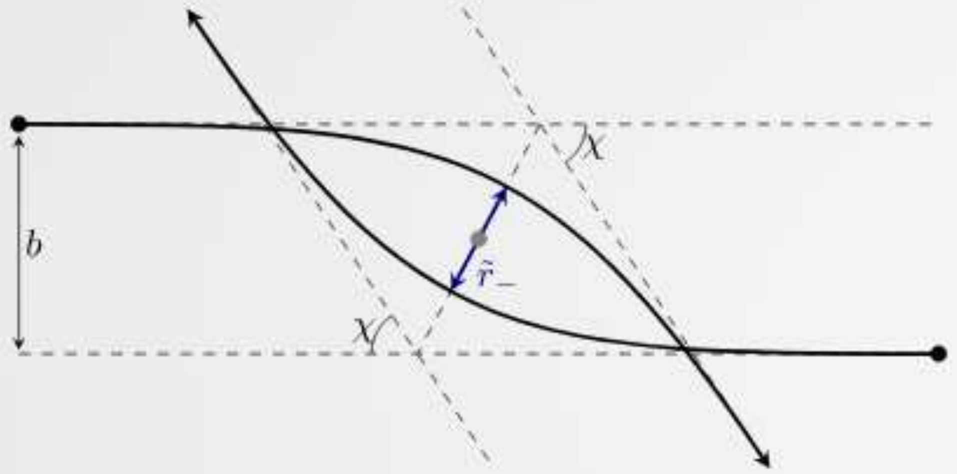
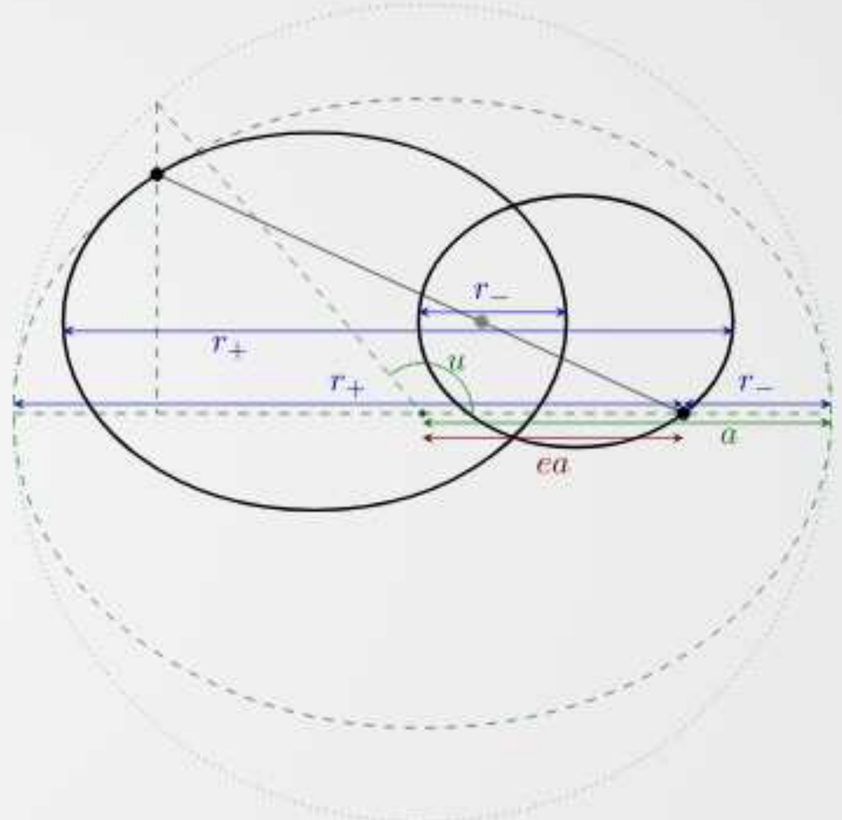
Post-Minkowskian EFT meets conservative potential at NNNLO

Continuation of work with R. Porto & Z. Liu in

[1910.03008] [1911.09130]
[2006.01184] [2007.04977]
[2008.06047]





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DESY.

