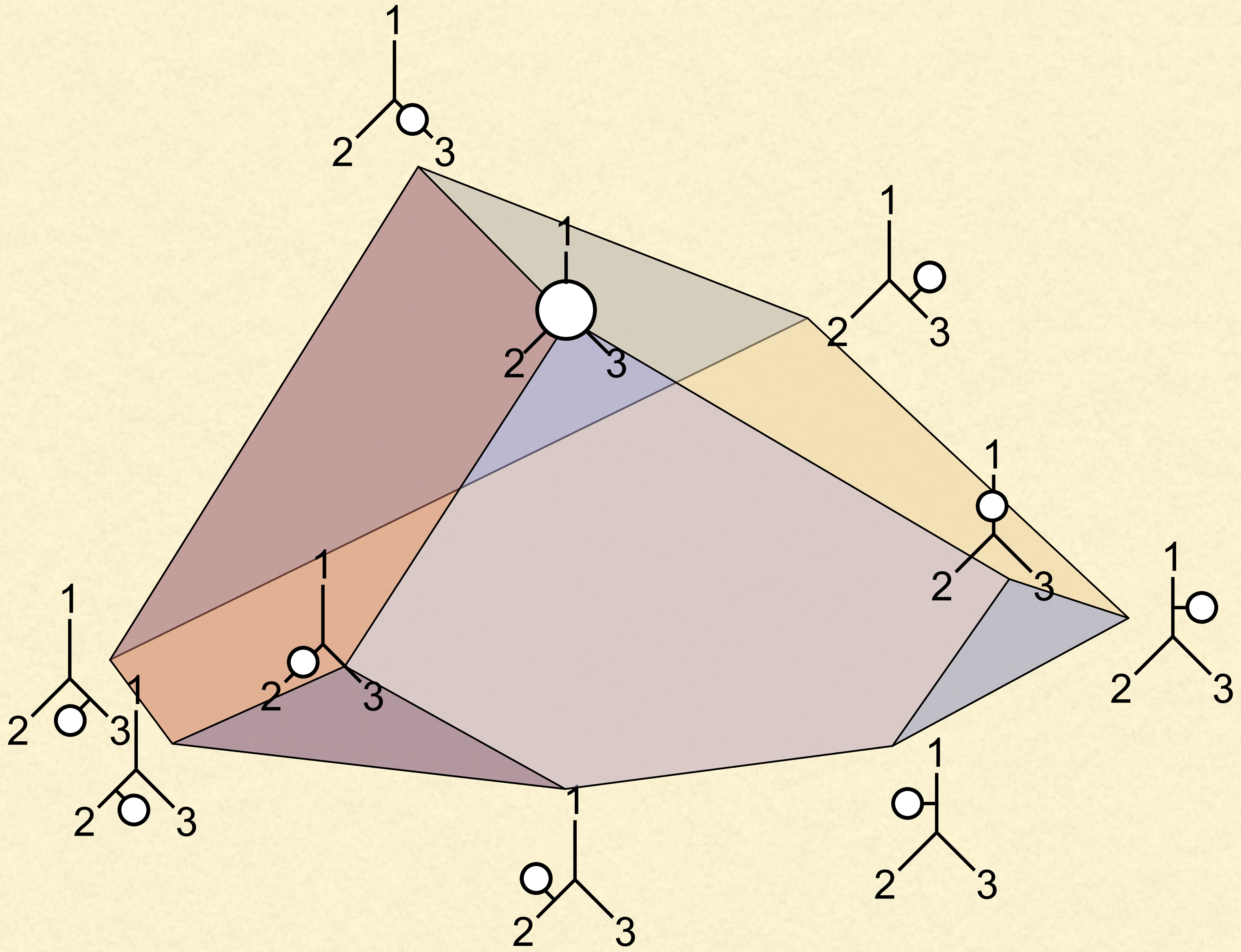
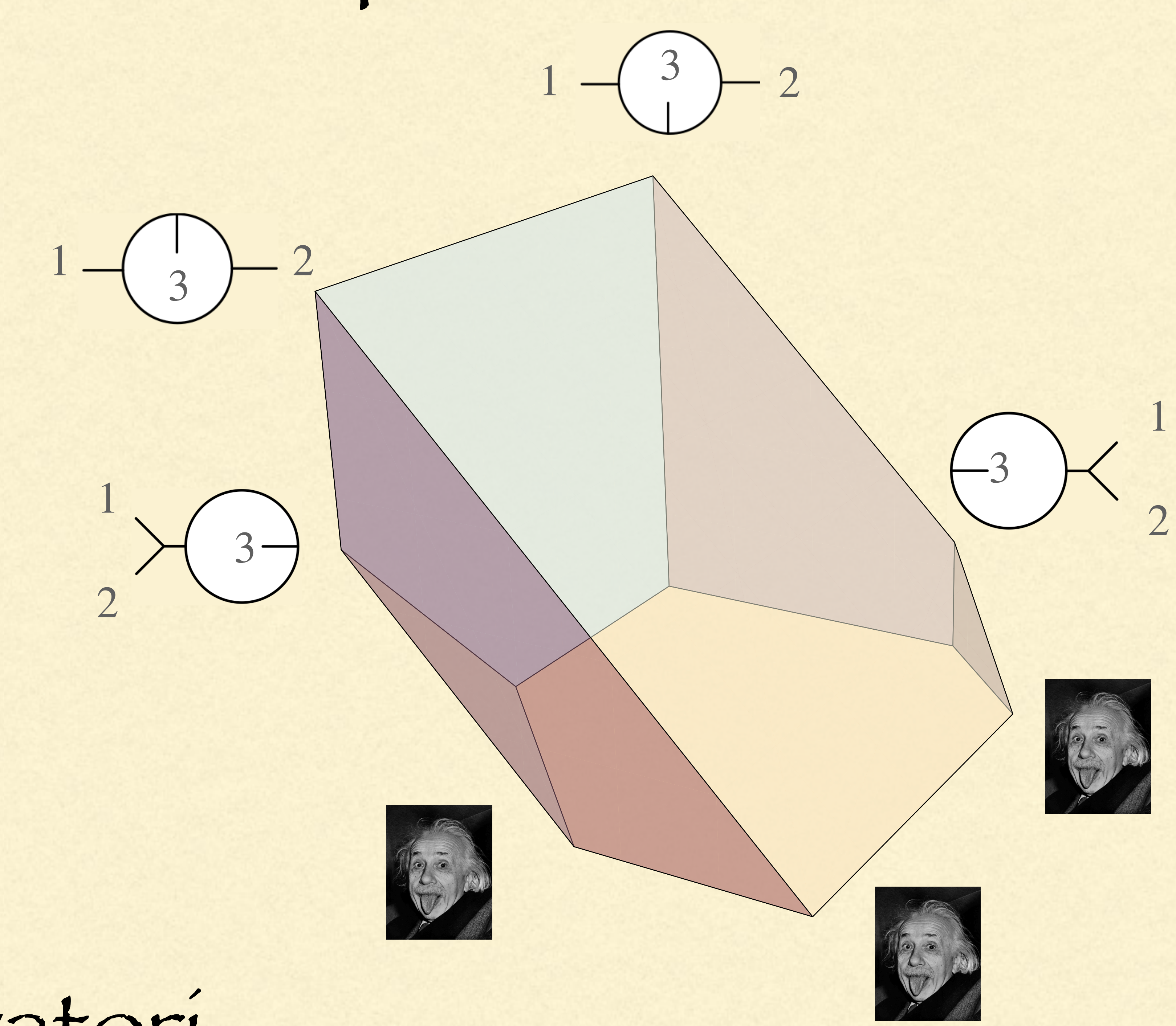
Positive geometries beyond the planar limit





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(Perimeter Institute)

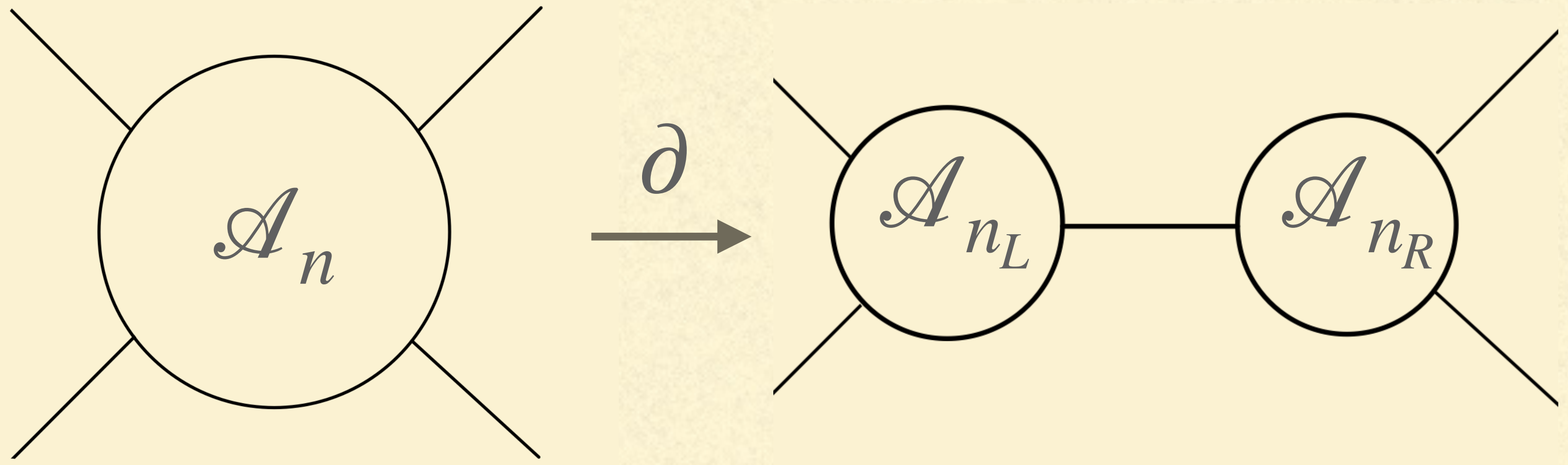
QCD meets Gravity 2020 @ Northwestern

Based on past & on-going collaborations with: Nima Arkani-Hamed, Song He, Hadleigh Frost, Hugh Thomas, Pierre-Guy Plamondon

Introduction

- *Positive Geometries* are regions in the kinematical space satisfying positivity conditions [Arkani-Hamed, Bai, Lam]

- The key feature of positive geometries is the recursive structure of their boundaries, which mimics analogous patterns in the singularities of scattering amplitudes.



- At loop level one consider *integrands* rather than amplitudes due to their simpler analytical properties: poles & residues rather than cuts & discontinuities

- A positive geometry uniquely defines a *Canonical form* which encodes the amplitude

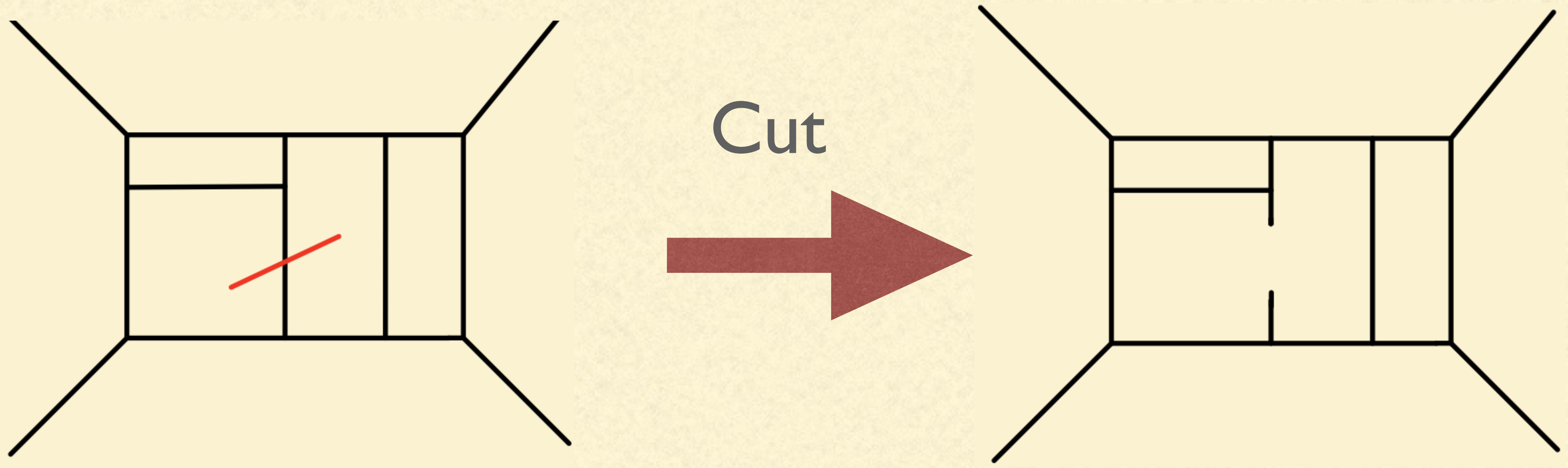
- The canonical form manifest hidden symmetries & yields new computational tools

Introduction

- The first positive geometry to be discovered was *the Amplituhedron* in the context of $\mathcal{N}=4$ SYM [Nima & Trnka]

- Later [ABHY 1711.09102] it was understood how *the Associahedron* plays the same role for tree level bi-adjoint theory. It was quickly generalized at 1-loop level [GS 1806.01842, AHST 1912.12948], since then a lot of effort in understanding higher loops.

- L>2 loops requires to go beyond the planar level: how to define integrands?



- The connection of amplitudes & positive geometries does not require SUSY, planarity, etc. etc.

