Integrals, Cuts and Monodromy

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Work in collaboration with Jacob Bourjaily, Hólmfriður Hannesdóttir, Andrew McLeod, Matthew Schwartz (arXiv:2007.13747).

Introduction

- Understand singularities of scattering amplitudes (or individual integrals). Simplest singularities are branch cuts in a given channel $s_{i,j,k,\dots}$.
- How does causality constrain singularities? Steinmann relations and sequential cuts.
- What about singularities on non-physical sheets?
- Provide the mathematical formalism to answer these questions.

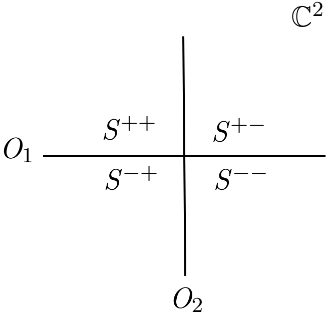
What I will say should apply very generally, including for non-planar integrals, as long as they are polylogarithmic. Some non-polylogarithmic generalization should exist as well.

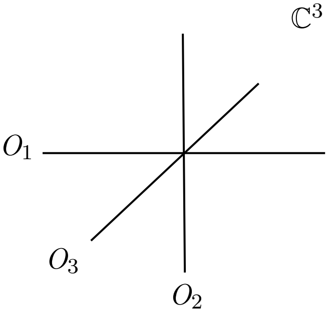
Sequential cuts

Steinmann relations (for S-matrix elements) [Lassalle, Stapp, Cahill]: cuts of a connected S-matrix element in overlapping[1] channels vanish.
Do Steinmann relations apply integral-by-integral or are they the result of a conspiracy in the sum of integrals contributing to an S-matrix element?

- Define precisely what we mean by sequential cuts.
- Describe how this connects to more recent mathematical literature.
- Use the answers for bootstrap (work in progress with [Bourjaily, Dixon, Hannesdóttir, McLeod, Schwartz]).

[1] With caveats, see below.





Stapp: keep momenta real, complexify the energy. Given a subset S of external particles, singularities possible when $\sum_{i\in S} E_i \to 0$, but the limit depends on the imaginary part. We define

$$M^{\pm} = \lim_{\sum_{i\in S} E_i \to 0, \Im \sum_{i\in S} E_i \gtrless 0} M.$$

Then, we define the

$$\mathrm{Disc}_S M = M^+ - M^-.$$

The sequential discontinuity is

$$\mathrm{Disc}_{S_1} \mathrm{Disc}_{S_2} M = (M^{++} - M^{+-}) - (M^{-+} - M^{--}).$$

One-loop triangle example

The one-loop triangle integral with momenta p_1, p_2 and p_3 is

$$T_1 = \frac{i\pi^2}{p_1^2}\frac{1}{z-\bar{z}}\left(2\,\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right),$$

where

$$z\bar{z} = \frac{p_2^2}{p_1^2}, \qquad (1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}.$$

As can be seen on this example, defining the discontinuities as described above is awkward in practice. Bootstrap practitioners use the symbol (iterated integral representation).

Discontinuities from symbols

A large number of integrals appearing in low-loop low-leg amplitudes can be written as iterated integrals[2]

$$M(z) = \int_{z_0}^{z} d\log f_1 \circ \cdots \circ d\log f_\ell + \cdots ,$$

where z parametrizes the kinematics space. So we can associate to $M(z)$ a *symbol*

$$\mathcal{S}(M(z)) = f_1(z) \otimes \cdots \otimes f_\ell(z) + \cdots .$$

Then, $M(z)$ has a branch point at the zero of f_ℓ with discontinuity $\operatorname{disc}_{f_\ell} M(z)$ such that

$$\mathcal{S}((2\pi i)^{-1} \operatorname{disc}_{f_1} M(z)) = f_2(z) \otimes \cdots \otimes f_\ell(z) + \cdots .$$

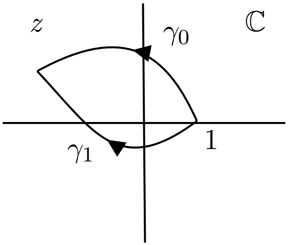
This notion of disc is not fully defined by this result. Also, its relation to Disc is unclear.

[2] The integral is independent on small variations of the contour.

The logarithm, a simple example

Consider the function

$$\log z = \int_1^z \frac{dw}{w}.$$



The logarithm is multivalued[3]

$$\log_{\gamma_0}(z) - \log_{\gamma_1}(z) = \oint_{\gamma_0 \circ \gamma_1^{-1}} \frac{dw}{w} = 2\pi i.$$

If we append a circle around $z = 0$, $\mathcal{C}(z, 0)$, we obtain the monodromy around $z = 0$

$$\mathscr{M}_{z=0} \log_{\gamma_0}(z) = \log_{\gamma \circ \mathcal{C}(z,0)}(z) = \log_{\gamma_0}(z) + 2\pi i.$$

[3]For the path composition $\gamma_a \circ \gamma_b$ we run along γ_a and then along γ_b.