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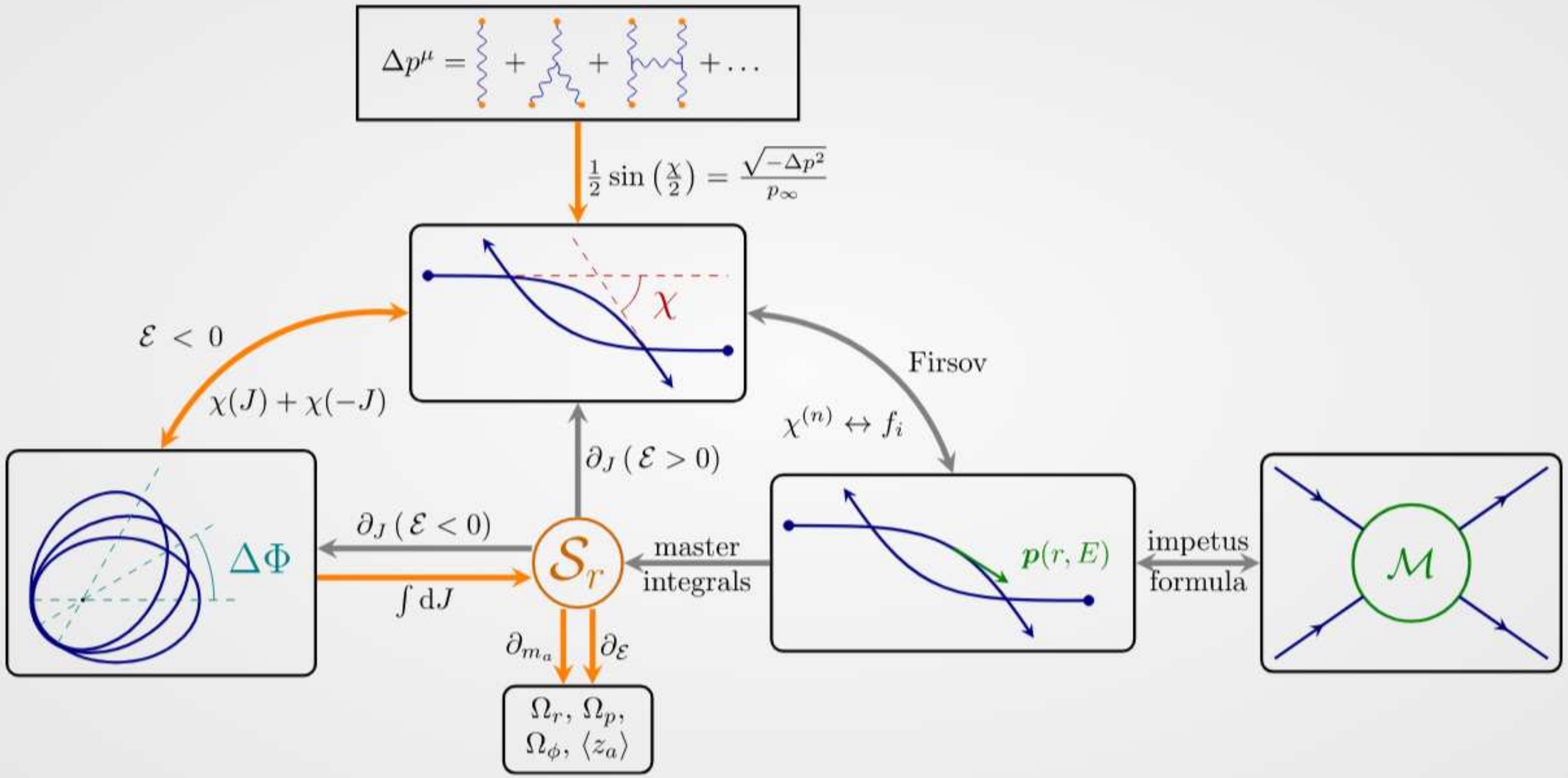
QCD meets Gravity VI

03.12.2020


$\Delta p^\mu = + + + \dots$
$\frac{1}{2}\sin\left(\frac{\chi}{2}\right) = \frac{\sqrt{-\Delta p^2}}{p_\infty}$
χ
$\mathcal{E} < 0$
$\chi(J) + \chi(-J)$
Firsov
$\chi^{(n)} \leftrightarrow f_i$
$\partial_J\,(\mathcal{E} > 0)$
$\partial_J\,(\mathcal{E} < 0)$
$\Delta\Phi$
$\int \mathrm{d}J$
$\mathcal{S}_r$
master integrals
$\boldsymbol{p}(r, E)$
impetus formula
$\mathcal{M}$
∂_{m_a}
$\partial_{\mathcal{E}}$
Ω_r, Ω_p, Ω_ϕ, $\langle z_a \rangle$