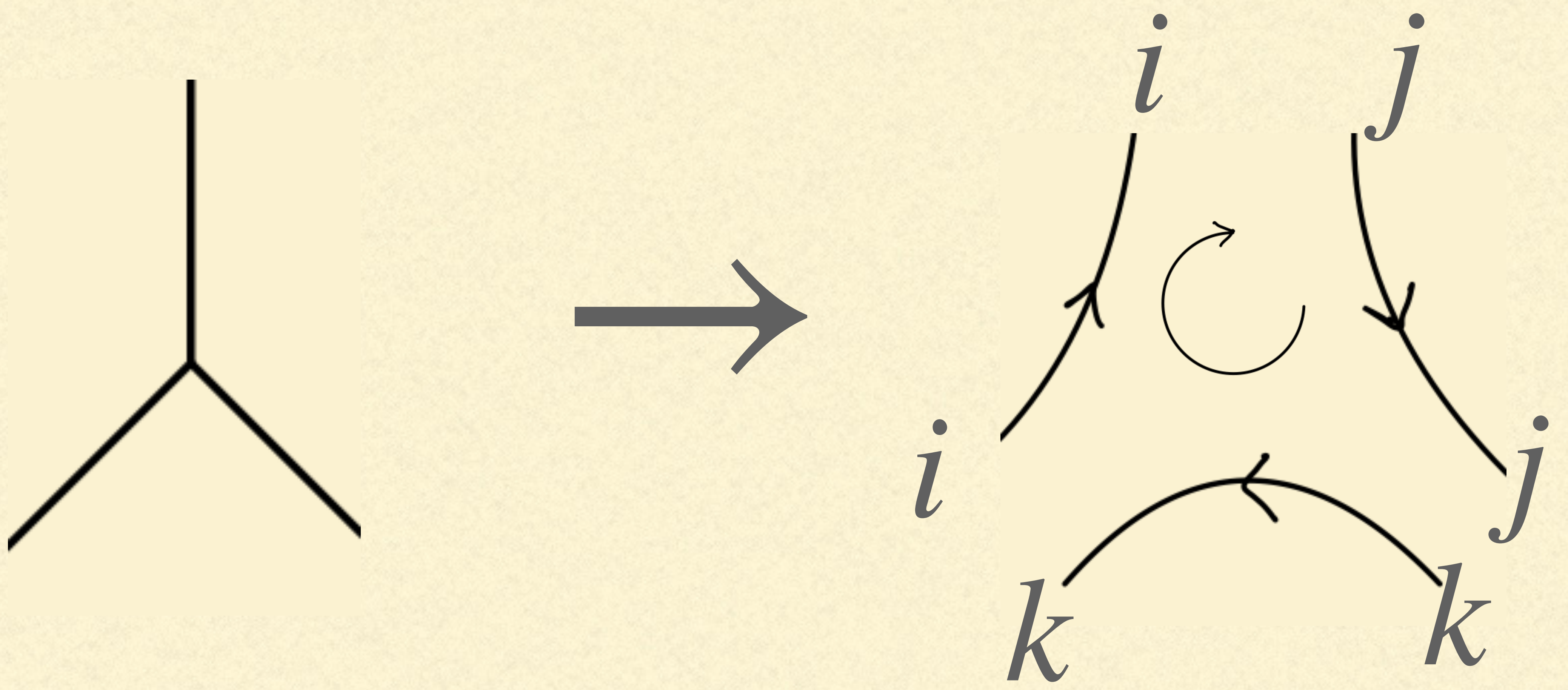
Outline

1. **Integrands beyond the planar limit**
2. Positive Geometries...and "gravity"
3. Conclusions

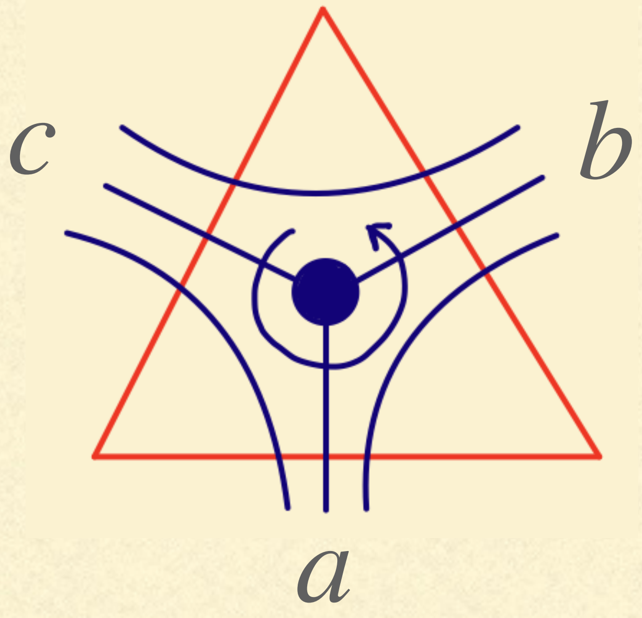
$\mathrm{Tr}(\phi^3)$ theory

$$\mathscr{L} = \mathscr{L}_{free} + g\ \mathrm{Tr}(\phi^3) \qquad \phi = \phi^i_j, \quad i = 1,\ldots,N$$

$$A_n = A_n(\overrightarrow{p} \mid \overrightarrow{ij})$$



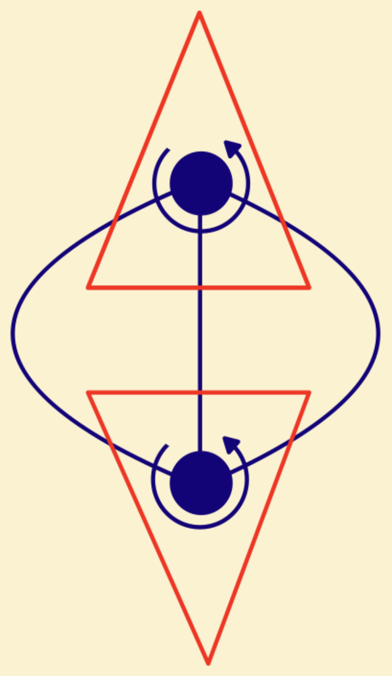
From fatgraphs to Riemann surfaces

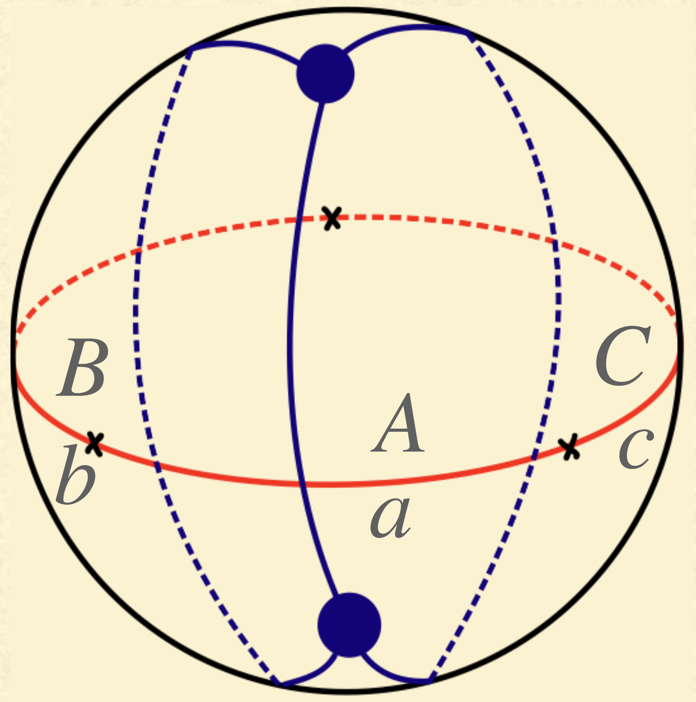


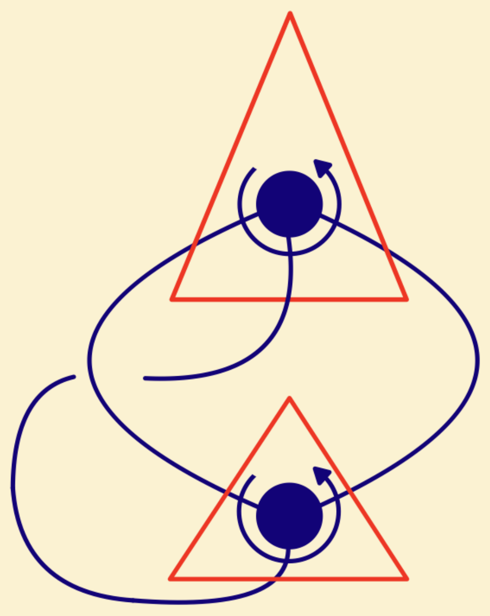
$$t = [a, b, c]$$

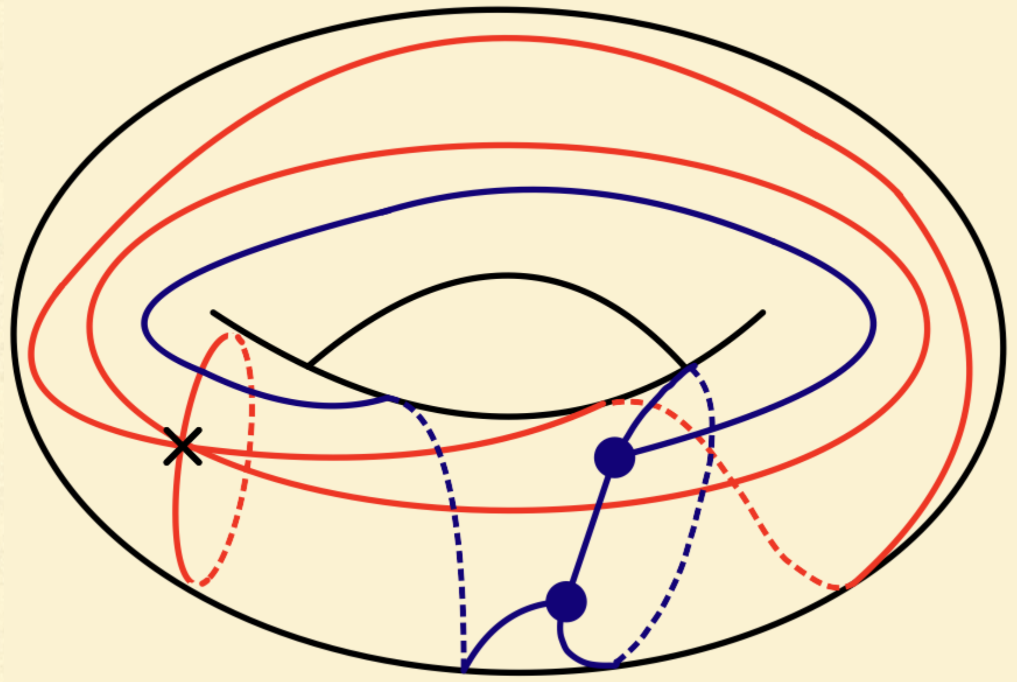
$$\Gamma \qquad S(\Gamma) = \bigcup_{t \in T} [a_t, b_t, c_t]$$

$\Gamma_{g,p}^{\vec{b}} = \{\Gamma \,|\, S(\Gamma)$ is of genus g, has p punctures and boundaries $\vec{b}\,\}$







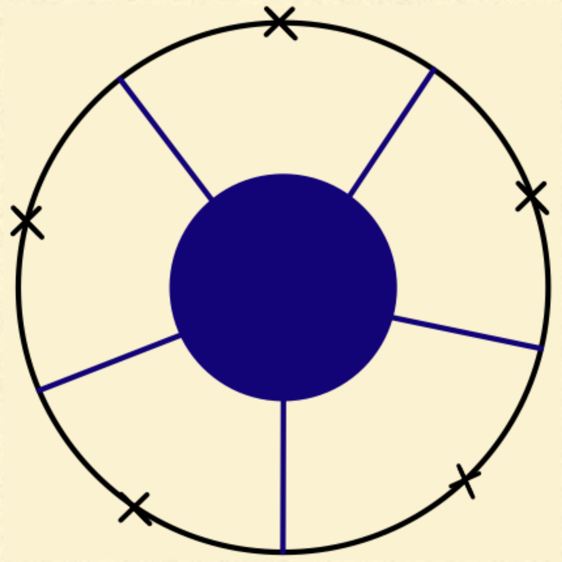


$S(\Gamma) = [a, b, c] \cup [A, C, B]$

$S(\Gamma) = [a, b, c] \cup [A, B, C]$

The color-ordered 1/N expansion

$$C_{\vec{b}} = \prod_{b \in \vec{b}} \prod_{k \in b} \delta^{i_k, j_{k+1}}$$



$$2g + p + b = L + 1$$

$$\lambda = g^2 N$$

$$A_n(\vec{p} \mid \vec{ij}) = (ig)^{n-2} \sum_{L=0}^{\infty} \lambda^L \sum_{\vec{b}} C_{\vec{b}} \sum_{p,g} \left(\frac{1}{N}\right)^{L-p} A^{\vec{b}}_{p,g}(\vec{p})$$

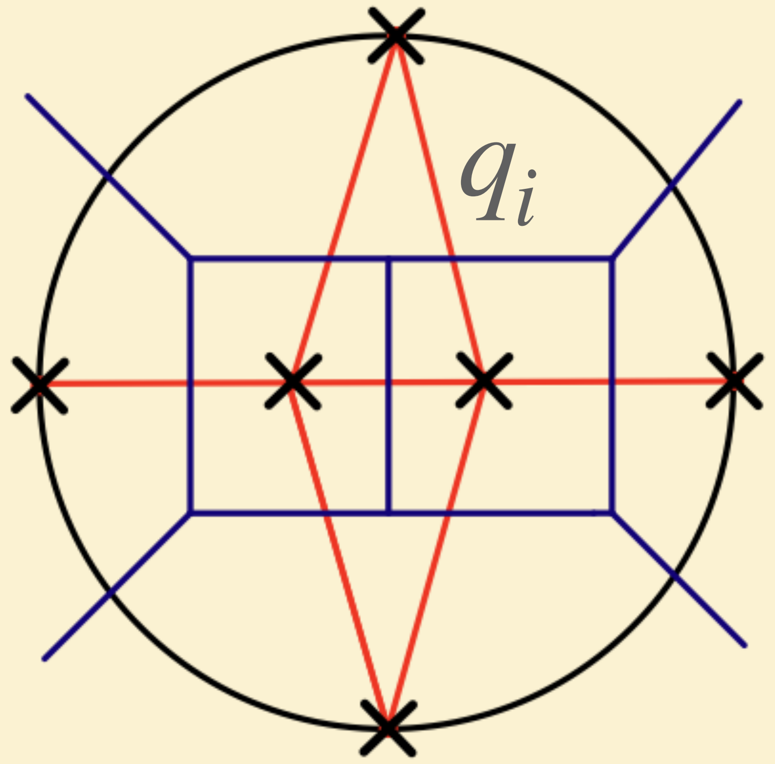
$$A^{\vec{b}}_{p,g} = \sum_{\Gamma \in \Gamma^{\vec{b}}_{p,g}} \int d^D \ell \; I_\Gamma(p, \ell) = \int d^D \ell \sum_{\Gamma \in \Gamma^{\vec{b}}_{p,g}} I_\Gamma(p, \ell) = \int d^D \ell \; \mathrm{I}^{\vec{b}}_{p,g}$$

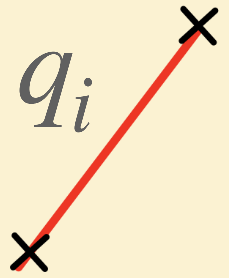
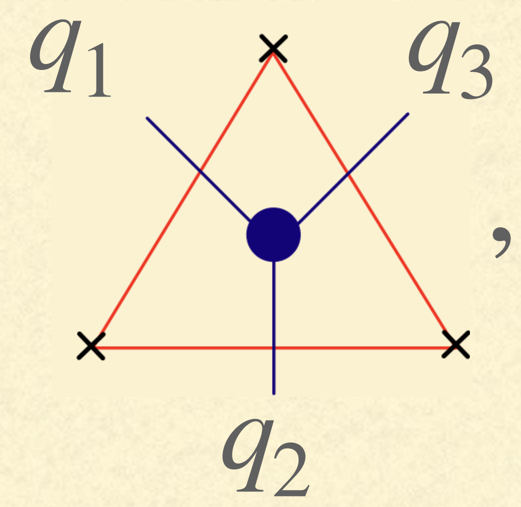
How to choose the loop momenta in a consistent way among all diagrams?

