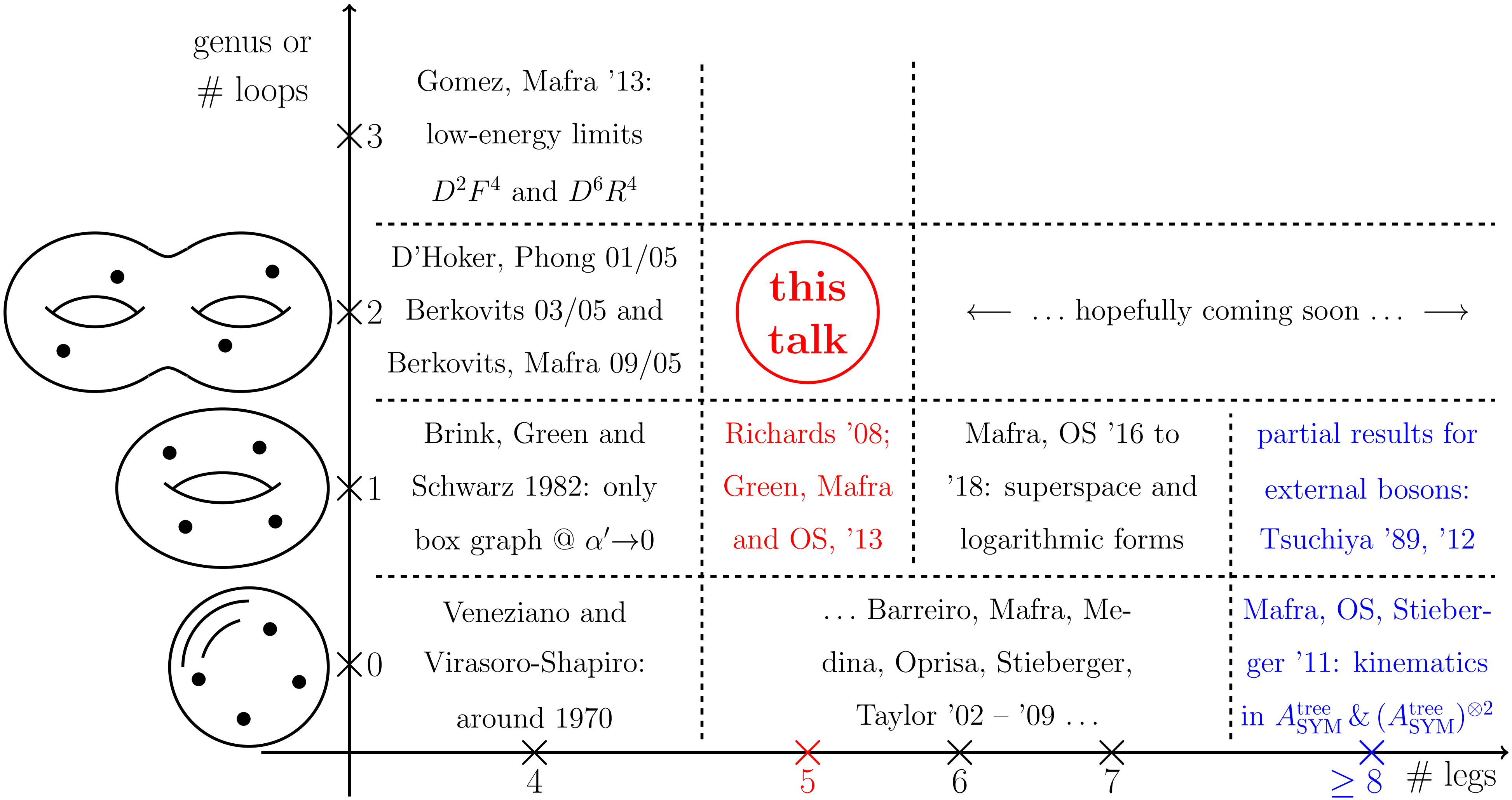
with Eric D'Hoker, Carlos Mafra and Boris Pioline

01.12.2020

Status report on explicit string amplitudeology

Overview: string amplitudes with explicitly evaluated/simplified integrands

(here: massless external type-II closed-string states in 10-dim Minkowski)



Motivation: Why 2-loop 5pt if we already know 2-loop 4pt?

- first showcase of loop momenta in chiral correlators / kin. numerators & BCJ moves between different diagrams (penta-box vs. double-box)
  [Carrasco, Johansson 1106.4711; Mafra, OS 1505.02746]

- simplest non-zero $U(1)$-violating amplitudes in type IIB and new eff. interactions D^6R^5 (without D^8R^4) $\Rightarrow$ lessons for S-duality
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- richer sets of modular graph functions at higher genus in α'-expansion, testing grounds for relations & generalizations to modular graph tensors

  [D'Hoker, Green, Pioline 1712.06135 & 1806.02691; D'Hoker, OS 2010.00924]

- zero-mode structure in the pure-spinor formalism similar to 3-loop 4pt

  [Gomez, Mafra 1308.6567]

Outline

I. Review of two-loop four points

[D'Hoker, Phong 0501197; Berkovits 0503197; Berkovits, Mafra 0509234]

II. Chiral splitting

[D'Hoker, Phong '88 & '89; Tourkine 1901.02432]

III. The two-loop five-point integrand

[D'Hoker, Mafra, Pioline, OS 2006.05270 & 2008.08687]