Scattering angle: state of the art

$$\frac{\chi}{2} = \sum_{n=1} \chi_b^{(n)} \left(\frac{GM}{b}\right)^n$$

3PM result: *QCD meets Gravity 2018* [BCRSSZ 18; Cheung, Solon 20; GK, Liu, Porto 20]

$$\frac{\chi_b^{(1)}}{\Gamma} = \frac{2\gamma^2 - 1}{\gamma^2 - 1}$$

$$\frac{\chi_b^{(2)}}{\Gamma} = \frac{3\pi}{8} \frac{5\gamma^2 - 1}{\gamma^2 - 1}$$

$$\frac{\chi_b^{(3)}}{\Gamma} = \frac{1}{(\gamma^2 - 1)^{3/2}} \Bigg[-\frac{4\nu}{3}\gamma\sqrt{\gamma^2 - 1}(14\gamma^2 + 25) \frac{(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)\Gamma^2}{3(\gamma^2 - 1)^{3/2}} - 8\nu(4\gamma^4 - 12\gamma^2 - 3)\,\mathrm{arcsinh}\sqrt{\frac{\gamma - 1}{2}} \Bigg]$$

with $\gamma = u_1 \cdot u_2 [= \sigma], \Gamma = \sqrt{1 + 2\nu(\gamma - 1)} [= h(\gamma, \nu)], \nu = m_1 m_2 / M^2$.
Directly feeds into the radial action ($J = p_\infty b = GM\mu j$):

$$i_r(j, \mathcal{E}) \equiv \frac{\mathcal{S}_r}{GM\mu} = \mathrm{sg}(\hat{p}_\infty)\chi_j^{(1)}(\mathcal{E}) - j\left(1 + \frac{2}{\pi}\sum_{n=1} \frac{\chi_j^{(2n)}(\mathcal{E})}{(1 - 2n) j^{2n}}\right)$$

What do we know about $\chi_b^{(4)}$?

Schwarzschild limit ($\nu = 0$):

$$\chi_b^{(4),\mathrm{Sch}} = \frac{105\pi\left(33\gamma^4 - 18\gamma^2 + 1\right)}{128(\gamma^2-1)^2}$$

Lots of PN data [Bini, Damour, Geralico 20; + Laporta, Mastrolia 20]

$$\chi^{\mathrm{tot}} = \chi^{\mathrm{local}} + \chi^{\text{non-local}}$$

$$\begin{aligned}
\tilde{\chi}_4^{\mathrm{nonloc,h}} = &\left(-\frac{63}{4} - \frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)\right)\pi\nu p_\infty^4 \\
&+ \left(-\frac{2753}{1120} - \frac{1357}{280}\ln\left(\frac{p_\infty}{2}\right) + \frac{63}{20}\nu\right)\pi\nu p_\infty^6\eta^2 \\
&+ \left(-\frac{27331}{10080}\nu^2 + \frac{199037}{40320}\nu - \frac{155473}{8960} - \frac{27953}{3360}\ln\left(\frac{p_\infty}{2}\right)\right)\pi\nu p_\infty^8\eta^4,
\end{aligned}$$