Global loop momenta

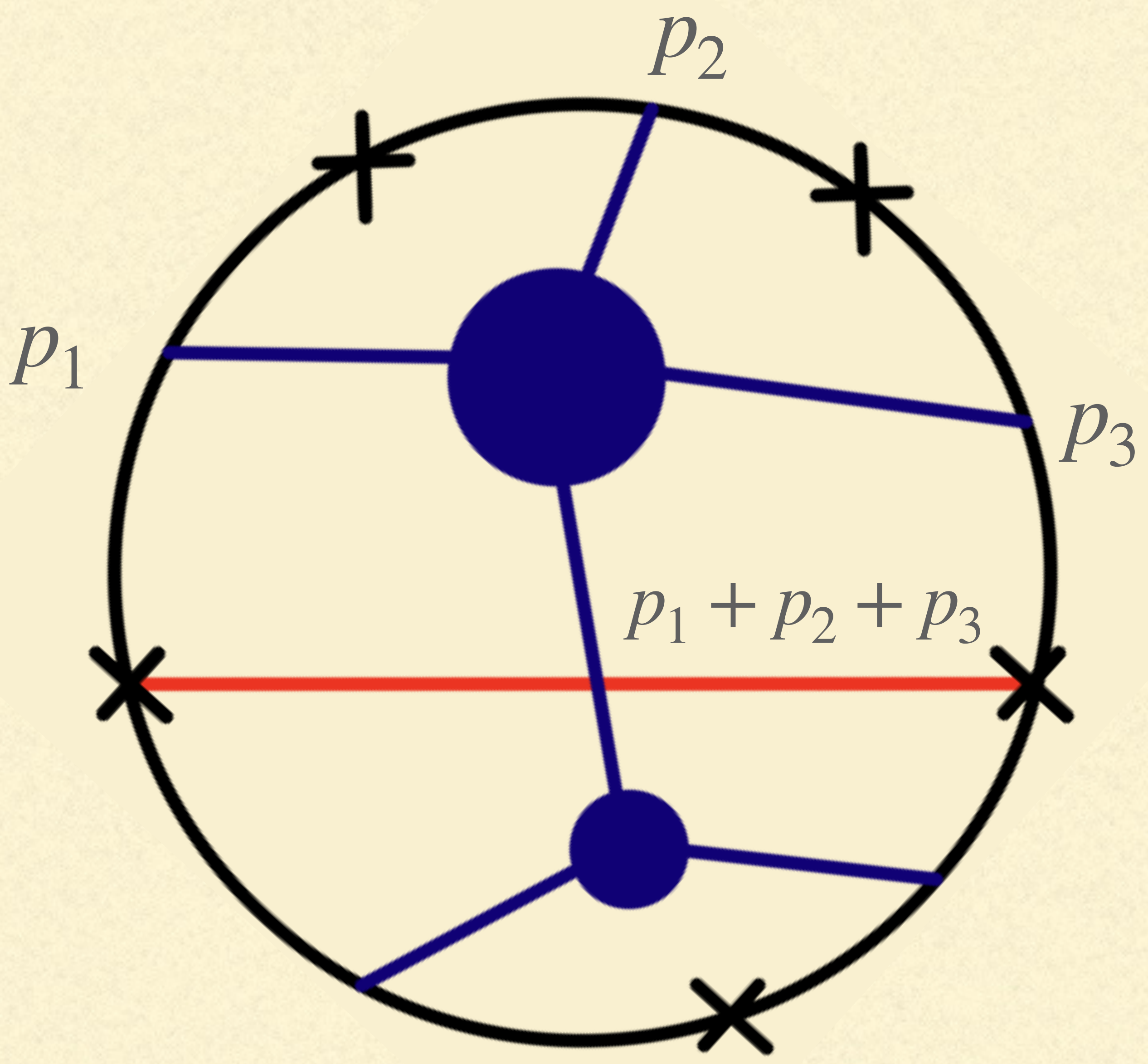
Other approaches to loop labelling: 1901.02432
1802.09395

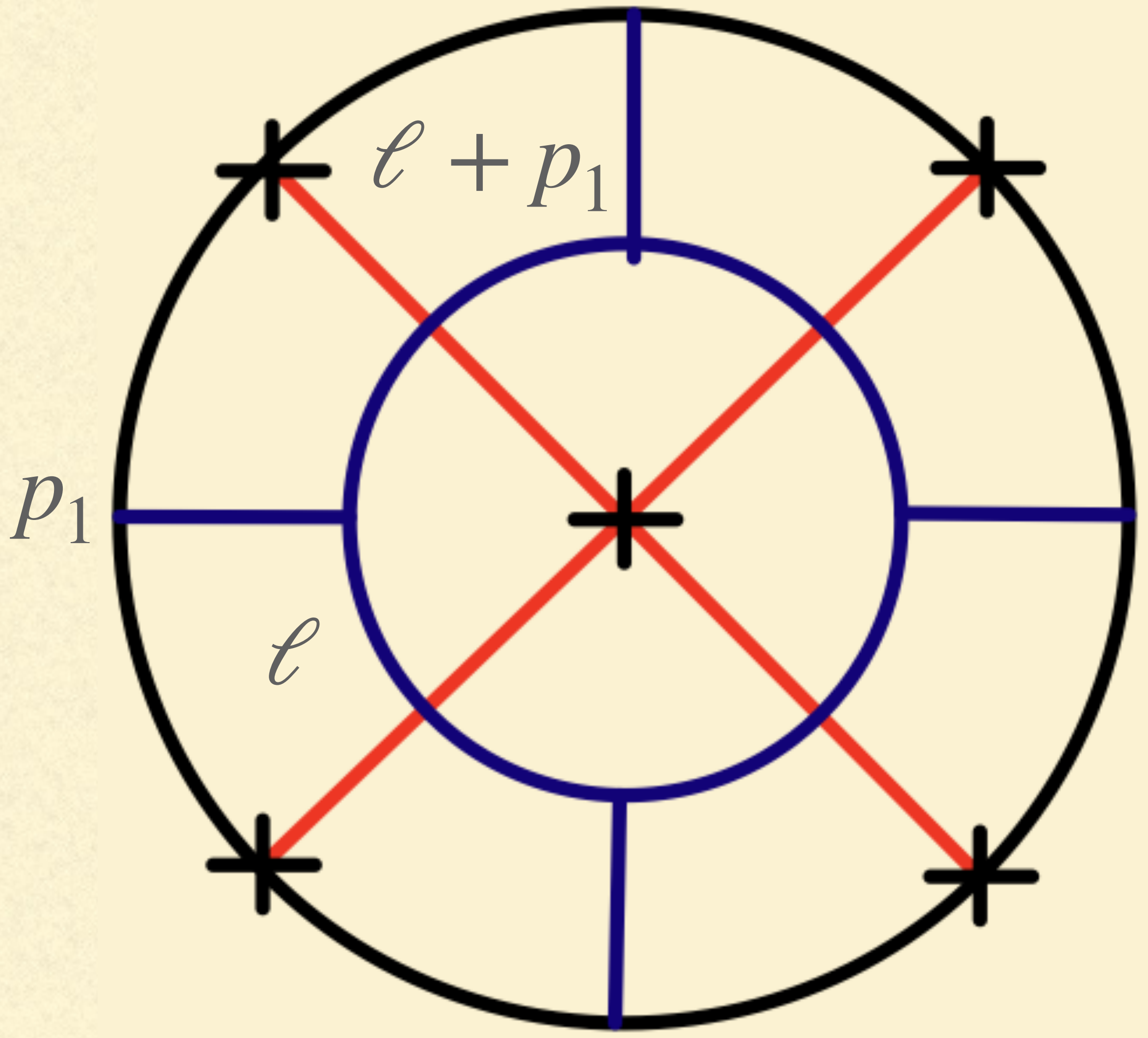
$V = \{\text{punctures}\} \cup \{\text{marked points}\} = \{\times\}$