Monodromy

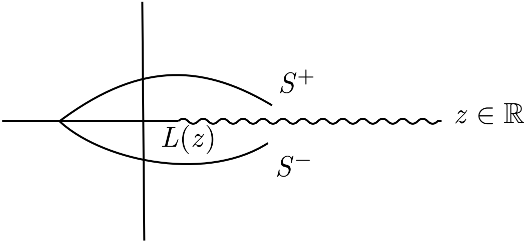
Using the iterated integral we can define an *analytic continuation* of the function. By appending to the integration path paths going around poles in the differentials appearing in the integrand, we go to "other sheets". [Deligne, Hain, Beilinson & Deligne, Goncharov & Deligne, Jianqiang Zhao]
The monodromy is the value of the function on these other sheets (which one, follows from the path). We define the discontinuity as the difference

$$\mathrm{disc}_{f_\ell} M(z) = (1 - \mathscr{M}_{f_\ell}) M(z).$$

This satisfies the symbol condition presented above.

Discontinuities

The discontinuities Disc are defined as follows (see Pham). There is a branch point (hypersurface) which corresponds to real kinematics Landau singularity. Suppose the equation of the Landau singularity is $L(z) = 0$ for real z. We choose the sign of L such that for $L < 0$ the phase space is empty (below threshold). The real z such that $L(z) > 0$ form the cut.
The complexified branch point is given by the complexified equation $\mathbf{L}(z)$ with complex z. The branch point can be avoided by contours with $\Im\mathbf{L}(z) \gtrless 0$. Disc is the difference[4] between above and below the cut.



[4]Sometimes more natural below minus above.

Discontinuities

The discontinuities are defined only for real z such that $L(z) > 0$ (they contain a $\theta(L(z))$ factor. In particular, they are *not* holomorphic or meromorphic.

The real structure plays a crucial role in their definition.

Can be computed from cuts (Cutkosky)

$$\mathrm{Cut}\, M = \int \prod_j d^d q_j \Big(\prod_{i \in \mathsf{cut}} (2i\pi) \delta^+(p_i^2 - m_i^2) \Big) \Big(\prod_{i \notin \mathsf{cut}} \frac{1}{p_i^2 - m_i^2} \Big).$$

The real structure also plays a crucial role in the computation of cuts.[5]

However, (perhaps up to a sign[6]), disc computed from monodromy is equal to Disc whenever Disc is defined.

[5] We need $p_i^0 \in \mathbb{R}$ and $p_i^2 \in \mathbb{R}$.

[6] Related to the signs of α in the solution of Landau equations.

Sequential discontinuities

Instead of attempting to analytically continue Disc (as [Abreu, Britto, Duhr, Gardi]), use disc arising from monodromy. Consider the Landau singularity loci f_1 and f_2 arising from two overlapping cuts and consider paths γ_1 around the locus $f_1 = 0$ and γ_2 around the locus $f_2 = 0$. Then the Steinmann relation can be written

$$(1 - \mathscr{M}_{\gamma_1})(1 - \mathscr{M}_{\gamma_2})M = 0.$$

A priori we have

$$(1 - \mathscr{M}_{\gamma_1})(1 - \mathscr{M}_{\gamma_2})M \neq (1 - \mathscr{M}_{\gamma_2})(1 - \mathscr{M}_{\gamma_1})M,$$

see below.

Computing monodromies

The computation of monodromies of polylogarithms has a well-developed mathematical theory which we describe now. The polylogarithmic functions satisfy differential equations which can be put in first order form. For example,

$$d\,\mathrm{Li}_2(z) = \frac{dz}{z} \times \mathrm{Li}_1(z), \tag{1}$$

$$d\,\mathrm{Li}_1(z) = \frac{dz}{1-z} \times 1. \tag{2}$$

In matrix form,

$$d\begin{pmatrix} 1 & \mathrm{Li}_1(z) & \mathrm{Li}_2(z) \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{Li}_1(z) & \mathrm{Li}_2(z) \end{pmatrix} \begin{pmatrix} 0 & \omega_1(z) & 0 \\ 0 & 0 & \omega_0(z) \\ 0 & 0 & 0 \end{pmatrix},$$

with $\omega_0(z) = \frac{dz}{z}$, $\omega_1(z) = \frac{dz}{1-z}$.

Connection

The linear system of differential equations defines a closed, flat connection

$$\omega = \begin{pmatrix} 0 & \omega_1(z) & 0 \\ 0 & 0 & \omega_0(z) \\ 0 & 0 & 0 \end{pmatrix},$$

$d\omega = 0,\ d\omega - \omega \wedge \omega = 0.$

By putting all the solutions of the the linear system in a matrix we obtain a *variation matrix*

$$\mathscr{M}(z) = \mathcal{P}\exp\int_0^z \omega = \begin{pmatrix} 1 & \mathrm{Li}_1(z) & \mathrm{Li}_2(z) \\ 0 & 1 & \log(z) \\ 0 & 0 & 1 \end{pmatrix},$$

where we dropped logarithmically divergent terms.[7]

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[7] Tangent basepoint regularization.