$$\begin{aligned}
\pi^{-1}\chi_4^{\mathrm{loc,f}} = &\left(\frac{105}{8} - \frac{15}{4}\nu\right) \\
&+ \left[\frac{315}{8} + \left(-\frac{109}{2} + \frac{123}{256}\pi^2\right)\nu + \frac{45}{8}\nu^2\right]p_\infty^2 \\
&+ \left[\frac{3465}{128} + \left(\frac{33601}{16384}\pi^2 - \frac{19597}{192}\right)\nu + \left(\frac{4827}{64} - \frac{369}{512}\pi^2\right)\nu^2 - \frac{225}{32}\nu^3\right]p_\infty^4 \\
&+ \left[\left(-\frac{1945583}{33600} + \frac{93031}{32768}\pi^2\right)\nu + \left(\frac{1937}{16} - \frac{94899}{32768}\pi^2\right)\nu^2 + \left(-\frac{2895}{32} + \frac{1845}{2048}\pi^2\right)\nu^3 + \frac{525}{64}\nu^4\right]p_\infty^6 \\
&+ \left[\left(\frac{3879719}{313600} + \frac{29201523}{33554432}\pi^2\right)\nu + \left(\frac{4843207}{89600} - \frac{469191}{131072}\pi^2\right)\nu^2 + \left(\frac{444975}{131072}\pi^2 - \frac{15875}{128}\right)\nu^3\right. \\
&+ \left.\left(\frac{104755}{1024} - \frac{4305}{4096}\pi^2\right)\nu^4 - \frac{4725}{512}\nu^5\right]p_\infty^8 + O(p_\infty^{10}).
\end{aligned}$$

PM-EFT for a worldline action coupled to GR

- Purely *classical* approach
- Systematic, extension to finite size and spin exists (see Rafael's talk)
- Perturbative expansion in G: can use particle physics/amplitudes toolbox
- Today: only *conservative* effects in the potential region

Full theory

$$S_{\mathrm{EH}} = -2M_{\mathrm{Pl}}^2 \int \mathrm{d}^4 x \sqrt{-g}\, R[g]$$

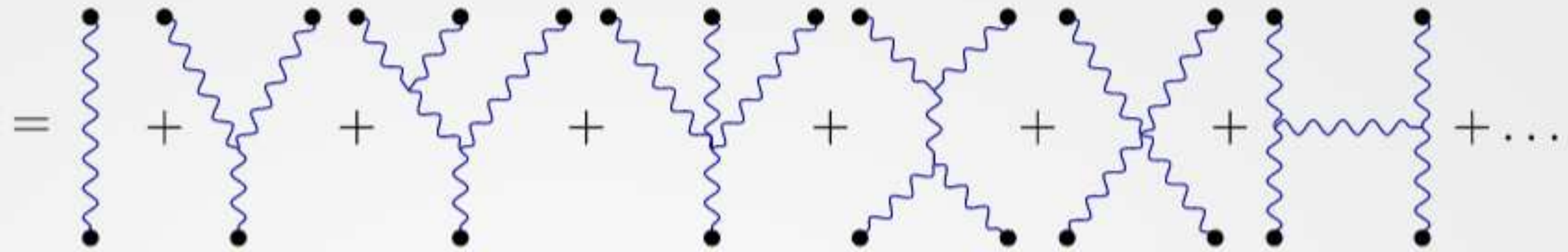
$$S_{\mathrm{pp}} = -\sum_a m_a \int \mathrm{d}\sigma_a \sqrt{g_{\mu\nu}(x_a^\alpha(\sigma))v_a^\mu(\sigma_a)v_a^\nu(\sigma_a)} + \ldots$$

$$\to -\sum_a \frac{m_a}{2} \int \mathrm{d}\tau_a\, g_{\mu\nu}(x_a(\tau_a))v_a^\mu(\tau_a)v_a^\nu(\tau_a) + \ldots$$

... = extensions to finite-size effects and spinning bodies

EFT action

$$e^{iS_{\text{eff}}[x_a]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\text{EH}}[h]+iS_{\text{GF}}[h]+iS_{\text{pp}}[x_a,h]}$$



We optimized the EH-Lagrangian by cleverly choosing gauge-fixing terms and adding total derivatives.

Without field redefinitions:

- 2-point Lagrangian: 2 terms
- 3-point Lagrangian: 6 terms
- 4-point Lagrangian: 18 terms ($\leftarrow$ 3PM)
- 5-point Lagrangian: 36 terms ($\leftarrow$ 4PM)

With field redefinitions:

- 2-point Lagrangian: 2 terms
- 3-point Lagrangian: 4 terms
- 4-point Lagrangian: 12 terms

We chose to not use field redefinitions to preserve the simple one-particle coupling to the WL. (Maybe we should for finite-size effects + spin?)

