# Generalizations of the KLT Formula

Callum R. T. Jones

*Based on*:

[20xx.xxxxx] H. Chi, H. Elvang, A. Herderschee, CRTJ, S. Paranjape

[2004.12948] L. A. Johnson, CRTJ, S. Paranjape

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- Present solutions to bootstrap equations together with (probably infinite parameter family) of open string-like solutions to the associated BCJ constraints and analyze the generalized double-copy.
- Comment on *massive deformations* [c.f. Shruti's talk], will understand why naive massive double-copy fails.

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$$\mathcal{A}_n^{L\otimes R} = \sum_{\alpha,\beta} \mathcal{A}_n^L[\alpha] m_n^{-1}[\alpha|\beta] \mathcal{A}_n^R[\beta]$$

where

$$\underbrace{\mathcal{A}_n^L = \sum_{\alpha} \mathrm{Tr}\left[T^{a_{\alpha(1)}} \dots T^{a_{\alpha(n)}}\right] \mathcal{A}_n^L[\alpha],}_{\text{Left single copy}} \qquad \underbrace{\mathcal{A}_n^R = \sum_{\beta} \mathrm{Tr}\left[\tilde{T}^{b_{\beta(1)}} \dots \tilde{T}^{b_{\beta(n)}}\right] \mathcal{A}_n^R[\beta],}_{\text{Right single copy}}$$

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- **Basic Question:** Does any bi-adjoint *zeroth copy* lead to a well-defined KLT formula? When does this produce physical scattering amplitude with *locality*, *factorization*, *Bose/Fermi symmetry*...

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- Restricted orderings $\implies$ *missing* factorization channels.

# Cubic Bi-Adjoint Scalar (BAS)

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- **Observation:** A miracle! No spurious singularities, determinant zeros always coincide with *missing* poles.

# Naive Deformation

**Optimistic Viewpoint:** will any deformation of BAS give a consistent double-copy?

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi^{aa'}\right)^2 - g\, f^{abc}\, \tilde{f}^{a'b'c'}\, \phi^{aa'}\, \phi^{bb'}\, \phi^{cc'} + \lambda\, d^{abcd} \tilde{d}^{a'b'c'd'}\, \phi^{aa'}\, \phi^{bb'}\, \phi^{cc'}\, \phi^{dd'}$$

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4-point scattering $\implies$ rank $= 2$

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Contains spurious (double) low-energy pole at $s + 2t = 0$.

*Not* a well-defined double-copy!

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- **Observation:** Rank does *not* change discontinuously as $\alpha' \to 0$.

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- **Procedure**:
  1. Construct a general Ansatz for $m_n$ and solve bootstrap equations order-by-order in deformation parameter.
  2. Construct a general Ansatz for single copies, solve *generalized BCJ relations*

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  3. Calculate *generalized double-copy*.

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- Form a general Ansatz:

$$f_2(s,t) = -\frac{g^2}{s} + \sum_{k=0}^{\infty}\sum_{r=0}^{k} \frac{a_{k,r}}{\Lambda^{2k}} s^r t^{k-r}$$

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- We also solve for 5-point kernel, 8 independent functions, more computationally intensive...

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$$m_4[1234|1243] =$$

$$-\frac{g^2}{s} + \frac{1}{\Lambda^2}(a_{1,0}t + a_{1,1}s) + \frac{a_{2,0}}{\Lambda^4}t(s+t) + \frac{1}{\Lambda^6}\left[a_{3,0}t^3 + a_{3,1}st^2 + a_{3,2}s^2t + a_{3,3}s^3\right] +$$

$$+ \frac{1}{\Lambda^8}\left[a_{4,0}t^4 + a_{4,1}st^3 + a_{4,2}s^2t^2 + (a_{4,0} - a_{4,1} + a_{4,2})s^3t\right]$$

$$+ \frac{1}{\Lambda^{10}}\left[a_{5,0}t^5 + a_{5,1}st^4 + a_{5,2}s^2t^3 + a_{5,3}s^3t^2\right]$$

$$+ \left[\frac{a_{1,0}a_{1,1}(a_{1,0} - a_{1,1})}{g^4\Lambda^6} + \frac{a_{1,1}(a_{3,1} - a_{3,2}) - a_{1,0}(a_{3,0} - a_{3,2} + a_{3,3})}{g^2\Lambda^8}\right.$$

$$\left. + \frac{a_{5,0} - a_{5,1} + a_{5,3}}{\Lambda^{10}}\right]s^4t + \frac{a_{5,5}}{\Lambda^{10}}s^5 + \ldots$$