IV. Conclusion

I. 1 Two-loop four-point amplitude of type-II superstrings

With four external supergravity states in $D = 10$:

$$\mathcal{A}^{2\,\text{loop}}_{4\,\text{pt}} \sim |t_8(f_1, f_2, f_3, f_4) + \text{SUSY}|^2 \int_{\mathcal{M}_2} \frac{\mathrm{d}^6\Omega}{(\det \operatorname{Im}\Omega)^5}$$

$$\times \int_{\Sigma^4} \big|\Delta(z_1, z_2)\Delta(z_3, z_4) k_2 \cdot k_3 + \Delta(z_2, z_3)\Delta(z_4, z_1) k_1 \cdot k_2\big|^2 \, \text{KN}^{2\,\text{loop}}_{4\,\text{pt}}$$

[D'Hoker, Phong 0501197; Berkovits 0503197; Berkovits, Mafra 0509234]

- polarizations from $f_j^{mn} = \epsilon_j^m k_j^n - \epsilon_j^n k_j^m$ factor out from scalar integral

$$t_8(f_1, f_2, f_3, f_4) = \text{Tr}(f_1 f_2 f_3 f_4) - \tfrac{1}{4}\text{Tr}(f_1 f_2)\text{Tr}(f_3 f_4) + \text{cyc}(2, 3, 4)$$

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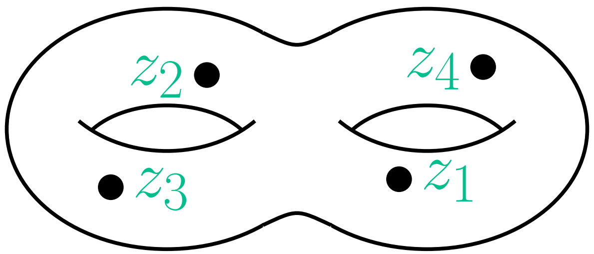
- integrate z_1, z_2, z_3, z_4 over four copies of genus-two surface Σ



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- integrate z_1, z_2, z_3, z_4 over four copies of genus-two surface Σ

- furthermore integrate over moduli space $\mathcal{M}_2$ of genus-two surfaces, parametrized by complex 2×2 period matrices $\Omega_{IJ} = \Omega_{JI}$

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[D'Hoker, Phong 0501197; Berkovits 0503197; Berkovits, Mafra 0509234]

- universal Koba-Nielsen factor (generalizing $|z_i - z_j|^{\alpha' k_i \cdot k_j}$ at tree level)

$$\text{KN}^{g\,\text{loop}}_{n\,\text{pt}} = \exp\Big(-\frac{\alpha'}{2} \sum_{1 \leq i < j}^{n} k_i \cdot k_j \underbrace{G_g(z_i, z_j | \Omega)}_{\uparrow} \Big)$$

Green fct. @ genus g

e.g. $G_{g=0}(z_i, z_j) = -\log|z_i - z_j|^2$

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- "$\mathrm{d}^2 z_j$" beyond genus 1: hol. differentials $\mathrm{d}z_j\to\omega_I(z_j)$, $I=1,2$, and

$$\Delta(z_a,z_b) = \varepsilon^{IJ}\omega_I(z_a)\omega_J(z_b) = \omega_1(z_a)\omega_2(z_b)-\omega_2(z_a)\omega_1(z_b)$$

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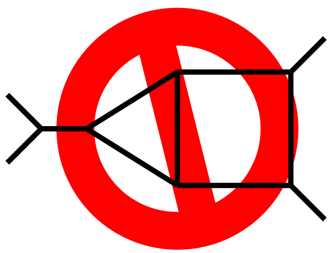
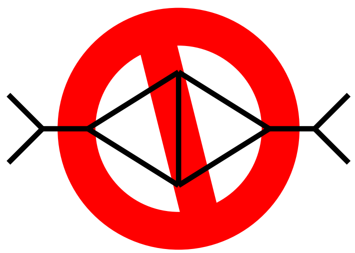
[D'Hoker, Phong 0501197; Berkovits 0503197; Berkovits, Mafra 0509234]

Particularly noteworthy what is NOT there in the integrand:

- no $(z_i-z_j)^{-1}$ singularities $\Rightarrow$ no $\displaystyle\frac{1}{\alpha' k_i\cdot k_j} = \int_{|z_i-z_j|<\epsilon} \frac{\mathrm{d}^2 z_i\,|z_i-z_j|^{\alpha' k_i\cdot k_j}}{2\pi\,|z_i-z_j|^2}$
- therefore, no reducible Feynman graphs in field-theory limit $\alpha'\to 0$:

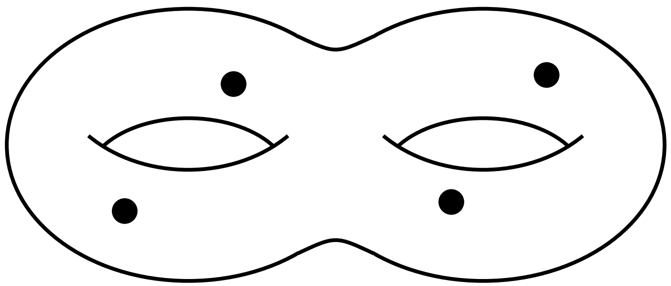
e.g.  and 

I. 2 Field-theory limit at two-loop four points

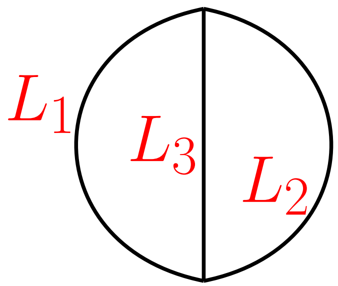
Supergravity graphs @ $\alpha' \to 0$ in "tropical degeneration" $\mathrm{Im}\,\Omega_{IJ} \to \infty$