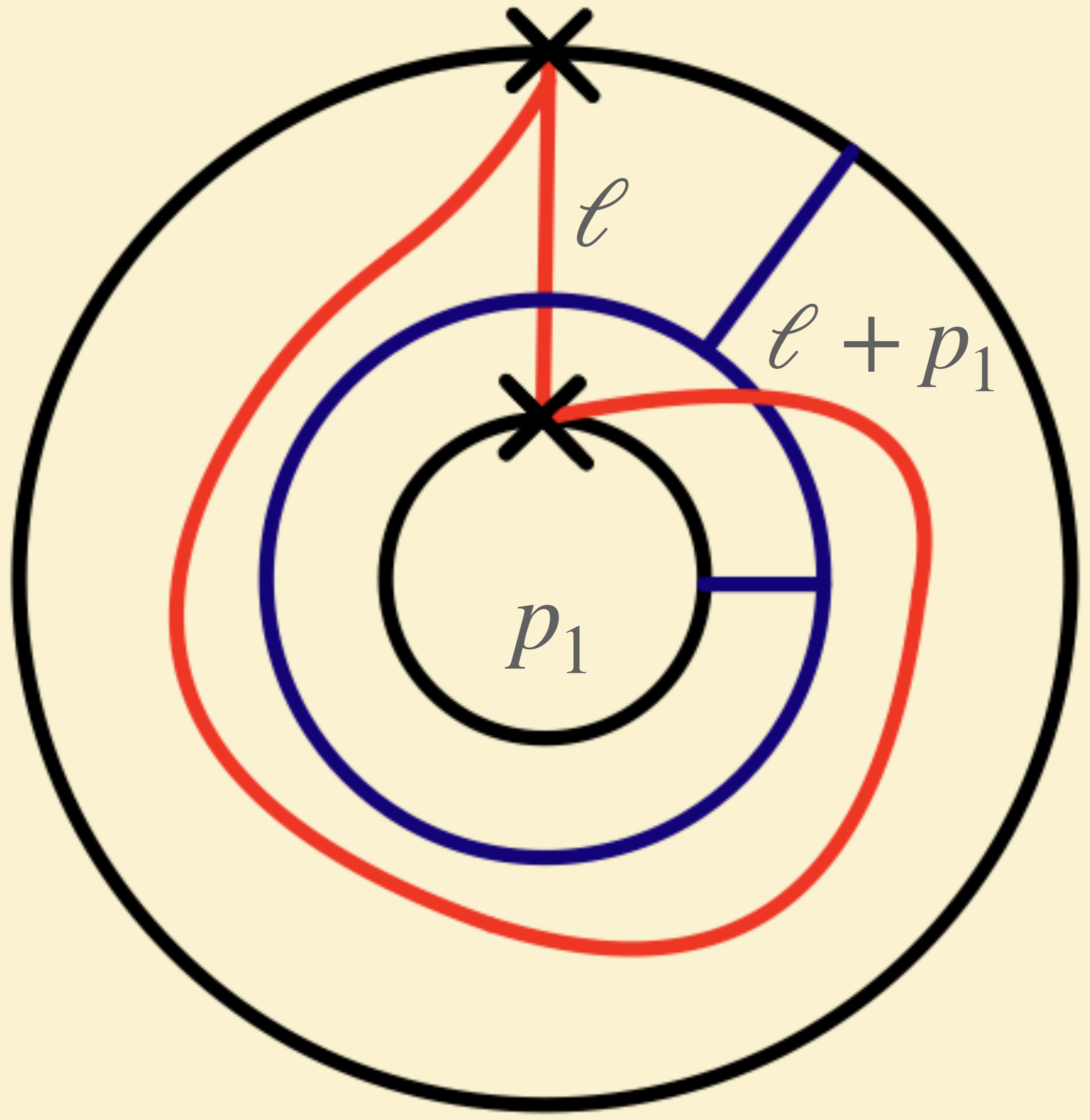


$\Big\{$  $\Big\} / \Big\{$  $, \sum_{i=1}^{3} q_i = 0 \Big\} = H_1(S, V)$

A global routing of momenta is equivalent to a choice of basis for $H_1(S, V)$



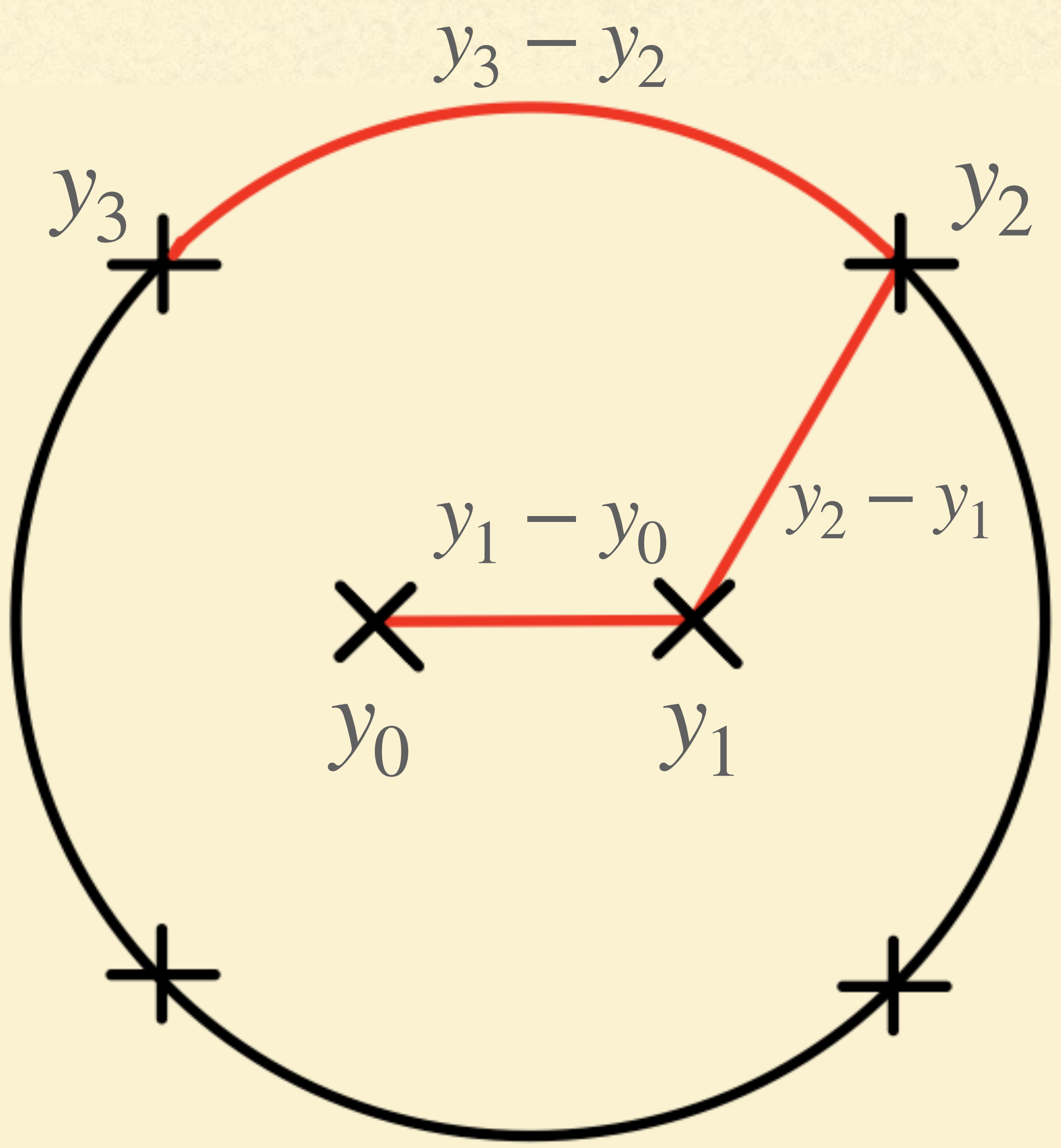


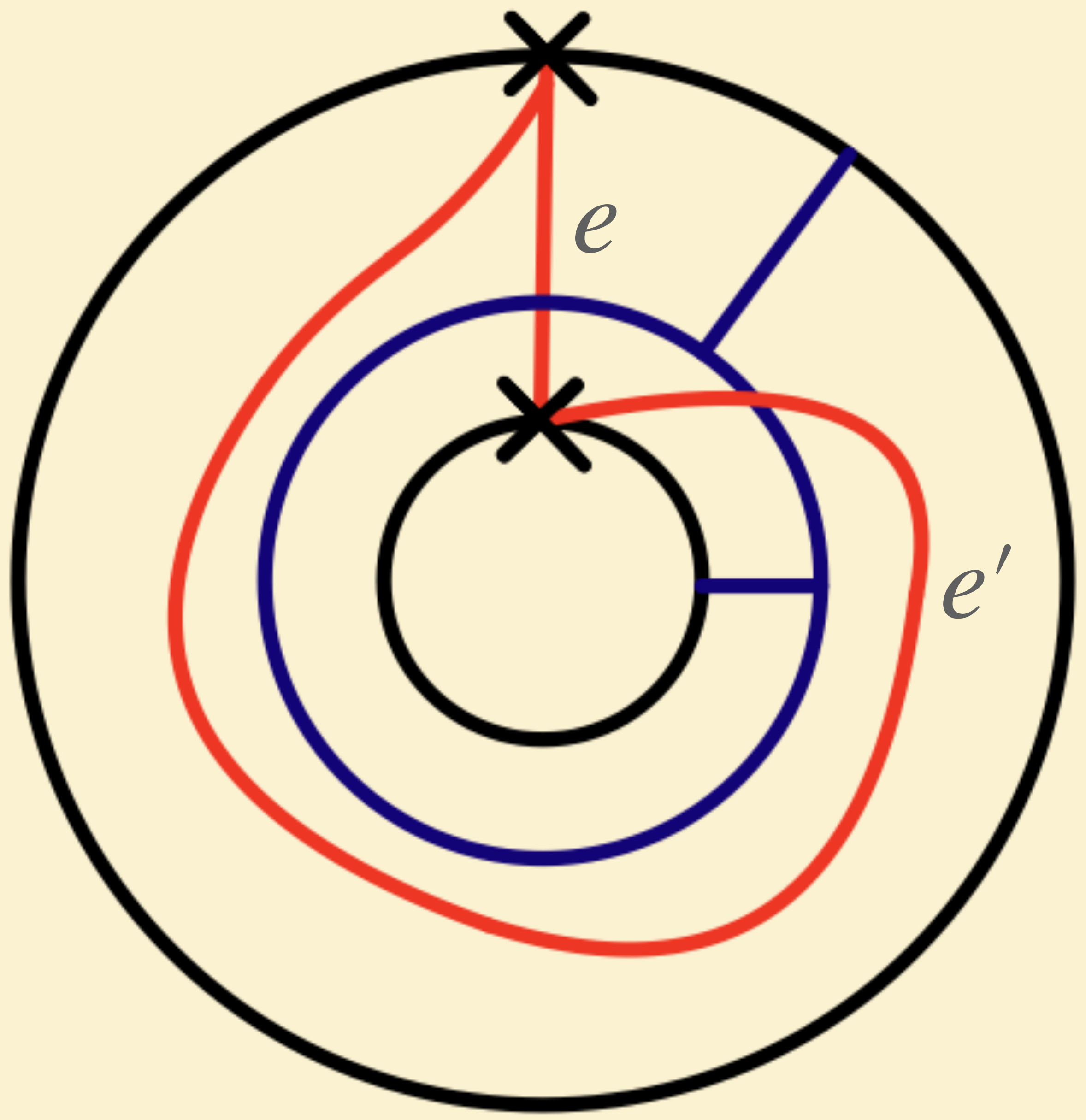


Global loop momenta

Other approaches to loop labelling: 1901.02432
1802.09395

In the planar case the class $[e_{ij}]$ is determined by the endpoints $\partial e_{ij} = v_i - v_j \Rightarrow$ assign dual momenta y_i to the vertices V and the momentum $y_i - y_j$ to to e_{ij}





$[e] \neq [e']$

No longer true for non-planar surfaces

Global loop momenta

Other approaches to loop labelling: 1901.02432
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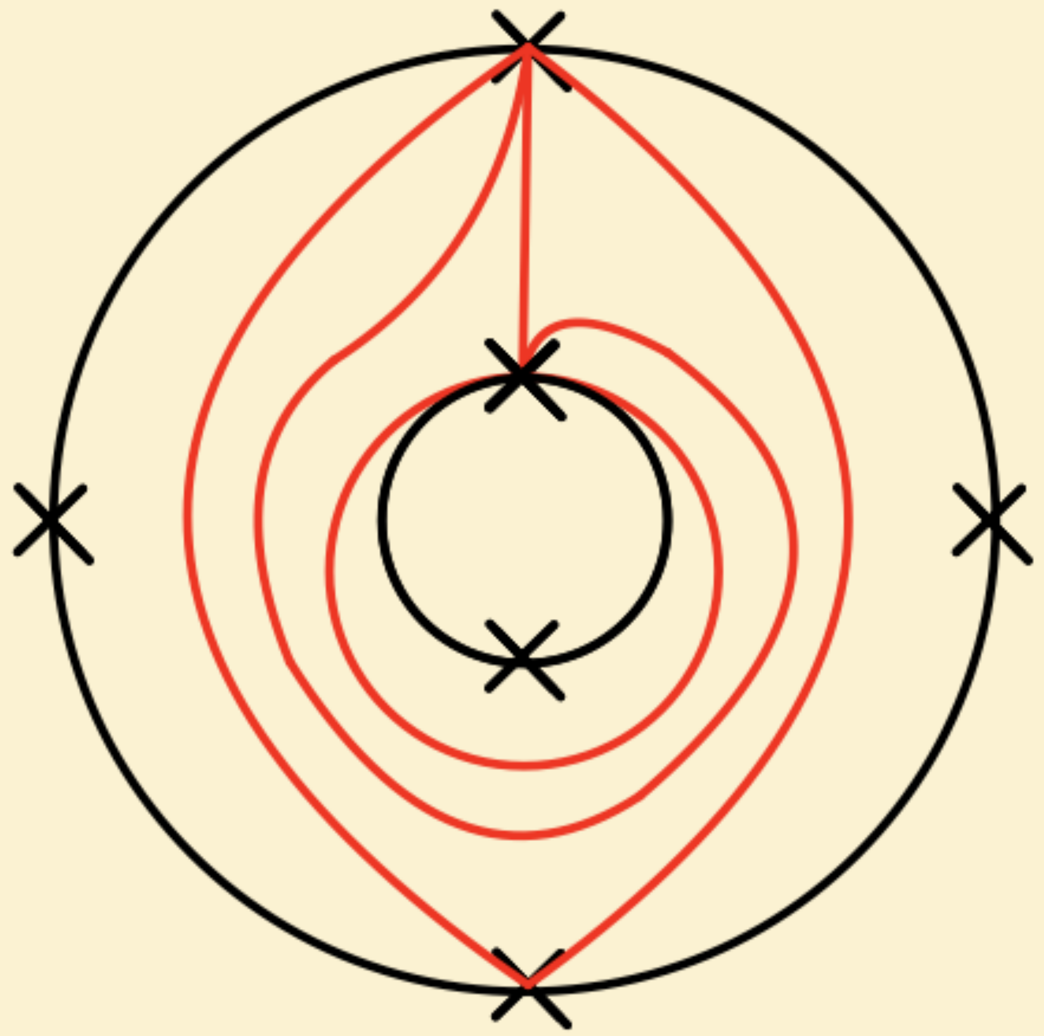
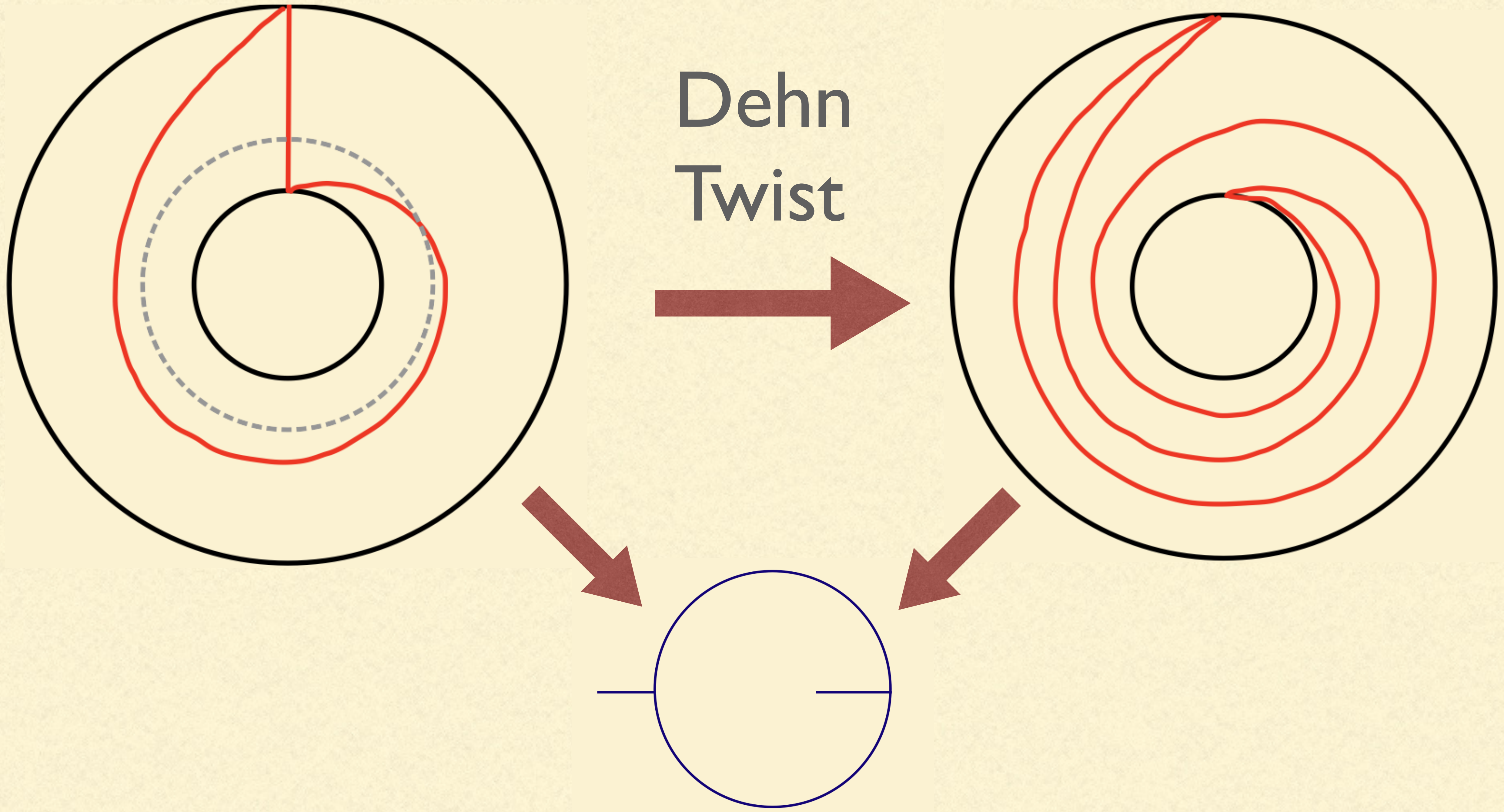
- The edges of any triangulation T contain a basis for $H_1(S, V)$: $[e'] = \sum_{e \in T} c_e \; [e]$
- Assign loop labels $q_{[e]}$ to the edges of T, every other arc is assigned $q_{[e']} = \sum_{e \in T} c_e \; q_{[e]}$
- The Feynman diagram Γ' dual to any triangulation T' is now assigned momenta