# Higher-Derivative Corrections to KLT Kernel

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$$-\frac{g^2}{s} + \frac{1}{\Lambda^2}(a_{1,0}t + a_{1,1}s) + \frac{a_{2,0}}{\Lambda^4}t(s+t) + \frac{1}{\Lambda^6}\left[a_{3,0}t^3 + a_{3,1}st^2 + a_{3,2}s^2t + a_{3,3}s^3\right] +$$

$$+ \frac{1}{\Lambda^8}\left[a_{4,0}t^4 + a_{4,1}st^3 + a_{4,2}s^2t^2 + (a_{4,0} - a_{4,1} + a_{4,2})s^3t\right]$$

$$+ \frac{1}{\Lambda^{10}}\left[a_{5,0}t^5 + a_{5,1}st^4 + a_{5,2}s^2t^3 + a_{5,3}s^3t^2\right]$$

$$+ \left[\frac{a_{1,0}a_{1,1}(a_{1,0} - a_{1,1})}{g^4\Lambda^6} + \frac{a_{1,1}(a_{3,1} - a_{3,2}) - a_{1,0}(a_{3,0} - a_{3,2} + a_{3,3})}{g^2\Lambda^8}\right.$$

$$\left. + \frac{a_{5,0} - a_{5,1} + a_{5,3}}{\Lambda^{10}}\right]s^4t + \frac{a_{5,5}}{\Lambda^{10}}s^5 + \ldots$$

Comparison with string kernel:

$$m_4^{\text{string}}[1234|1243] = -\frac{1}{\pi\alpha' s} - \frac{\pi\alpha'}{6}s - \frac{7\pi^3\alpha'^3}{360}s^3 - \frac{31\pi^5\alpha'^5}{15120}s^5 + \ldots$$

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- Have solved generalized BCJ conditions for $\mathcal{A}_5[1^+2^+3^+4^+5^+]$ up to $D^6F^5$.

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Using kernel $m_n$ together with $\mathcal{A}_4^L$ and $\mathcal{A}_4^R$ we can form the *generalized double-copy*:

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**Observation:** For all operators, shifting $a_{k,l}$ can always be *compensated* by a shift in $e_{k,l}$. In particular, can always choose *minimal kernel* $a_{k,l}=0$, and still generate most general double-copy. Remains true at 5-point up to $D^8R^5$.

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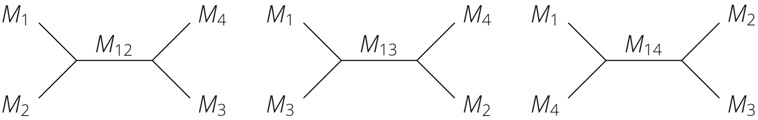

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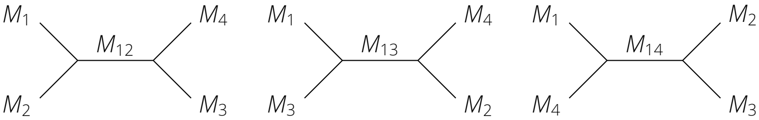


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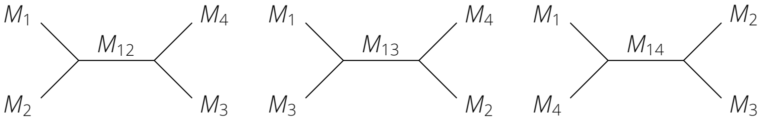


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- **Open Problem:** Other solutions to these constraints, ex: *massive string states*...

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Thank You!

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