$$\text{at finite } \alpha'\,\mathrm{Im}\,\Omega = \begin{pmatrix} L_1+L_3 & -L_3 \\ -L_3 & L_2+L_3 \end{pmatrix}$$

genus two worldsheet



$\xrightarrow[\alpha' \to 0]{\text{trop}}$

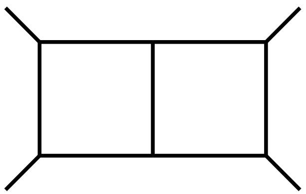


worldline skeleton

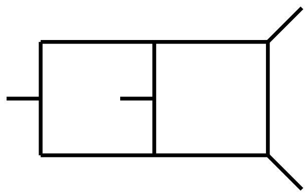
Get Schwinger parametrization of planar and non-planar double boxes

$$\mathcal{A}^{2\,\text{loop}}_{4\,\text{pt}} \to |t_8(f_1,f_2,f_3,f_4)+\text{SUSY}|^2 \int_0^\infty \frac{\mathrm{d}L_1\,\mathrm{d}L_2\,\mathrm{d}L_3}{(L_1L_2+L_1L_3+L_2L_3)^{D/2}}$$

$$\times \int_{\Sigma^4} \big|\Delta(z_1,z_2)\Delta(z_3,z_4)k_2\cdot k_3 + \Delta(z_2,z_3)\Delta(z_4,z_1)k_1\cdot k_2\big|^2\, \mathrm{KN}^{2\,\text{loop}}_{4\,\text{pt}}\Big|_{\text{trop}}$$



and



[Tourkine 1309.3551]

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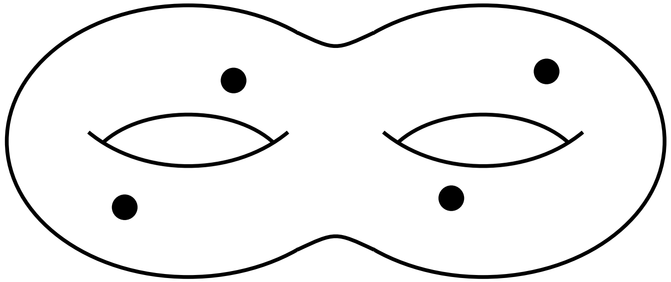

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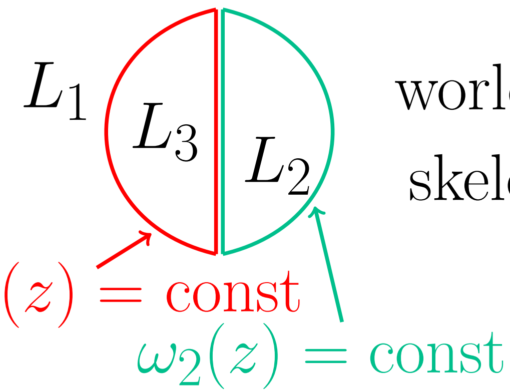

worldline skeleton

Refined look – why **only** double-boxes?

$$\mathcal{A}^{2\,\text{loop}}_{4\,\text{pt}} \ \to \ |t_8(f_1,f_2,f_3,f_4)+\text{SUSY}|^2 \int_0^\infty \frac{\mathrm{d}L_1\,\mathrm{d}L_2\,\mathrm{d}L_3}{(L_1L_2+L_1L_3+L_2L_3)^{D/2}}$$

$$\times \int_{\Sigma^4} \big|\underbrace{\Delta(z_1,z_2)}_{=\,-\Delta(z_2,z_1)}\Delta(z_3,z_4)k_2\cdot k_3 + \Delta(z_2,z_3)\Delta(z_4,z_1)k_1\cdot k_2\big|^2 \,\mathrm{KN}^{2\,\text{loop}}_{4\,\text{pt}}\Big|_{\text{trop}}$$