Monodromy matrices

Under a monodromy around $z = 0$ we have $\mathscr{M}(z) \to \mathscr{M}_0 \mathscr{M}(z)$ and under a monodromy around $z = 1$ we have $\mathscr{M}(z) \to \mathscr{M}_1 \mathscr{M}(z)$, with

$$\mathscr{M}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\pi i \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathscr{M}_1 = \begin{pmatrix} 1 & -2\pi i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

We have $(1 - \mathscr{M}_0)(1 - \mathscr{M}_1)\mathscr{M}(z) \neq (1 - \mathscr{M}_1)(1 - \mathscr{M}_0)\mathscr{M}(z)$, so the monodromies *do not commute*.

Variation matrix

For the purposes of understanding the monodromy it is very useful to consider not only the function we're interested in, but a whole matrix of functions:

$$\mathscr{M}(z) = \mathcal{P} \exp \int^{z} \omega.$$

The "beyond the symbol" ambiguity is encoded in the choice of the starting point of the integral. The variation matrix also contains the coproduct.
Some similarities (but also some differences) with matrices appearing in the work of Bloch & Kreimer.

Commutativity

There is an interesting tension between the comm

$$\mathrm{Disc}_{S_1}\,\mathrm{Disc}_{S_2}\,M = (M^{++}-M^{+-})-(M^{-+}-M^{--}) = \mathrm{Disc}_{S_2}\,\mathrm{Disc}_{S_1},$$

but

$$(1-\mathscr{M}_{\gamma_1})(1-\mathscr{M}_{\gamma_2})M \neq (1-\mathscr{M}_{\gamma_2})(1-\mathscr{M}_{\gamma_1})M.$$

The paths in the monodromy approach are allowed to wander off to other sheets while the analytic continuations used in the computations of Disc are much more restricted.
The reality conditions and the physical region play a crucial role in the computation of Disc and in the computation of cuts. There doesn't seem to be a corresponding ingredient in the mathematical approach to these functions.

One-loop triangle example

$$\omega = \begin{pmatrix} 0 & \omega_0 + \bar{\omega}_0 & \omega_1 + \bar{\omega}_1 & 0 \\ 0 & 0 & 0 & -\omega_1 + \bar{\omega}_1 \\ 0 & 0 & 0 & \omega_0 - \bar{\omega}_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathscr{M}(z,\bar{z}) = \mathcal{P}\exp\int^{z,\bar{z}} \omega = \begin{pmatrix} 1 & \log z\bar{z} & \mathrm{Li}_1(z)+\mathrm{Li}_1(\bar{z}) & \Phi_1(z,\bar{z}) \\ 0 & 1 & 0 & -\mathrm{Li}_1(z)+\mathrm{Li}_1(\bar{z}) \\ 0 & 0 & 1 & \log\frac{z}{\bar{z}} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are four monodromy matrices for going around $z=0$, $z=1$, $\bar{z}=0$ and $\bar{z}=1$. Monodromies in z and $\bar{z}$ commute.

Monodromy matrices

$$\mathscr{M}_{z=0} = \begin{pmatrix} 1 & 2\pi i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi i \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathscr{M}_{z=1} = \begin{pmatrix} 1 & 0 & -2\pi i & 0 \\ 0 & 1 & 0 & 2\pi i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

The monodromy $\mathscr{M}_{z=1}$ is computed as a product of contributions $0 \to 1$, circle around $z = 0$ and $1 \to 0$.

A two-loop example

For the two-loop box-triangle integral, we obtain the connection:

$$\omega = \left(\begin{array}{c|cc|ccc|cc|c} 0 & -\omega_1-\overline{\omega}_1 & \omega_0+\overline{\omega}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\overline{\omega}_0 & \omega_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 & -\overline{\omega}_1 & \omega_0+\overline{\omega}_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \omega_0-\overline{\omega}_0 & -\omega_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\overline{\omega}_0 & -\omega_0+\overline{\omega}_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_1 & \overline{\omega}_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\omega}_0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad (5)$$

The monodromy matrices $\mathscr{M}_{z=0}$, $\mathscr{M}_{z=1}$, $\mathscr{M}_{\bar{z}=0}$ and $\mathscr{M}_{\bar{z}=1}$ can be computed by integrating ω along various paths (and dropping divergent terms).

Structure of the connection

The connection has a block structure with non-vanishing entries in the blocks above the diagonal blocks.
The sizes of the blocks encode detailed information about an underlying Hodge structure.
One can only take a limited number of iterated cuts

$$(1-\mathscr{M}_{z=0})^3\mathscr{M}(z,\bar{z})=0, \qquad (1-\mathscr{M}_{z=1})^2\mathscr{M}(z,\bar{z})=0.$$

Work to be done...

- Understand a better the "physical contours" to be used for reaching the physical region (above and below the cuts).
- Related, find how the reality structure underlying the computations of cuts and discontinuities fits with the mathematical methods.
- Use this formalism to do bootstrap. Understand the adjacency conditions in the symbol.
- Compute the S-matrix on *all* sheets and understand the (complex) singularities outside the physical region. Does causality play any role for complex kinematics?

Thank You!