PM deflection

Having integrated out potential gravitons we have:

$$S_{\text{eff}} = \sum_{n=0}^{\infty} \int \mathrm{d}\tau_1 \, G^n \mathcal{L}_n[x_1(\tau_1), x_2(\tau_2)]$$

with

$$\mathcal{L}_0 = -\frac{m_1}{2} \eta_{\mu\nu} v_1^\mu(\tau_1) v_1^\nu(\tau_1)$$

E.o.m. from variation of the action

$$-\eta^{\mu\nu} \frac{\mathrm{d}}{\mathrm{d}\tau_1} \left(\frac{\partial \mathcal{L}_0}{\partial v_1^\nu} \right) = m_1 \frac{\mathrm{d}v_1^\mu}{\mathrm{d}\tau_1} = -\eta^{\mu\nu} \left(\sum_{n=1}^{\infty} \frac{\partial \mathcal{L}_n}{\partial x_1^\nu(\tau_1)} - \frac{\mathrm{d}}{\mathrm{d}\tau_1} \left(\frac{\partial \mathcal{L}_n}{\partial v_1^\nu} \right) \right)$$

allows us to compute the trajectories order by order:

$$x_a^\mu(\tau_1) = b_a^\mu + u_a^\mu \tau_a + \sum_n G^n \delta^{(n)} x_a^\mu(\tau_a)$$

with $b = b_1 - b_2$ the impact parameter and u_a the incoming velocty at infinity, fulfilling

$$u_1 \cdot u_2 = \gamma \,, \quad u_a \cdot b = 0 \,.$$

Scattering angle.

First we compute the deflection using above trajectories:

$$\Delta p_1^\mu = m_1 \Delta v_1^\mu = -\eta^{\mu\nu} \sum_n \int_{-\infty}^{+\infty} \mathrm{d}\tau_1 \frac{\partial \mathcal{L}_n}{\partial x_1^\nu} \,,$$

At 3PM:

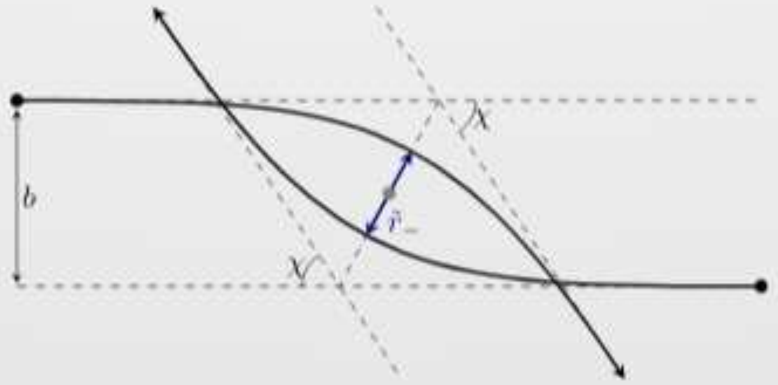
- $\mathcal{L}_1 \left[b_a + u_a\tau_a + \delta^{(1)}x_a + \delta^{(2)}x_a\right]$
- $\mathcal{L}_2 \left[b_a + u_a\tau_a + \delta^{(1)}x_a\right]$
- $\mathcal{L}_3 \left[b_a + u_a\tau_a\right]$

At 4PM:

- $\mathcal{L}_1 \left[b_a + u_a\tau_a + \delta^{(1)}x_a + \delta^{(2)}x_a + \delta^{(3)}x_a\right]$
- $\mathcal{L}_2 \left[b_a + u_a\tau_a + \delta^{(1)}x_a + \delta^{(2)}x_a\right]$
- $\mathcal{L}_3 \left[b_a + u_a\tau_a + \delta^{(1)}x_a\right]$
- $\mathcal{L}_4 \left[b_a + u_a\tau_a\right]$

Physical scattering angle is then simply

$$2\sin\left(\frac{\chi}{2}\right) = \frac{|\Delta \mathbf{p}_{1\mathrm{cm}}|}{p_\infty} = \frac{\sqrt{-\Delta p_1^2}}{p_\infty}$$



Integration

For an alternative derivation of the same integrand using a dynamical WL see Gustav's talk.