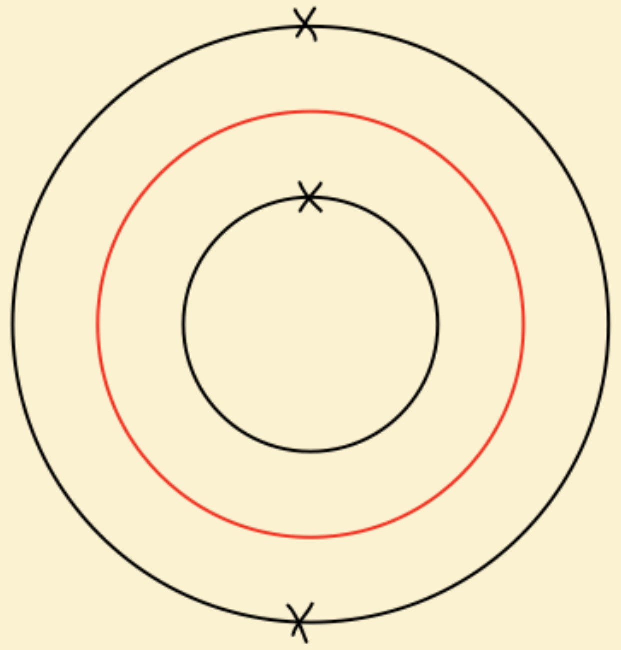


Problem:

Many triangulations correspond to the same diagram

The mapping class group

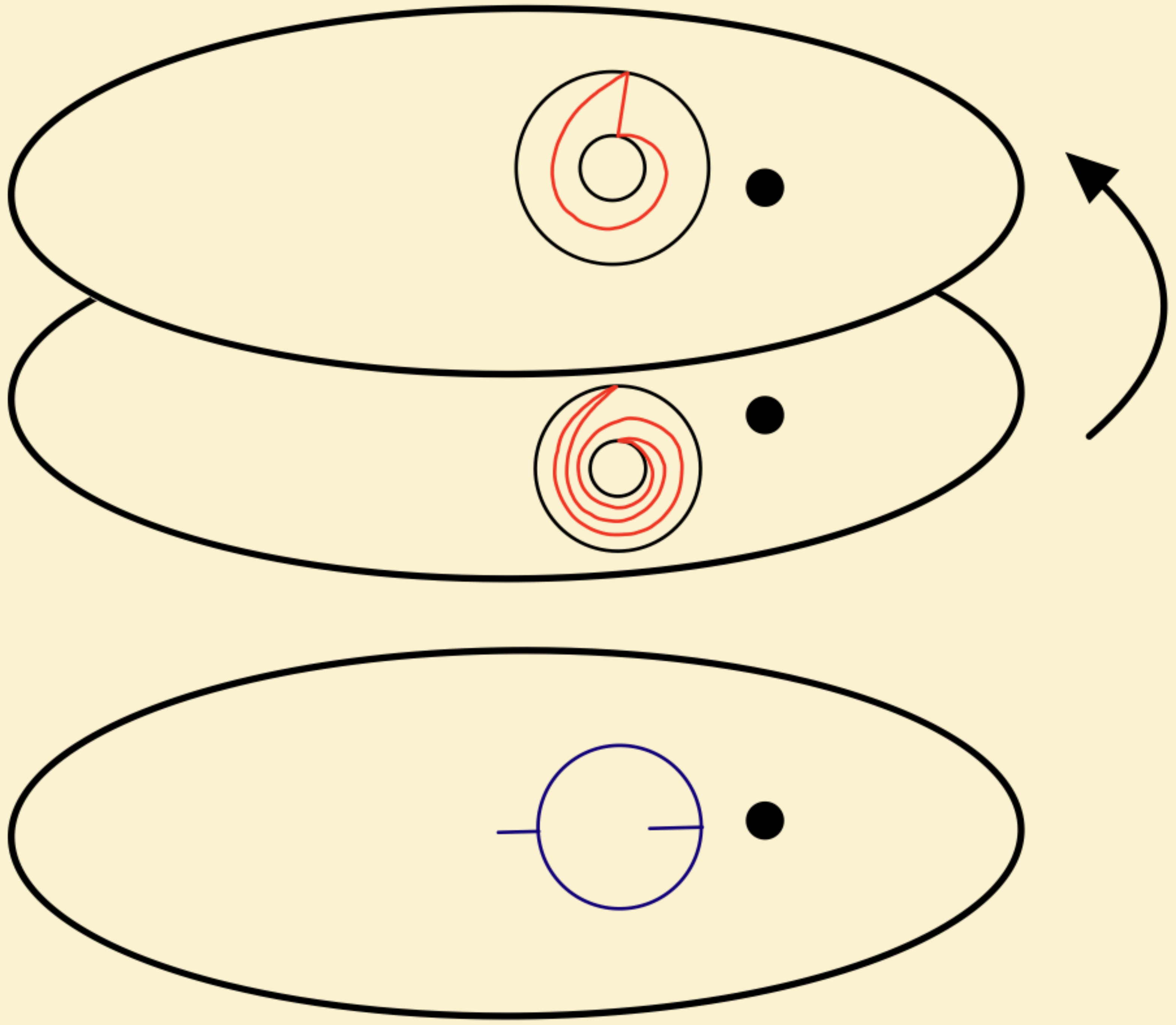


The surface  has non trivial *mapping class group*

The mapping class group

Triangulations

Feynman (fat)graphs



$\mathrm{MCG}(S)$

Solution:

Choose a *fundamental domain* for the action of the MCG

The integrand

- Choose a set of triangulations $\mathcal{T}$ dually containing every Feynman diagram once.
- Label every arc e with a variable X_e
- Define the rational function:

$$\boxed{I_S^{\mathcal{T}}(X) = \sum_{T \in \mathcal{T}} \frac{1}{\prod_{e \in T} X_e}}$$

We obtain the physical integrand by substituting $X_e \to (q_{[e]}^2 - m^2)$

The master integrand

The integrands $I_S^{\mathcal{T}}$ satisfy recursive properties which descend from those of the *master integrand*

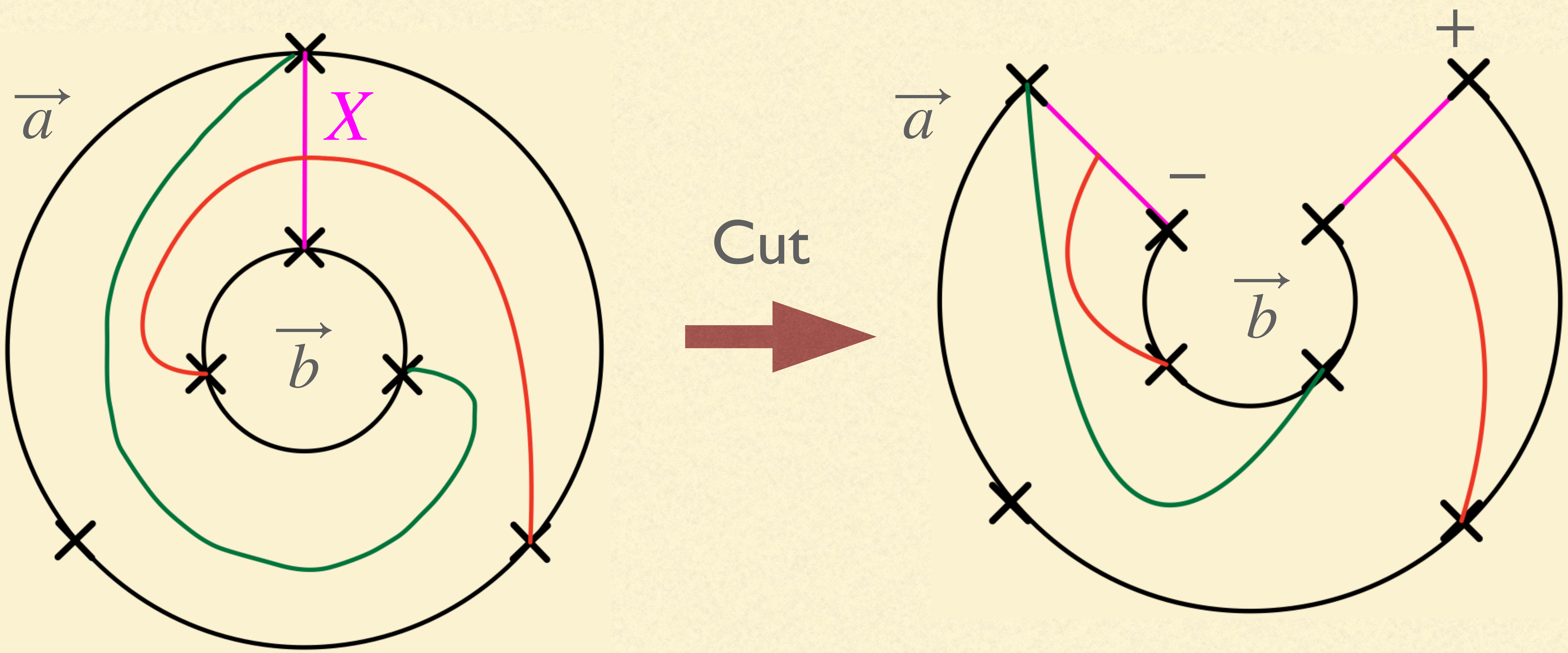
$$\boxed{I_S(X) = \sum_T \frac{1}{\prod_{e\in T} X_e}}$$

- It is a formal object: "$\int I \, d^D\ell = \infty \times A$"
- Uniform recursive behavior, while that of $I_S^{\mathcal{T}}$ is slightly affected by the choice of $\mathcal{T}$

$$\text{Res } I_S|_X = I_{S'}(X')$$

The surface S' is obtained by cutting S along X

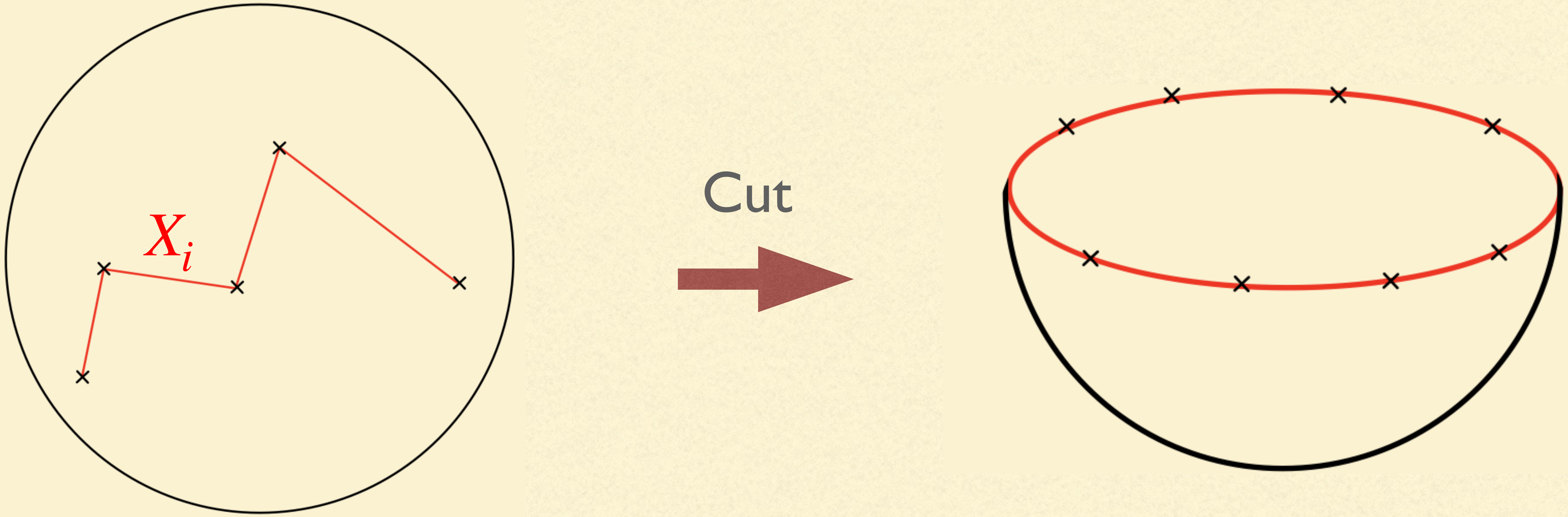
Singularities of the integrand



A pole with residue given by a forward limit tree amplitude

$$\mathrm{Res}\ I_{0,0}^{(\vec{a})(\vec{b})}\Big|_X = I_{0,0}^{(\vec{a}_L,-,\vec{b},+,\vec{a}_R)}$$