$= -\Delta(z_2,z_1)$, vanishes if $z_1, z_2 \in$ same worldline edge

$\Delta(z_1,z_2)\to 0$
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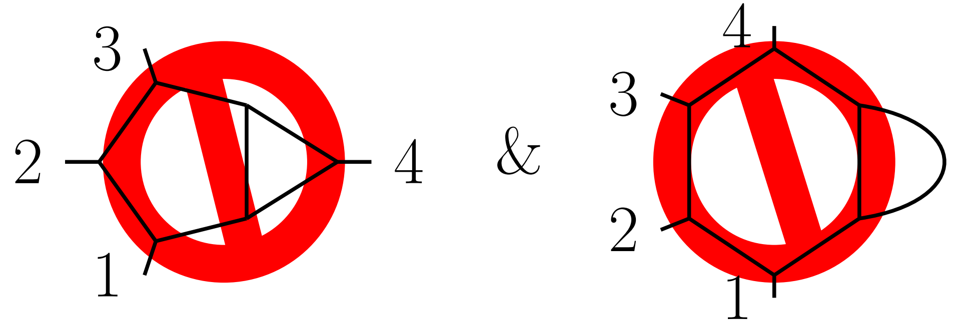
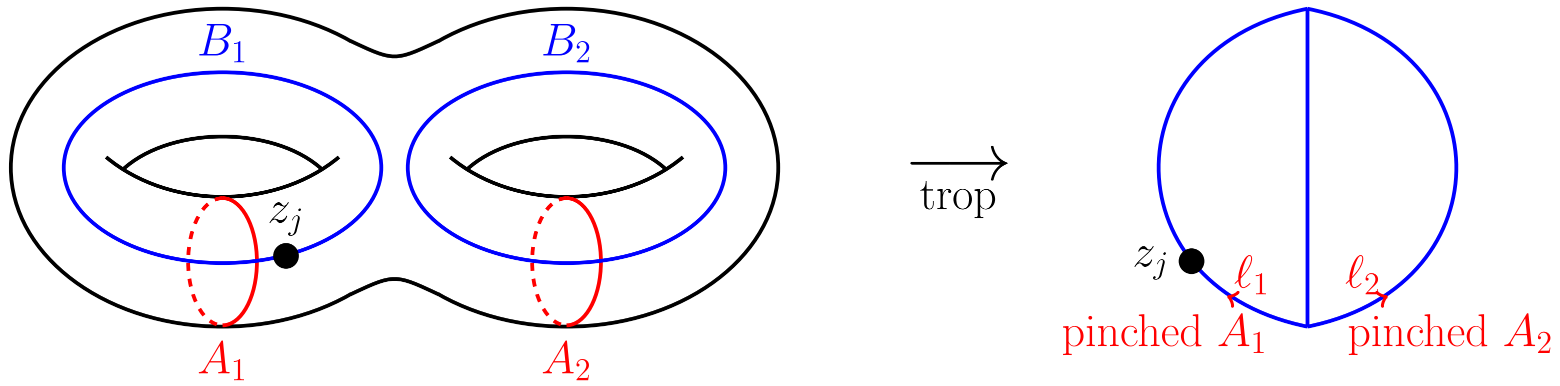


all $\Delta(z_a,z_b) \to 0$

II. 1 Chiral splitting and global loop integrands from strings

Loop momenta ℓ_I in string theory = zero modes w.r.t. A_I cycles



$$\ell_I^m = \underbrace{\frac{1}{2\pi}\oint_{A_I} \partial_z X^m = \frac{1}{2\pi}\oint_{A_I} \partial_{\bar z} X^m}_{\text{shared between left- and right-movers}}\,, \qquad I = 1, 2, \ldots, g$$

[D'Hoker, Phong '88, '89]

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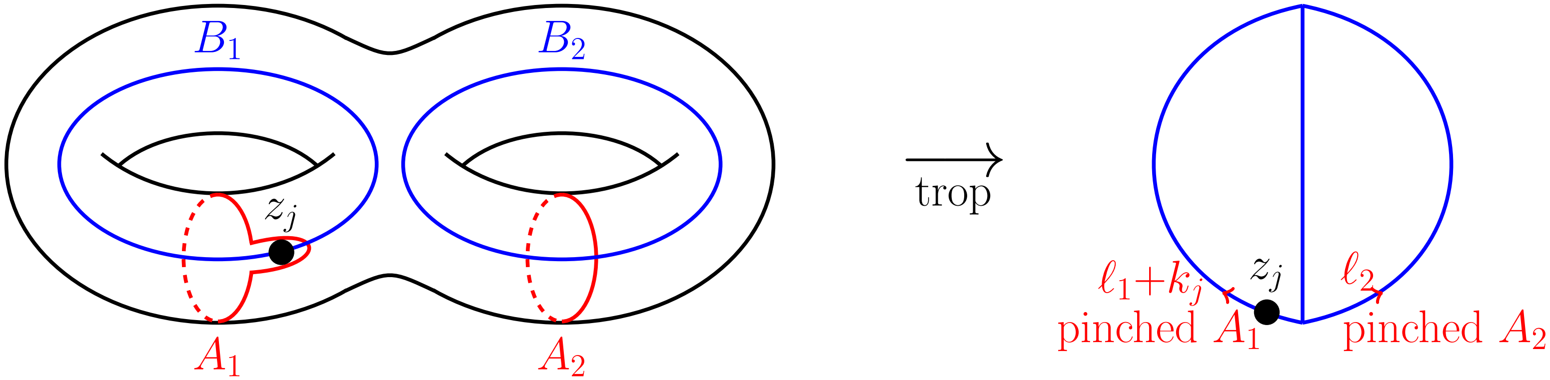

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[D'Hoker, Phong '88, '89]

Loop momentum jumps when transporting punctures around B_I cycles

$$z_j \to z_j + B_1 \quad \Longrightarrow \quad A_1 \to A_1 + \underset{z_j}{\odot} \quad \Longrightarrow \quad \ell_1 \to \ell_1 + k_j$$

Reminiscent of "labelling problem" in field-theory graphs – how to select

ℓ_1 versus $\ell_1 + k_j$, or more generally, how to globally define loop integrand?