Generic structure:

$$\int \mathrm{d}^D q \frac{\delta(q\cdot u_1)\delta(q\cdot u_2)e^{ib\cdot q}}{(q^2)^m} \underbrace{\int \mathrm{d}^D\ell_1\cdots \mathrm{d}^D\ell_L \frac{\delta(\ell_1\cdot u_{a_1})\cdots\delta(\ell_L\cdot u_{a_L})}{(\ell_1\cdot u_{b_1}\pm i0)^{i_1}\cdots(\ell_L\cdot u_{b_L}\pm i0)^{i_L}(\text{sq. props})}}_{\text{Cut Feynman integrals with linear and square propagators}}$$

- Always one delta function per loop momentum of the form $\delta(\ell\cdot u_a)$.
- The other combination might appear as a linear propagator in iterations.
- Automatically land in soft classical integrals (see Michael's, Julio's and Carlo's talk)
- 3PM: a single set of square propagators captures all integrals (i.e. the H-family).
  - Need a subset of integrals discussed in [BCRSSZ 18; Parra-Martinez, Ruf, Zeng 20]
  - In total 876 different integrals (including different $\pm i0$ prescriptions)
- 4PM: we were able to embed all integrals using two families of square propagators.
  - In total 79332 different integrals (including different $\pm i0$ prescriptions)

IBP relations

[Chetyrkin, Tkackov 81; Gehrmann, Remiddi 2000]

- Very efficient algorithms using integration by part and Lorentz invariance identities to reduce to small subset of independent integrals, e.g. [Laporta 00, Lee 10]
- Implemented in many public packages, e.g. LiteRed, FIRE, Kira, Reduze, AIR.
- Delta functions behave like linear propagators under IBPs (exponent = derivative). Additionally, integrals with negative power delta functions vanish.
  - Use `CutDS->{...}` in LiteRed
  - Use `RESTRICTIONS={...}` in FIRE
  - Significant speed-up!
- We use a combination of LiteRed+FIRE6, 3PM: ~5 minutes, 4PM: ~7 hours (4 cores, 32GB RAM)
- 3PM: 7 master integrals without linear propagators, 2 master integrals with linear propagators.
- 4PM: so far we brought the system down to 149 master integrals. (But we know that we are missing some symmetries...)

DEQs and the canonical form @ 3PM

Compute the master integrals using differential equations and their canonical form [Kotikov, Remiddi, Gehrmann 91, 98, 99; Henn 13, 14].

- This method for our integrals was discussed in [Parra-Martinez, Ruf, Zeng 20].
- Single scale $\gamma = u_1 \cdot u_2 = (x^2 - 1)/(2x)$: $\partial/\partial x \vec{I} = \mathbb{M} \cdot \vec{I}$
- Having found a canonical form, series exansion in ϵ is very simple
- Boundary conditions are 3D (due to delta functions) static integrals, already familiar from PN-EFT.

Final Fourier transform is known to all orders:

$$\Delta p_a^\mu = \int \mathrm{d}^D q (A \overbrace{q^\mu}^{-i\frac{\partial}{\partial b_\mu}} + \underbrace{B u_1^\mu + C u_2^\mu}_{\substack{\text{bootstrap from} \\ \text{on-shell condition} \\ (\Delta p_a + p_a)^2 = p_a^2}}) \frac{\delta(q \cdot u_1)\delta(q \cdot u_2) e^{ib\cdot q}}{(q^2)^{m+n}}$$

(NB: using on-shell constraints we only need terms $\sim \partial/\partial b^\mu$. Eikonal?)

Where are we at NNNLO?

[work with Z. Liu, G. Mogull, R. Porto]

- Simple 5-pt GR Feynman rules ✔
- Integrand ✔
- Map to two basic integral families ✔
- IBP reduction + symmetries ✔ (can we do better?)
- Solve DEQs: different approaches under consideration
  - Directly solve the DEQs
  - Find canonical form (many packages: epsilon, Fuchsia, Canonica, INITIAL)
  - Numerics + reconstruction
- Boundary conditions: in progress
  - Masters without linear propagators ✔
  - Masters with linear propagators: can be reduced to 2D integrals using symmetrization trick (see e.g. [Cheng, Wu 87; Saotome, Akhoury 13; Parra-Martinez, Ruf, Zeng 20])

Analytic continuation and Firsov's formula

Let us get a feeling of the analytic continuation and some funny games we can play!

$$i_r(j,\mathcal{E}) \equiv \frac{\mathcal{S}_r}{GM\mu} = \mathrm{sg}(\hat{p}_\infty)\chi_j^{(1)}(\mathcal{E}) - j\left(1 + \frac{2}{\pi}\sum_{n=1}\frac{\chi_j^{(2n)}(\mathcal{E})}{(1-2n)j^{2n}}\right)$$

What about $\chi_j^{(3)}$?