Singularities of the integrand



A tree level amplitude from a multiple cut of a vacuum integrand

$$\mathrm{Res}\ I^{\varnothing}_{0,p}\,|_{X_i} = I^{(1,\ldots,p+3)}_{0,0}$$

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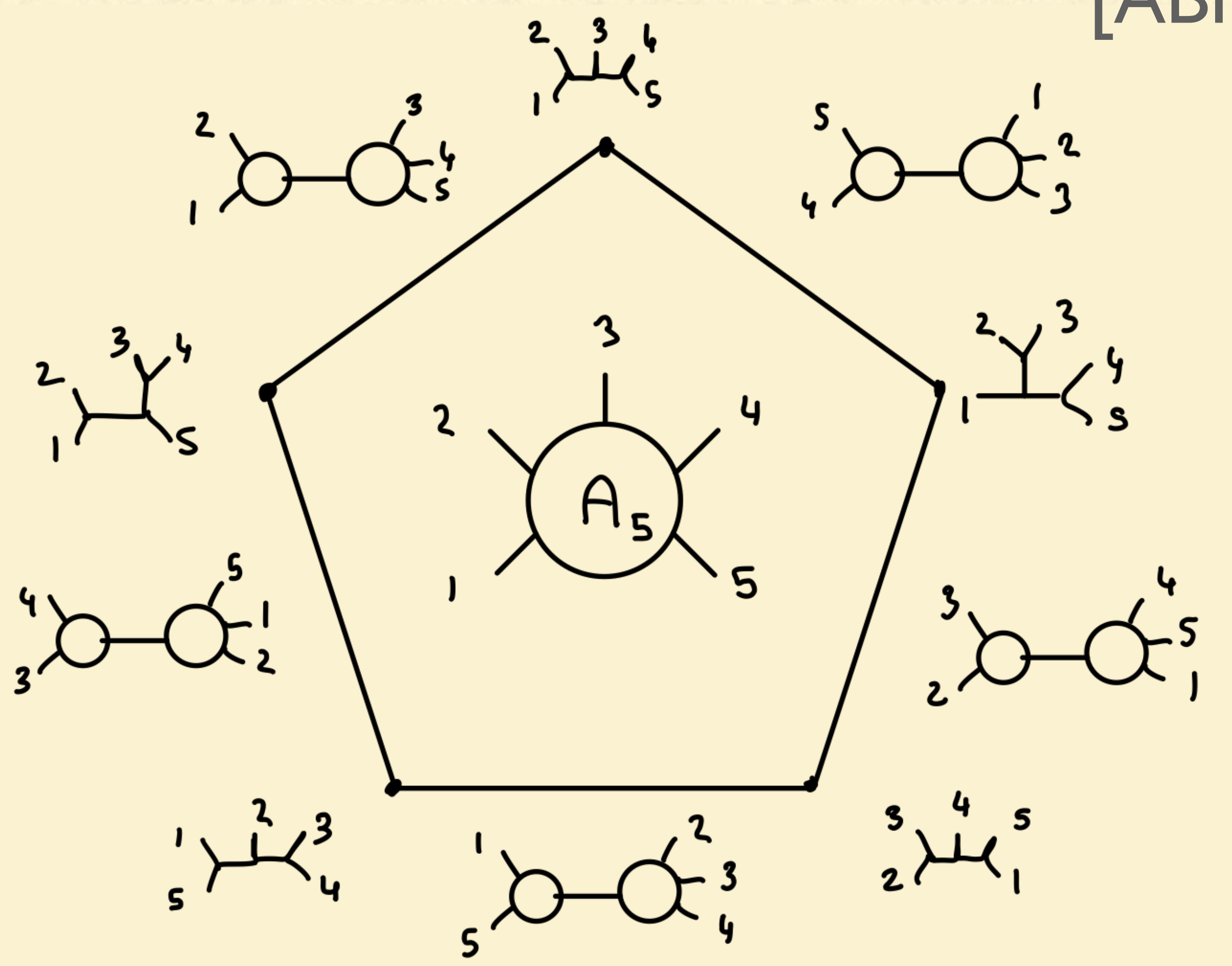
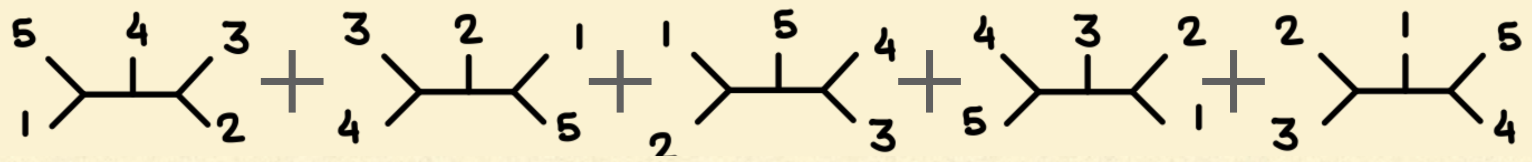
Politopality, Projectivity & Completeness

[ABHY]



$$\Omega = A_n \, dX \; = $$

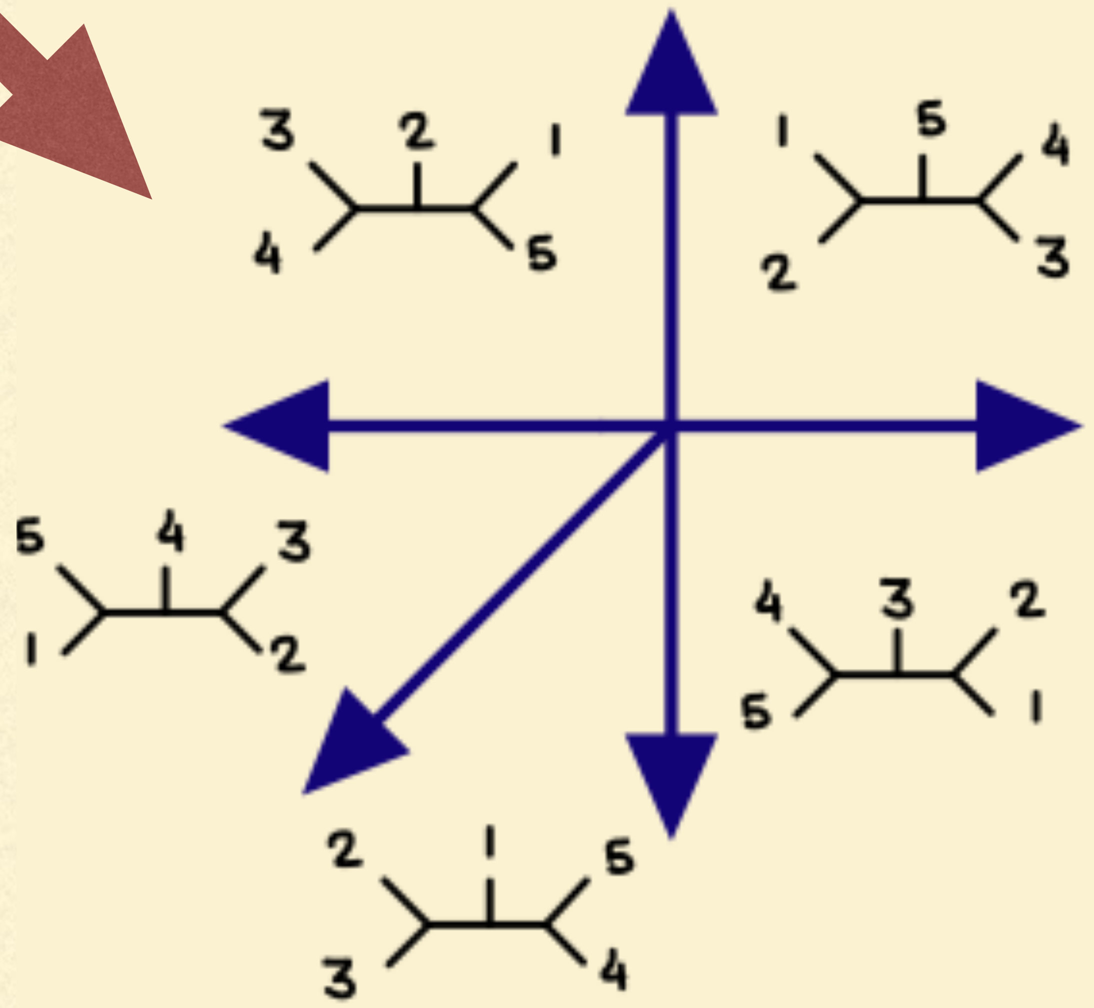


$$= (X_{13})(X_{14}) - (X_{13})(X_{35}) + (X_{25})(X_{35}) - (X_{25})(X_{24}) + (X_{14})(X_{24})$$

$(x) := d\log(x)$

Ω is *projective:* invariant under $X \to \alpha(X)\; X$.
Not true for the individual Feynman diagrams
Manifest from BCFW-like formulae:

$$\Omega = \left(\frac{X_{24}}{X_{13}}\right)\left(\frac{X_{14}}{X_{25}-X_{24}}\right) + \left(\frac{X_{25}}{X_{13}}\right)\left(\frac{X_{35}}{X_{24}-X_{25}}\right)$$

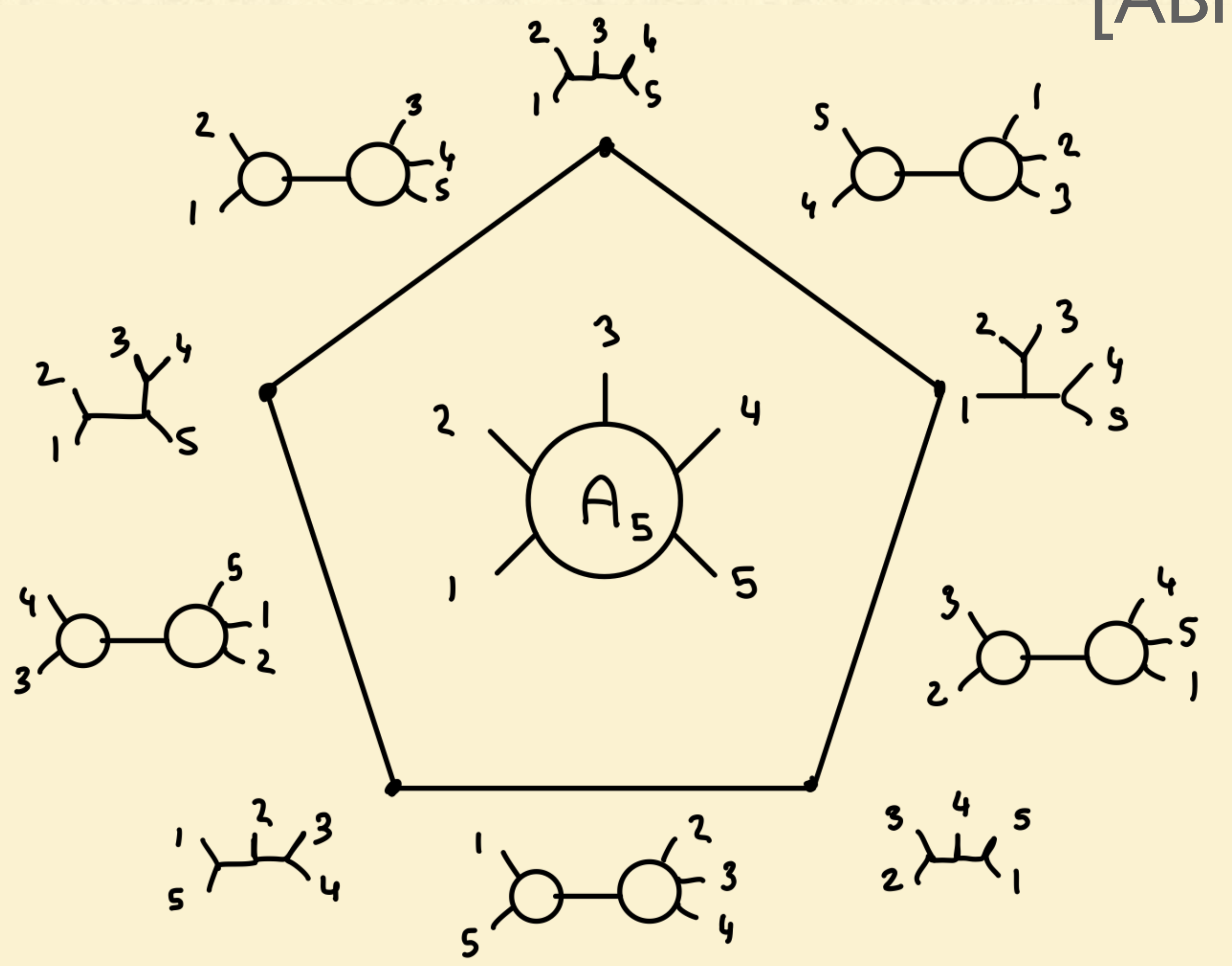


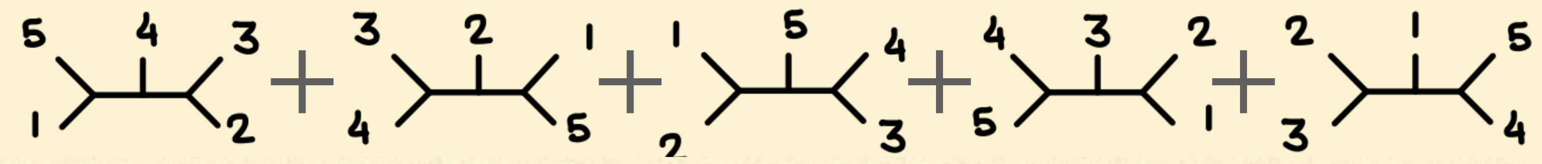
Orientation of
Feynman Diagrams
indicated by cones

Good starting point to go beyond tree level

Politopality, Projectivity & Completeness

[ABHY]