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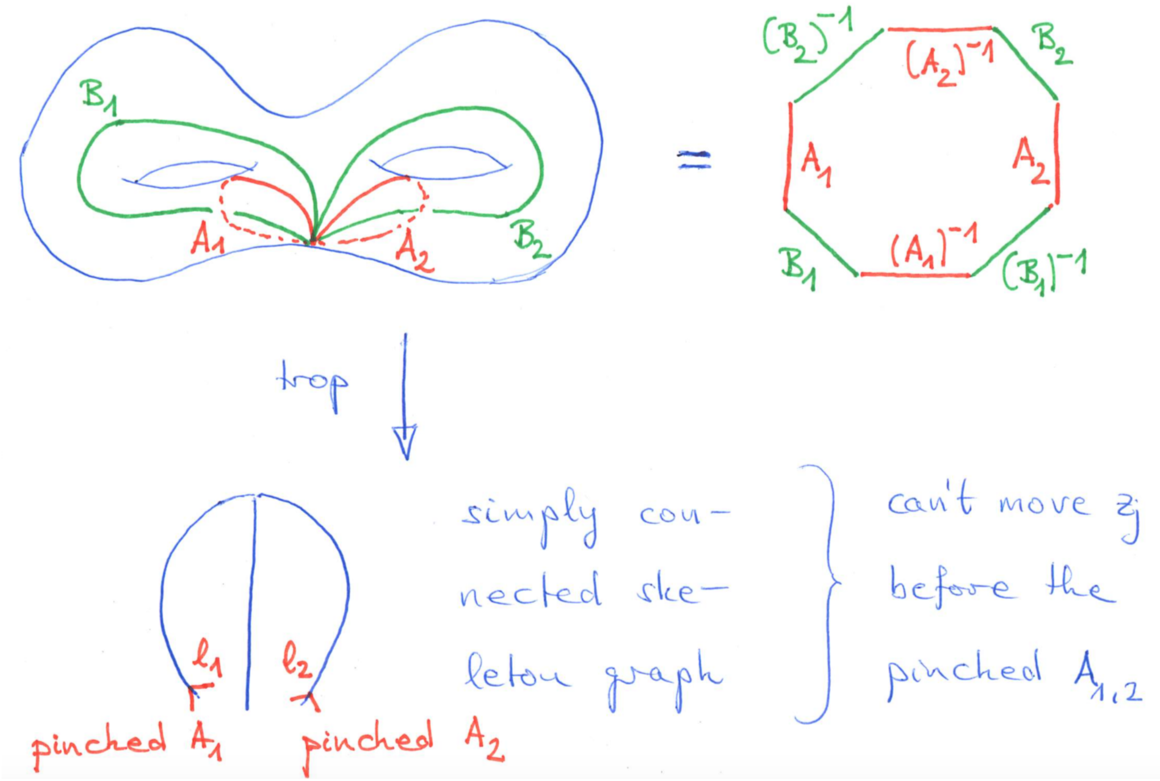
Loop momentum jumps when transporting punctures around B_I cycles

$$z_j \to z_j + B_1 \quad \Longrightarrow \quad A_1 \to A_1 + \underset{z_j}{\circlearrowleft\!\!\odot} \quad \Longrightarrow \quad \ell_1 \to \ell_1 + k_j$$

Resolution: Forbid B_I-cycle transport of z_j via "canonical dissection"



$\Longrightarrow$ globally defined loop momenta w.r.t. reference dissection.

[Tourkine 1901.02432]

II. 2 Chiral splitting – cradle of double copy

Left- and right-movers of closed strings **only** coupled via joint zero mode

$$\ell_I^m = \frac{1}{2\pi}\oint_{A_I} \partial_z X^m = \frac{1}{2\pi}\oint_{A_I} \partial_{\bar z} X^m \, , \qquad I = 1,2,\ldots,g$$

Outsource finite-dim zero-mode integral $\int \mathrm{d}^{gD}\ell$ from path integral $\int \mathcal{D}[X]$

$\Longrightarrow$ closed-string integrands at fixed loop momentum ℓ_I factorize into

$$\text{chiral amplitudes} \ \ \mathcal{F}_n(\underbrace{\epsilon,\chi}_{\text{gauge polarizations}},k,\ell|\underbrace{z,\Omega}_{\text{mero-morphic}})$$

$$\mathcal{A}_{n\,\mathrm{pt}}^{2\,\mathrm{loop}} \sim \int_{\mathbb{R}^{2D}} \mathrm{d}^{2D}\ell \int_{\mathcal{M}_2} \mathrm{d}^6\Omega \int_{\Sigma^n} \mathcal{F}_n(\epsilon,\chi,k,\ell|z,\Omega)\,\overline{\mathcal{F}_n(\tilde\epsilon,\tilde\chi,k,\ell|z,\Omega)}$$

[D'Hoker, Phong '88, '89]

Leftover task in this talk: Compute the chiral amplitude $\mathcal{F}_5$ at five points

$\longrightarrow$ solve a simpler "open-string" problem to be viewed as "single-copy"

II. 2 Chiral splitting – cradle of double copy

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Chiral amplitude in the above four-point example

$$\mathcal{F}_4(\epsilon,\chi,k,\ell|z,\Omega) \;=\; t_8(f_1,f_2,f_3,f_4)\Big(\Delta(z_1,z_2)\Delta(z_3,z_4)k_2{\cdot}k_3 + (1\leftrightarrow 3)\Big)\,\mathcal{I}_4$$

with chiral Koba-Nielsen factor

$$\mathcal{I}_n \;=\; \exp\left(i\pi\Omega_{IJ}\ell^I\cdot\ell^J + 2\pi i\sum_{j=1}^n k_j\cdot\ell^I\int_*^{z_j}\omega_I + \frac{\alpha'}{2}\sum_{1\le i<j}^n k_i\cdot k_j \overbrace{\log E(z_i,z_j|\Omega)}^{\text{"prime form", } \log(z_i-z_j) \text{ at higher genus}}\right)$$

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“prime form”, $\log(z_i - z_j)$ at higher genus

$$\mathcal{I}_n = \exp\left(i\pi\Omega_{IJ}\ell^I\cdot\ell^J + 2\pi i\sum_{j=1}^n k_j\cdot\ell^I\int_*^{z_j}\omega_I + \frac{\alpha'}{2}\sum_{1\leq i<j}^n k_i\cdot k_j\,\overbrace{\log E(z_i,z_j|\Omega)}\right)$$