Firsov's formula

Let's do a detour to the c.o.m. momentum along the trajectory:

$$H(r,\mathbf{p}) = E \Rightarrow \mathbf{p}(r,E)$$

Relation to angle most easily extracted from Firsov's formula

$$\bar{\mathbf{p}}^2(r,E) = \exp\left[\frac{2}{\pi}\int_{r|\bar{\mathbf{p}}(r,E)|}^{\infty} \frac{\chi_b(\tilde{b},E)\mathrm{d}\tilde{b}}{\sqrt{\tilde{b}^2 - r^2\bar{\mathbf{p}}^2(r,E)}}\right]$$

In PM language:

$$\mathbf{p}(r,E) = p_\infty^2\left(1+\sum_{n=1}^{\infty} f_n(E)\left(\frac{GM}{r}\right)^n\right), \quad \frac{\chi}{2} = \sum_{n=1}^{\infty}\chi_b^{(n)}(E)\left(\frac{GM}{b}\right)^n$$

$$f_n = \sum_{\sigma\in\mathcal{P}(n)} g_\sigma^{(n)} \prod_\ell \left(\widehat{\chi}_b^{(\sigma_\ell)}\right)^{\sigma^\ell}$$

$$\widehat{\chi}_b^{(n)} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\chi_b^{(n)}, \quad g_\sigma^{(n)} = \frac{2(2-n)^{\Sigma^\ell-1}}{\prod_\ell(2\sigma^\ell)!!}$$

Radial action at 4PM

$$i_r(j,\mathcal{E}) = -j + \frac{\hat{p}_\infty^2}{\sqrt{-\hat{p}_\infty^2}}\frac{f_1}{2} + \frac{\hat{p}_\infty^2}{2j}f_2 + \frac{\hat{p}_\infty^4}{8j^3}\left(f_2^2 + 2f_1 f_3 + 2f_4\right) + \cdots$$

where in turn

$$f_1 = \frac{2\chi_j^{(1)}}{\hat{p}_\infty}\,,$$

$$f_2 = \frac{4\chi_j^{(2)}}{\pi\hat{p}_\infty^2}\,,$$

$$f_3 = \frac{\left(\chi_j^{(1)}\right)^3}{3\hat{p}_\infty^3} - \frac{4\chi_j^{(1)}\chi_j^{(2)}}{\pi\hat{p}_\infty^3} + \frac{\chi_j^{(3)}}{\hat{p}_\infty^3}\,,$$

$$f_4 = \frac{8\chi_j^{(4)}}{3\pi\hat{p}_\infty^4} + \ldots \quad (\leftarrow \text{That's our missing guy})$$

Resummation

Let us truncate our theory at given order n, i.e. $\mathcal{M}_m = f_m = 0$ for $m \geq n$.

We can try to resum contributions to all orders in G, e.g. for the scattering angle:

$$\frac{\chi[f_1]}{2} = \arctan(y/2)$$

$$\frac{\chi[f_{1,2}] + \pi}{2} = \frac{1}{\sqrt{1 - \mathcal{F}_2 y^2}} \left(\frac{\pi}{2} + \arctan\left(\frac{y}{2\sqrt{1 - \mathcal{F}_2 y^2}} \right) \right)$$

with $y \equiv GMf_1/b$ and $\mathcal{F}_2 \equiv f_2/f_1^2$

- Resummation of $\Delta\phi$ works similar to χ.
- We can resum parts of $\mathcal{S}_r$.
- We can resum $f_{1,2}$ contributions for r_{min} and $r_{\pm}$. ("closed form" for real positive roots of arbitrary order polynomial?)
- Difficult for $f_{1,2,3}$. Anyone can do it?

Conclusions and outlook

- We have a systematic and efficient setup to study the gravitational 2-body problem in its full glory.
- Integration is the bottleneck, but there is a lot of new technology around to help us.
- Can we improve our setup by making contact to the eikonal?
- 4PM is not far!
- Using Firsov, we can resum certain contributions to all orders in G.
  - Will also help us for analytic continuation of radiation, radiation-reaction