$\Omega = A_n\, dX =$ 

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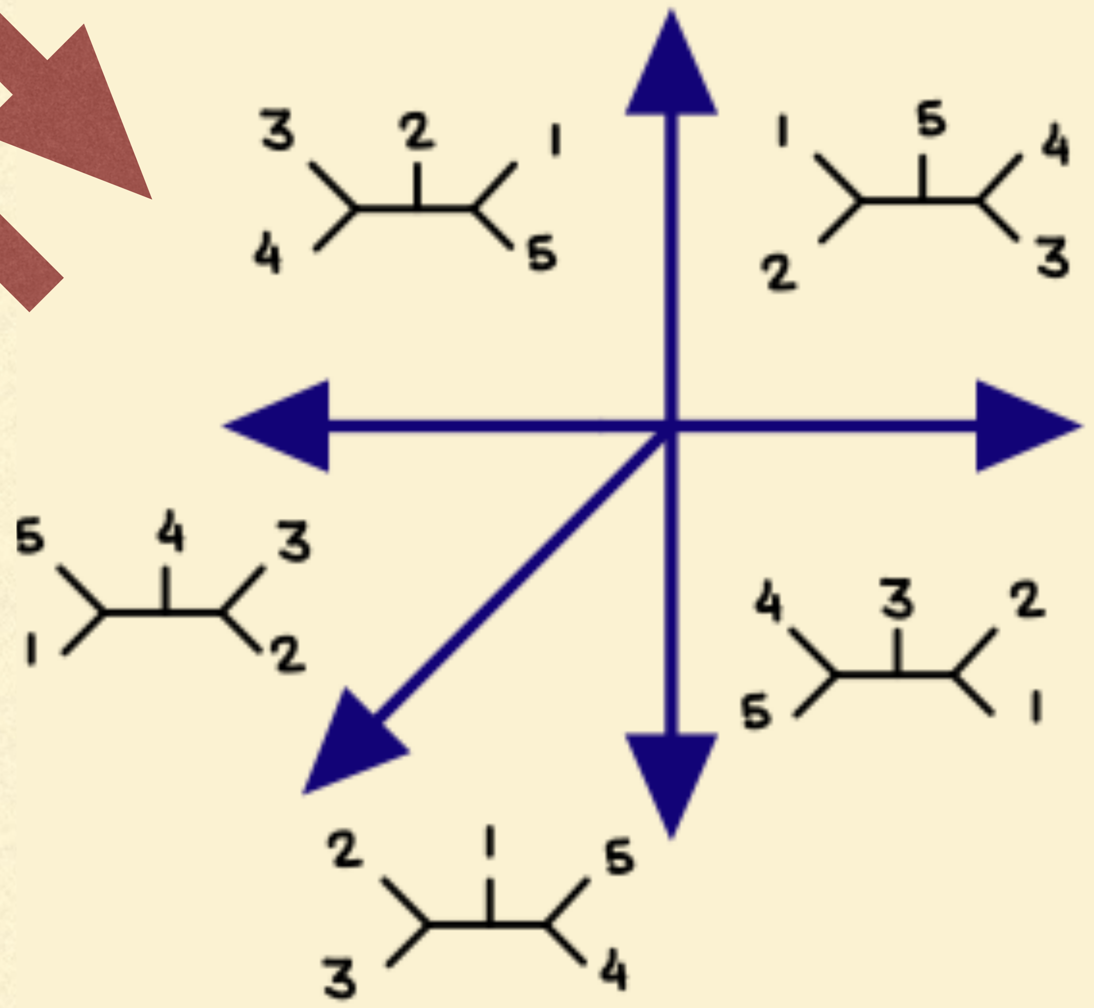
2d space-time
(See Nima's @
Zoomplitudes)



Orientation of
Feynman Diagrams
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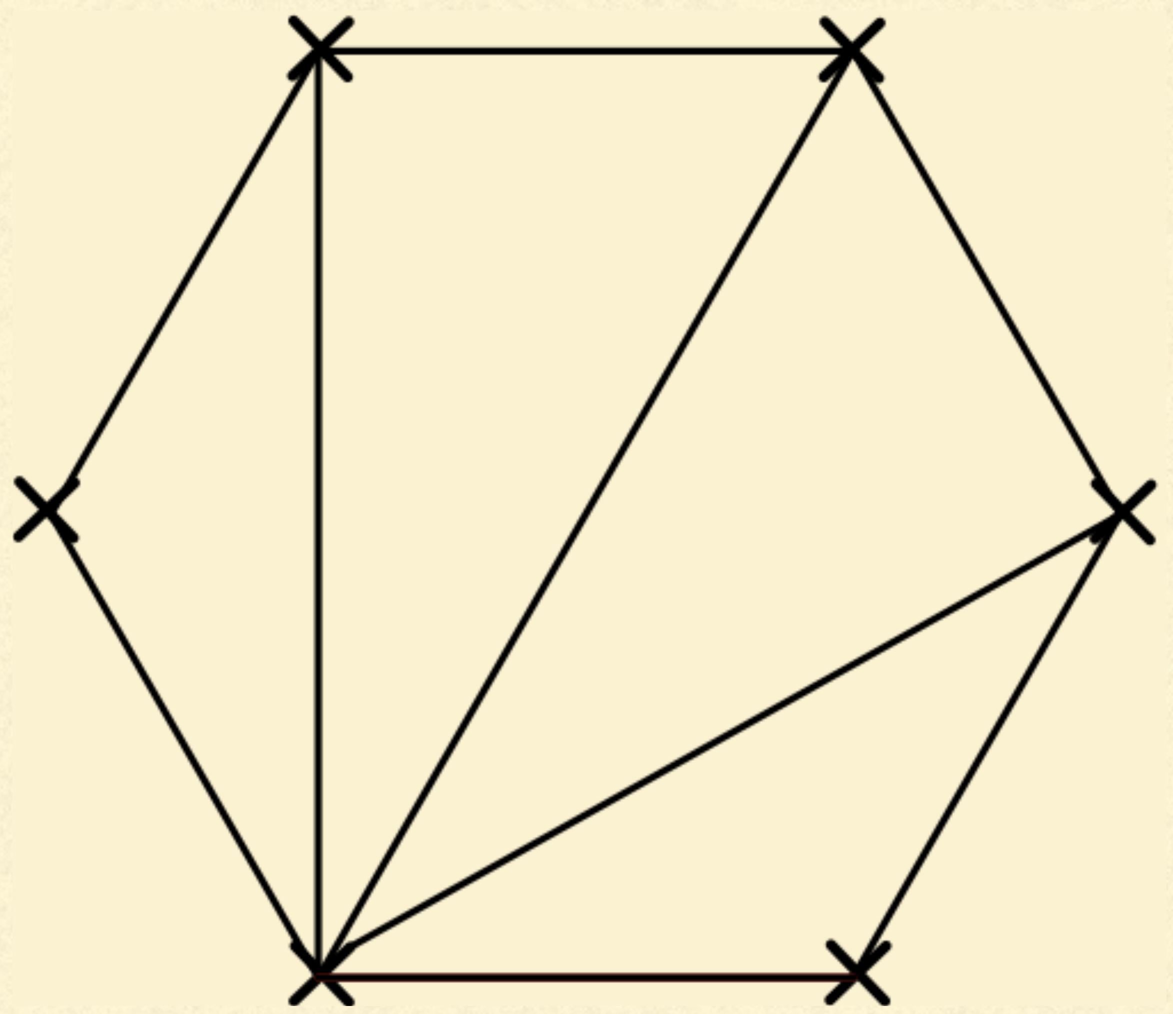
Good starting point to go beyond tree level

Tropical Teichmüller Space

[Fock & Goncharov arXiv:math/0510312]

- Choose a reference triangulation T
- To every arc e on S is associated a geometric intersection vector $\mathrm{geom}(e) = \{i(e, e'), e' \in T\}$
- Project through $b \in \partial S$ ("on-shell condition"), obtaining new vectors $\chi(e)$ (a.k.a. *g-vectors*).
- Add cones $\sigma = \mathrm{Conv}(\{\chi(e), e \in T'\})$ for every triangulation T'



The collection of cones σ form a fan Σ_S^T which is known as *Tropical Teichmüller space* or *Cluster fan*

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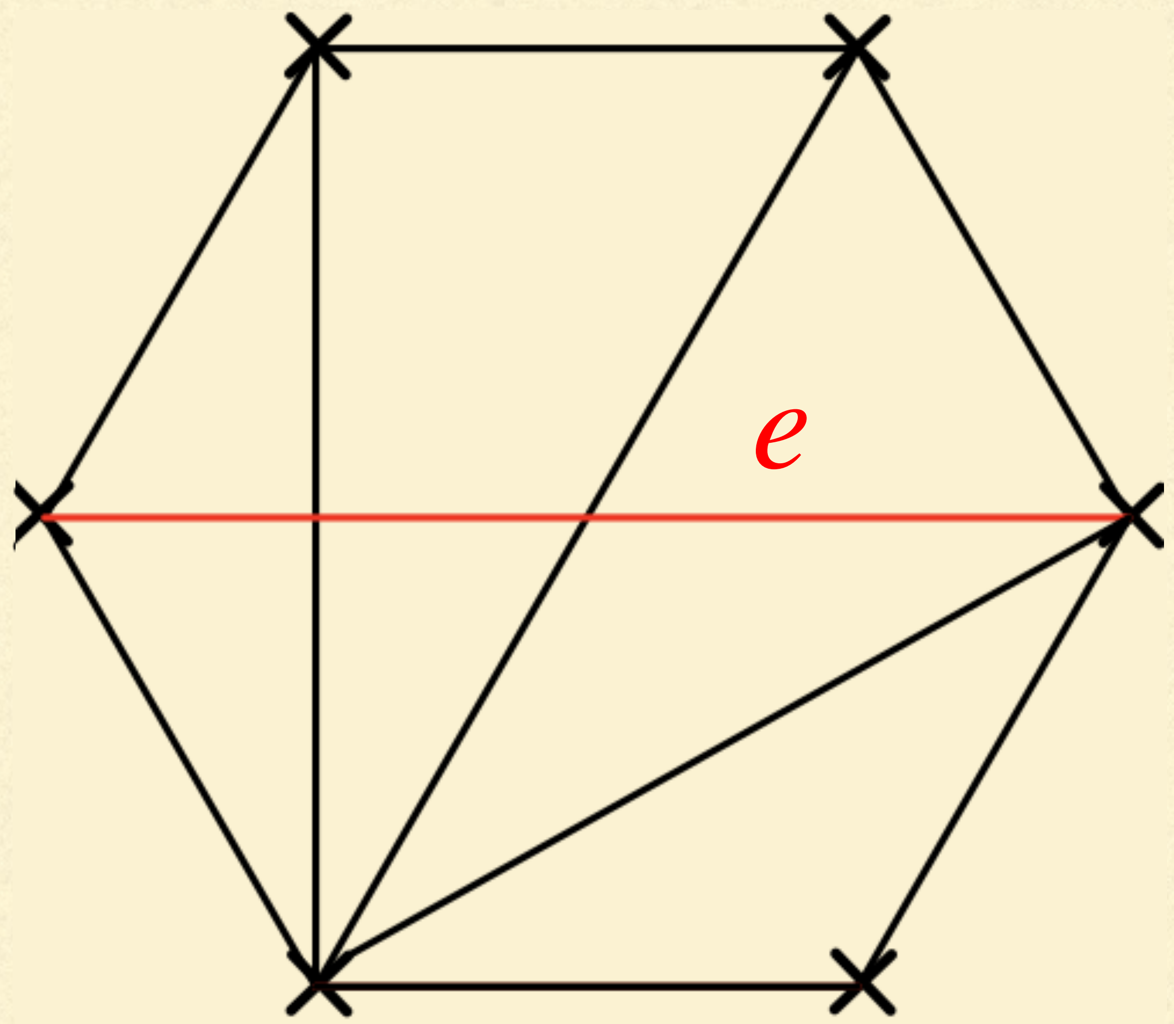

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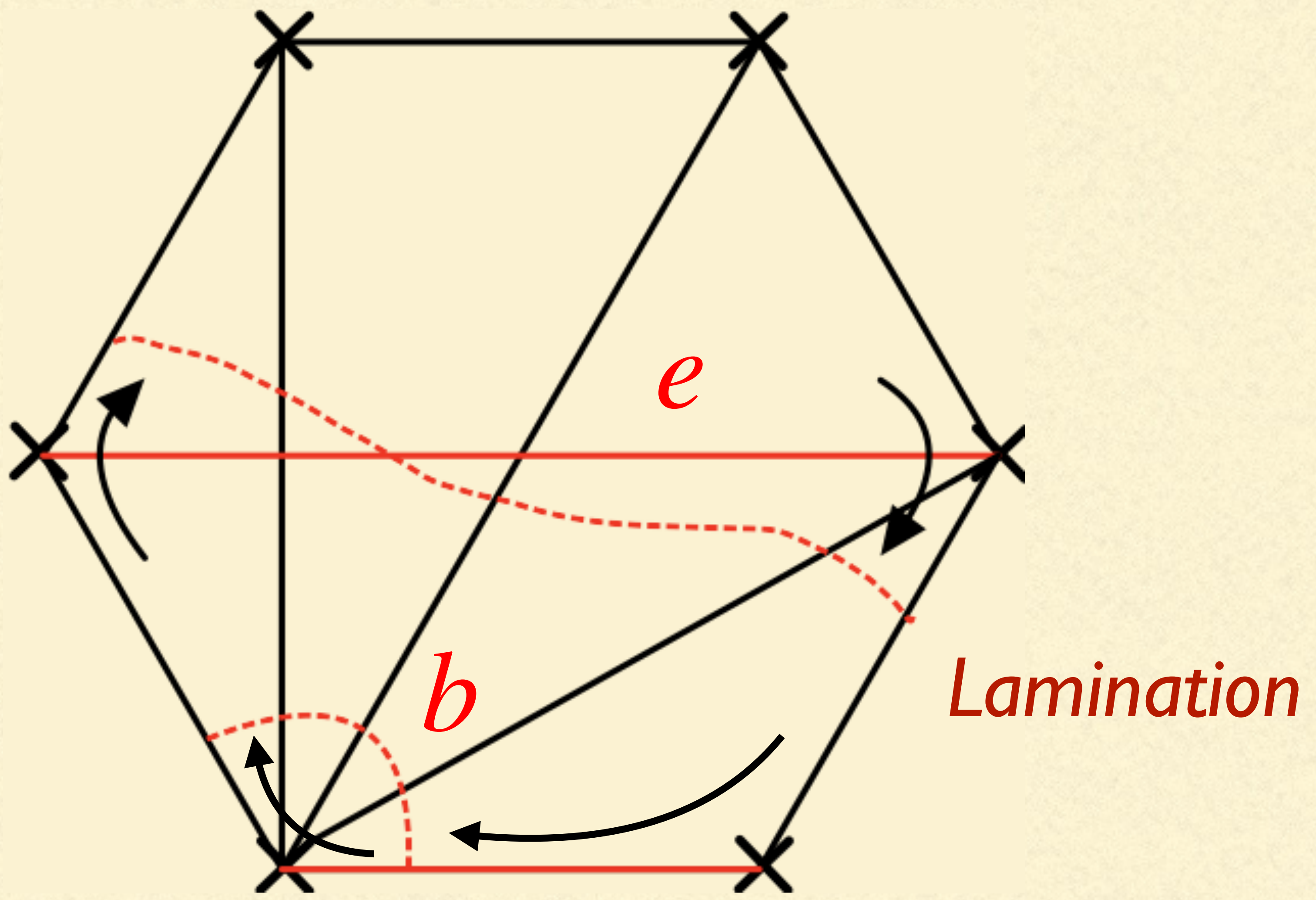
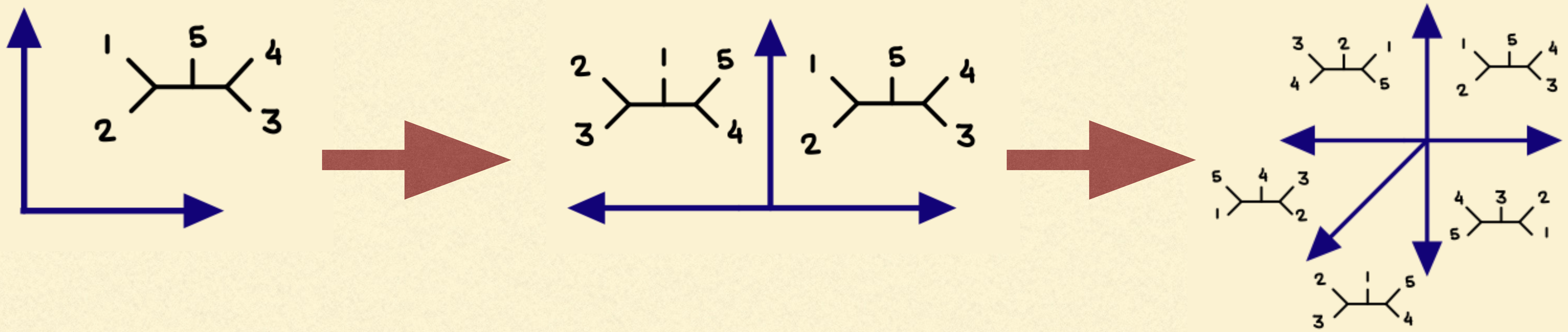