Recover traditional KN_n with modular invariant Green function from $\int_\ell$

$$\underbrace{\mathrm{d}^{2D}\ell\,|\mathcal{I}_n|^2}_{\text{Gaussian}} \sim \text{KN}_n = \underbrace{\exp\left(-\frac{\alpha'}{2}\sum_{1\leq i<j}^n k_i\cdot k_j\,G_g(z_i,z_j|\Omega)\right)}_{\text{not expressible as holomorphic}\,\times\,\text{cplx. conj.}}$$

III. 1 Structure of the five-point integrand

- $\mathcal{F}_4(z,\Omega) \sim \Delta(z_a,z_b)\Delta(z_c,z_d)$ completely fixed by fermionic zero modes

  in the pure-spinor formalism **[Berkovits 0503197; Gomez, Mafra 1003.0678]**

- zero modes entering $\mathcal{F}_5(z,\Omega)$ admit one $\ell^I \omega_I(z_a)$ or OPE singularity

$$\underbrace{\partial_{z_a} \log E(z_a,z_b|\Omega)}_{\frac{1}{z_a-z_b} \text{ @ higher genus}} = \omega_I(z_a) \underbrace{\frac{\partial}{\partial \zeta_I} \log \theta_\nu(\zeta|\Omega) \Big|_{\zeta_I = \int_{z_a}^{z_b} \omega_I}}_{g^I_{a,b} = -g^I_{b,a}} + \begin{pmatrix} \text{terms} \\ \text{that} \\ \text{cancel} \end{pmatrix}$$

- remaining zero-mode dynamics at 5pt: $\Rightarrow \Delta(z_b,z_c)\Delta(z_d,z_e)$ as @ 4pt

III. 1 Structure of the five-point integrand

- $\mathcal{F}_4(z,\Omega) \sim \Delta(z_a,z_b)\Delta(z_c,z_d)$ completely fixed by fermionic zero modes in the pure-spinor formalism [Berkovits 0503197; Gomez, Mafra 1003.0678]
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- remaining zero-mode dynamics at 5pt: $\Rightarrow \Delta(z_b,z_c)\Delta(z_d,z_e)$ as @ 4pt
- group theory $\Rightarrow$ perms$\{\omega_I(z_a)\Delta(z_b,z_c)\Delta(z_d,z_e)\} \in$ cyclic 5-dim basis

$$\mathcal{F}_5(z,\Omega) = \mathcal{I}_5 \Big\{ \omega_I(z_1)\Delta(z_2,z_3)\Delta(z_4,z_5) \underbrace{\mathcal{K}^I_{5,1,2|3,4}} + \text{cyc}(1,2,3,4,5) \Big\}$$

"subcorrelator": all polarizations and linear in ℓ^I and $g^I_{a,b}$

III. 2 Subcorrelators from first principles

$$\mathcal{A}_{5\,\mathrm{pt}}^{2\,\mathrm{loops}} \sim \int_{\mathbb{R}^{20}} \mathrm{d}^{20}\ell \int_{\mathcal{M}_2} \mathrm{d}^6\Omega \int_{\Sigma^5} \mathcal{F}_5(\epsilon, \chi, k, \ell | z, \Omega) \, \overline{\mathcal{F}_5(\tilde{\epsilon}, \tilde{\chi}, k, \ell | z, \Omega)}$$

$$\mathcal{F}_5 = \mathcal{I}_5 \left\{ \omega_I(z_1) \Delta(z_2, z_3) \Delta(z_4, z_5) \, \mathcal{K}^I_{5,1,2|3,4} + \mathrm{cyc}(1,2,3,4,5) \right\}$$

Three principles to determine chiral amplitude $\mathcal{F}_5$ & subcorrelator $\mathcal{K}^I_{5,1,2|3,4}$

- locality: $\mathcal{K}^I_{5,1,2|3,4}$ has no poles in $\ell_I \cdot k_j$ or $k_i \cdot k_j$ before $\int_z$ or $\int_\Omega$
- BRST invariance modulo $\frac{\mathrm{d}}{\mathrm{d}z_j}$ – linearized gauge invariance $\oplus$ SUSY
- "homology invariance": B-cycle monodromies cancelled by shift in ℓ_I

$$\mathcal{F}_5(z_j \to z_j + B_I) = \mathcal{F}_5(\ell_I \to \ell_I + k_j)$$

Together with zero-mode counting of pure-spinor formalism $\Longrightarrow$ $\mathcal{K}^I_{5,1,2|3,4}$

III. 3 Kinematics at two-loop five points

Solution to locality, BRST & homology invariance in terms of

2 types of pure-spinor superfields $T^m_{1,2,3|4,5}$, $S_{1;2|3|4,5} \ni \{\epsilon_i, \chi_i, k_i, \theta\}$

$$\mathcal{K}^I_{5,1,2|3,4} = 2\pi \ell^I_m T^m_{5,1,2|3,4}$$

$\partial_{\zeta_I} \log \theta_\nu(\zeta|\Omega)$ at $\zeta_I = \int_{z_1}^{z_2} \omega_I$

$$+ \{ g^I_{3,1} S_{1;3|4|2,5} + g^I_{4,1} S_{1;4|3|2,5} + g^I_{2,1} (S_{1;2|5|3,4} - S_{2;1|5|3,4}) + \text{cyc}(5,1,2) \}$$

Bosonic components of $T^m_{1,2,3|4,5}$, $S_{1;2|3|4,5}$ $\rightarrow$ effectively t_8 and ε_{10} tensors

$$R_{1;2|3,4,5} = (\epsilon_1 \cdot k_2) t_8(f_2, f_3, f_4, f_5) - \tfrac{1}{2} t_8([f_1, f_2], f_3, f_4, f_5)$$

$$T^m_{1,2,3|4,5}\big|_{\text{bos}} = 2k_4 \cdot k_5 \big[\epsilon^m_1 t_8(f_2, f_3, f_4, f_5) + (1 \leftrightarrow 2,3,4,5) \big] - k_4 \cdot k_5 \varepsilon^m_{10}(\epsilon_1, f_2, f_3, f_4, f_5)$$
$$+ i \big[k^m_1 (R_{1;2|3,4,5} + R_{1;3|2,4,5}) + (1 \leftrightarrow 2,3) \big] + 2i \big[k^m_4 R_{4;5|1,2,3} + (4 \leftrightarrow 5) \big]$$

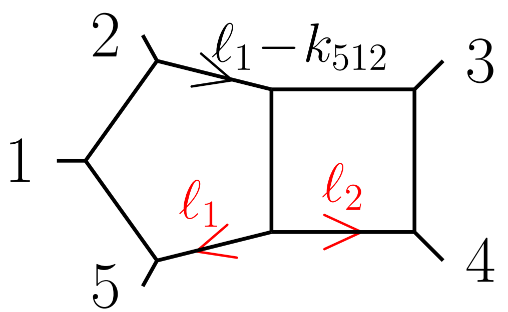
$$S_{1;2|3|4,5}\big|_{\text{bos}} = (2k_4 \cdot k_5 - k_1 \cdot k_2) R_{1;2|3,4,5} - k_1 \cdot k_2 R_{1;3|2,4,5}$$
$$+ 2 \big[k_3 \cdot k_4 R_{4;5|1,2,3} - k_4 \cdot k_5 R_{4;3|1,2,5} + (4 \leftrightarrow 5) \big]$$

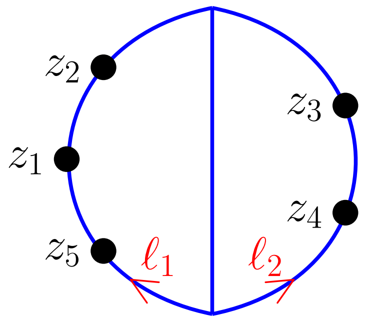
III. 4 Field-theory limit at two-loop five points

In field-thy limit $\mathrm{Im}\,\Omega \to \infty$, set $\Delta(z_a, z_b) \to 0$ if $z_a, z_b \in$ same worldline

$$\mathcal{A}_{5\,\mathrm{pt}}^{2\,\mathrm{loops}} \sim \int_{\mathbb{R}^{20}} \mathrm{d}^{20}\ell \int_{\mathcal{M}_2} \mathrm{d}^6\Omega \int_{\Sigma^5} \mathcal{F}_5(\epsilon,\chi,k,\ell|z,\Omega)\,\overline{\mathcal{F}_5(\tilde\epsilon,\tilde\chi,k,\ell|z,\Omega)}$$

$$\mathcal{F}_5 \;=\; \mathcal{I}_5 \Big\{ \underbrace{\omega_I(z_1)\Delta(z_2,z_3)\Delta(z_4,z_5)}_{=\,const\ \text{on penta-box below}}\, \mathcal{K}^I_{5,1,2|3,4} + \underbrace{\mathrm{cyc}(1,2,3,4,5)}_{=\,0\ \text{on penta-box below}} \Big\}$$





$$\Delta(z_5, z_1) = 0$$
$$\Delta(z_1, z_2) = 0$$
$$\Delta(z_3, z_4) = 0$$

III. 4 Field-theory limit at two-loop five points

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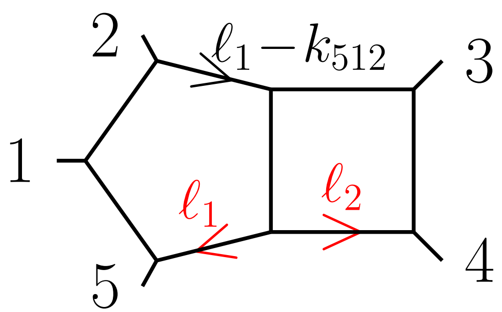
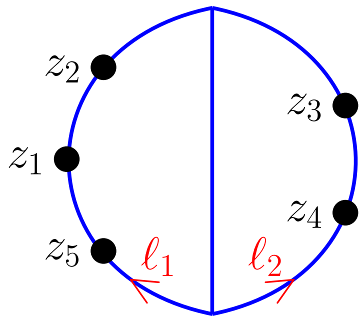
$$\mathcal{F}_5 = \mathcal{I}_5 \Big\{ \underbrace{\omega_I(z_1)\Delta(z_2, z_3)\Delta(z_4, z_5)}_{=\,const \text{ on penta-box below}} \mathcal{K}^I_{5,1,2|3,4} + \underbrace{\text{cyc}(1,2,3,4,5)}_{=\,0 \text{ on penta-box below}} \Big\}$$





$$\Delta(z_5, z_1) = 0$$
$$\Delta(z_1, z_2) = 0$$
$$\Delta(z_3, z_4) = 0$$

Hence, penta-box numerator = subcorrelator @ depicted skeleton graph Γ

$$\mathcal{K}^I_{5,1,2|3,4} = 2\pi \ell^I_m T^m_{5,1,2|3,4}$$

all $g^I_{a,b} \to \pm i\pi$ @ $\operatorname{Im}\Omega \to \infty$
with $\pm$ depending on Γ

$$+ \big\{ g^I_{3,1} S_{1;3|4|2,5} + g^I_{4,1} S_{1;4|3|2,5} + g^I_{2,1} (S_{1;2|5|3,4} - S_{2;1|5|3,4}) + \text{cyc}(5,1,2) \big\}$$

III. 4 Field-theory limit at two-loop five points

Penta-box numerator $N_{5,1,2|3,4}(\ell)$ = subcorrelator @ depicted skeleton Γ

$$\mathcal{K}^I_{5,1,2|3,4} = 2\pi \ell^I_m T^m_{5,1,2|3,4}$$

all $g^I_{a,b} \to \pm i\pi$ @ $\mathrm{Im}\,\Omega \to \infty$
with $\pm$ depending on Γ

$$+ \{ g^I_{3,1} S_{1;3|4|2,5} + g^I_{4,1} S_{1;4|3|2,5} + g^I_{2,1} \underbrace{(S_{1;2|5|3,4} - S_{2;1|5|3,4})}_{T_{12,5|3,4}} + \mathrm{cyc}(5,1,2) \}$$

Working out signs, normalization factors @ kin. identities:

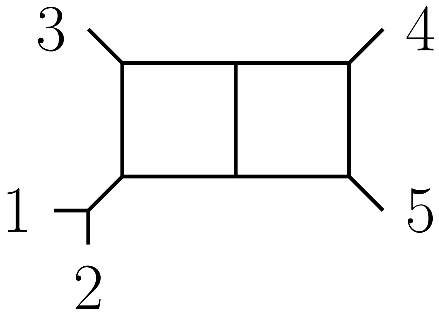
$$\Rightarrow \quad N_{5,1,2|3,4}(\ell) = i \Big[\ell_1^m - \tfrac{1}{2}(k_5 + k_1 + k_2)^m \Big] T^m_{5,1,2|3,4} + \tfrac{1}{2}\left(T_{12,5|3,4} + T_{51,2|3,4} + T_{52,1|3,4}\right)$$

Reproduces earlier numerator proposed via locality & BRST invariance.

[Mafra, OS 1505.02746]

III. 4 Field-theory limit at two-loop five points

Similarly, extract double-box numerators from $|\partial \log E(z_1, z_2)|^2$ in $|\mathcal{F}_5|^2$



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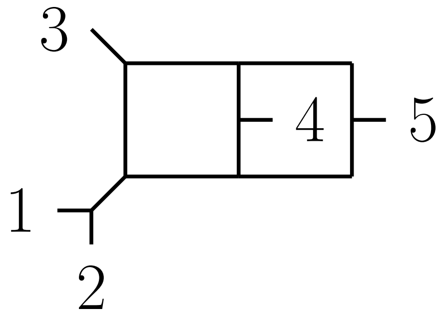

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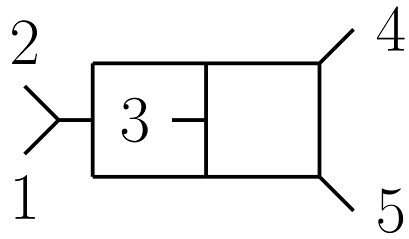
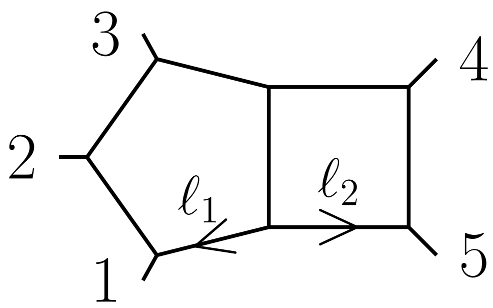


$$\longrightarrow \quad \text{same numerator } T_{12,3|4,5} = S_{1;2|3|4,5} - S_{2;1|3|4,5}$$

Consistent with kinematic Jacobi relation



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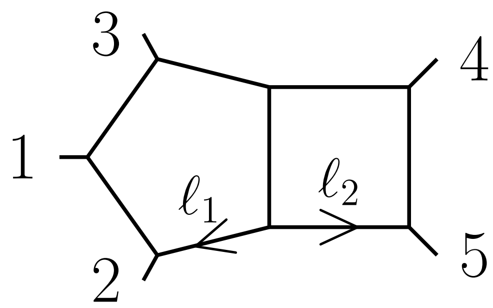

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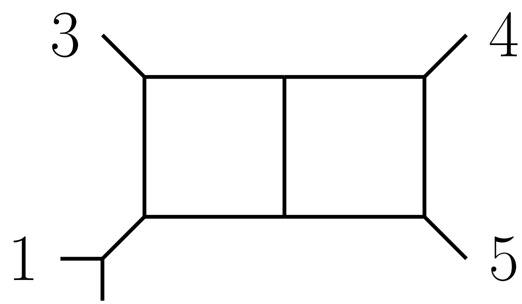

$$N_{1,2,3|4,5}(\ell) \quad - \quad N_{2,1,3|4,5}(\ell) \quad = \quad 2 \quad T_{12,3|4,5}$$

Similarly, non-planar penta-boxes from $\alpha' \to 0$ obey color-kinematics!

[Carrasco, Johansson 1106.4711]

IV. Conclusion

- chiral splitting: at fixed loop momentum, closed-string integrands are double copies of meromorphic open-string quantities "chiral amplitudes"
- determined chiral amplitudes at two-loop five points from locality, BRST invariance and homology invariance $\Longrightarrow$ t_8 & ε_{10}-tensors + SUSY
- field-theory limit "tropical degeneration" of closed-string amplitude $\Longrightarrow$ BCJ representation for two-loop five-point integrand of supergravity

Thank you for your attention !