The collection of cones σ form a fan Σ_S^T which is known as *Tropical Teichmüller space* or *Cluster fan*

Tropical Teichmüller Space [FG, STZ]

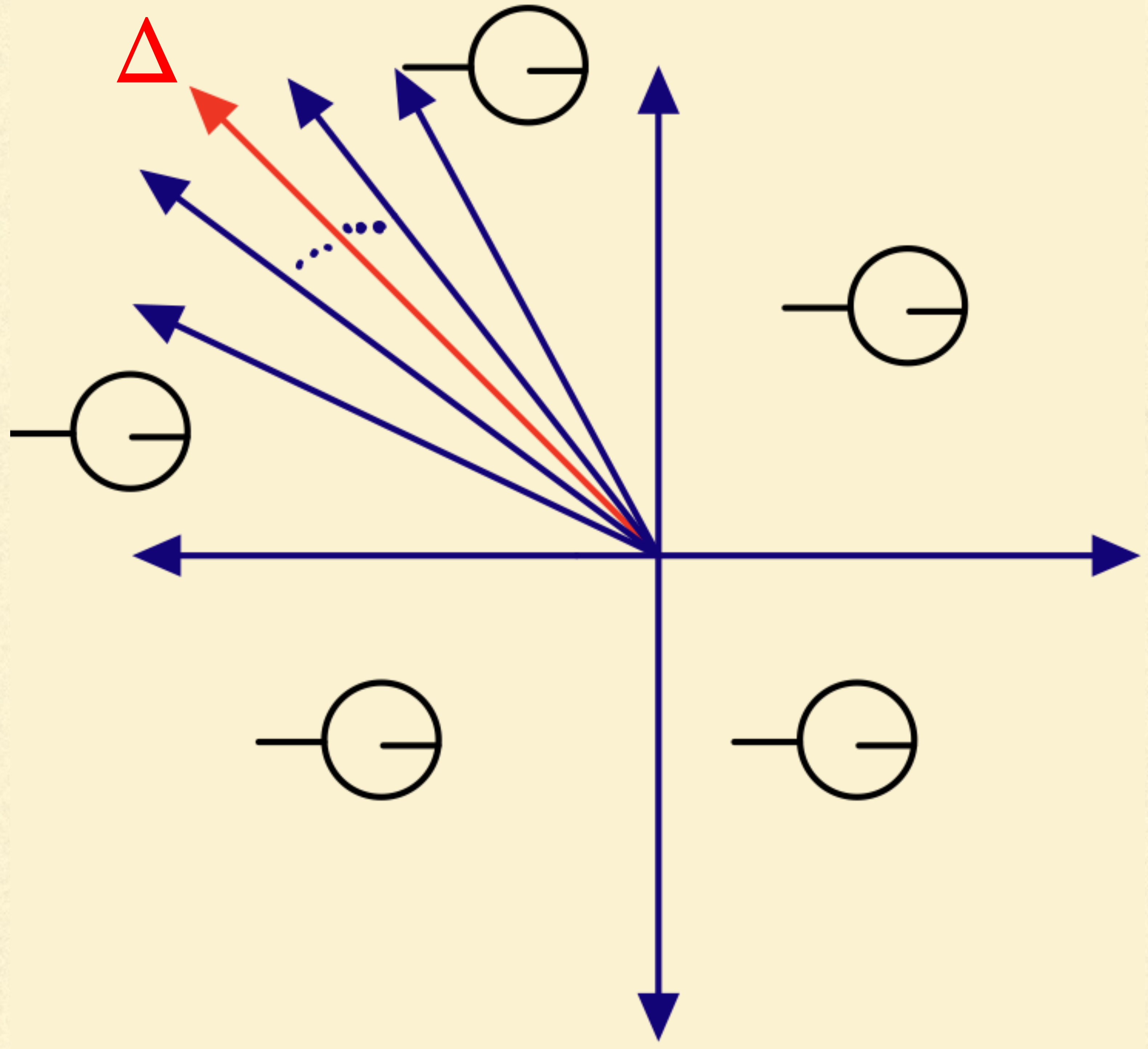
For surfaces with trivial mapping class group, the fan is complete:

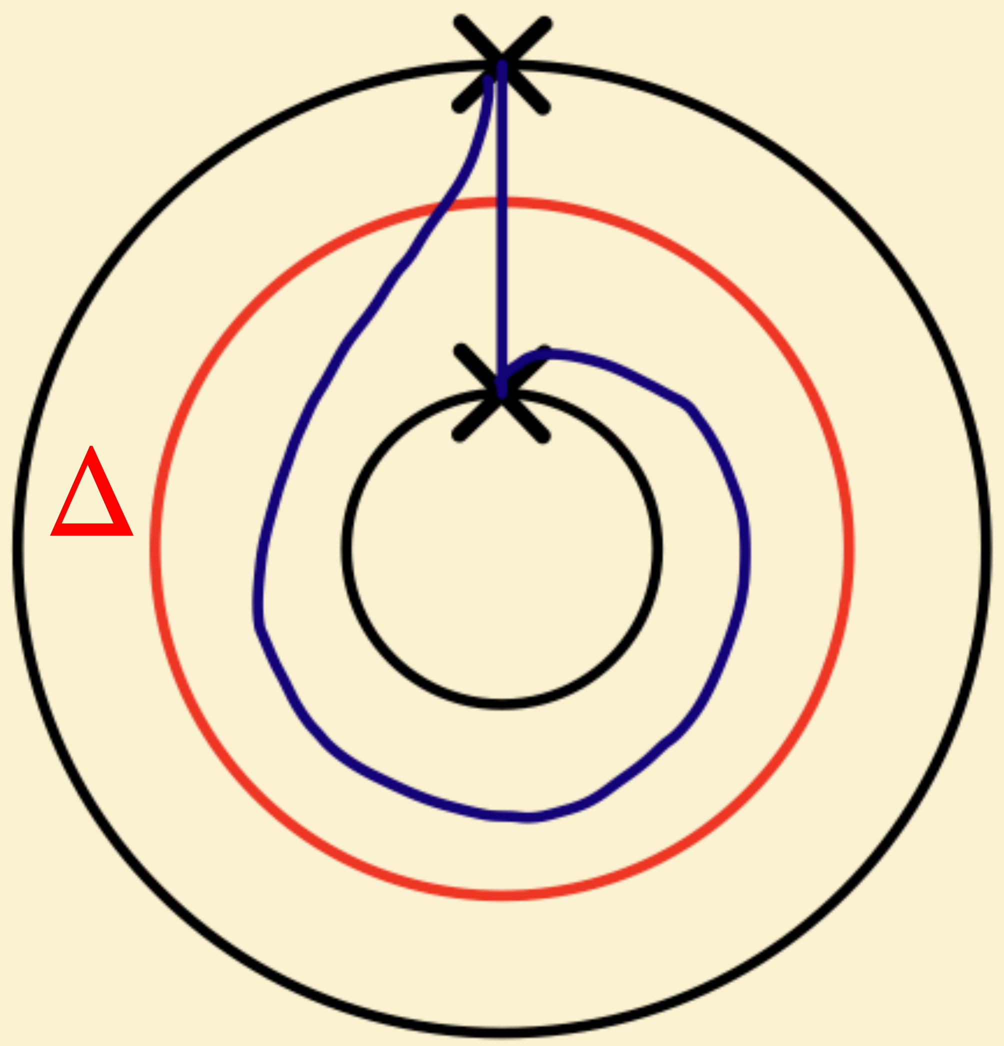


Implied by the existence of ABHY polytope (equivalent via the 2d space-time picture)

Tropical Teichmüller Space [FG, STZ]

If $\mathrm{MCG(S)}$ is non trivial the fan Σ_S *is not complete*:

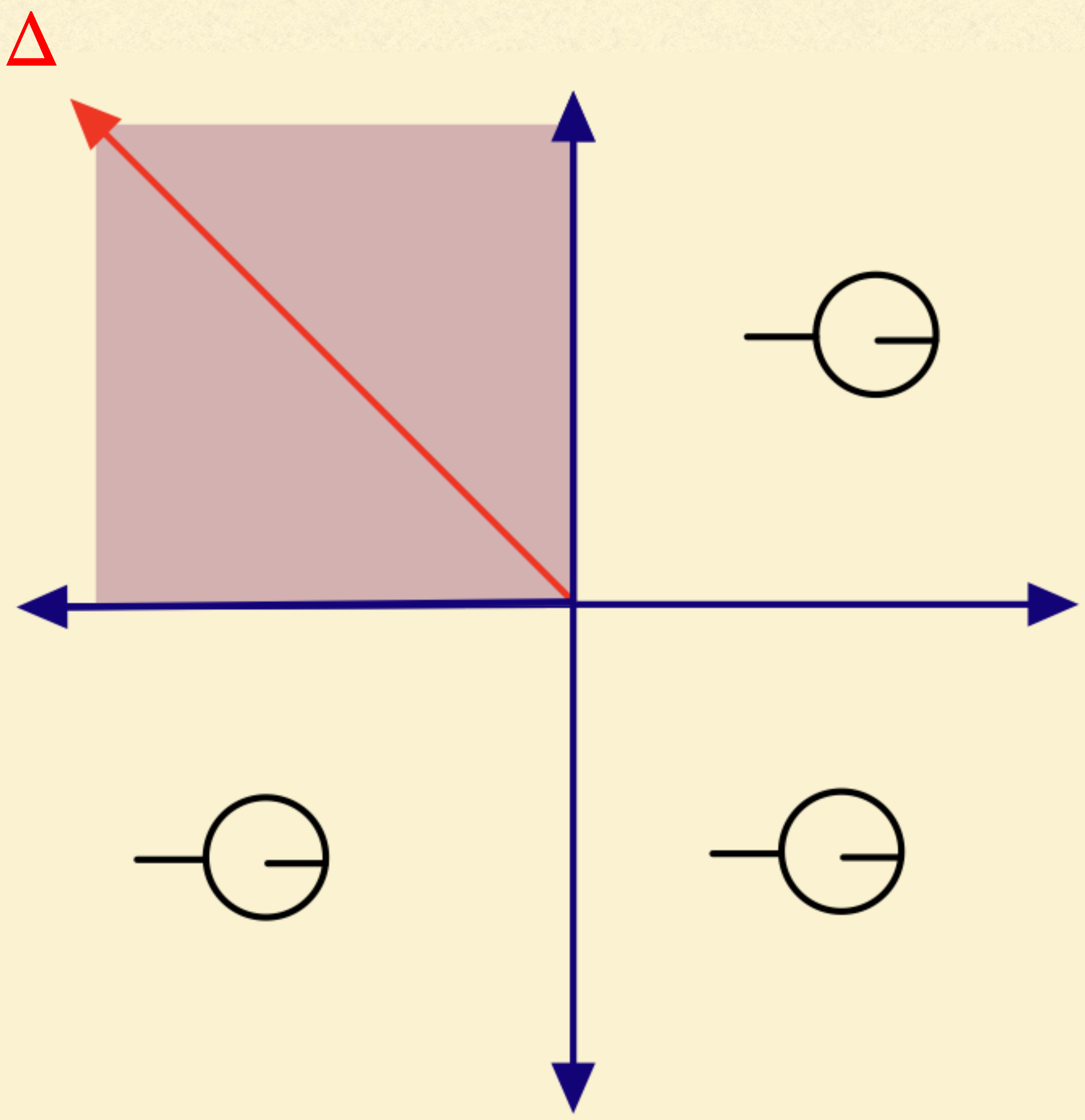




It is impossible to find a polytope associated to Σ_S, a projective canonical form nor an integrand

Completing the fan

We complete the fan by keeping M copies of the integrand and then adding Δ-*cones:*



We obtain a new complete fan, a polytope and a projective form

$$\Omega_S = \left(M \times I_s + \frac{1}{\Delta} \underline{\Omega}_\Delta \right) dX$$

The integrand is recovered via the limit $\Delta \to \infty$

Projectivity holds for the full Ω_S form only....

What is Δ*?*

Completing the fan

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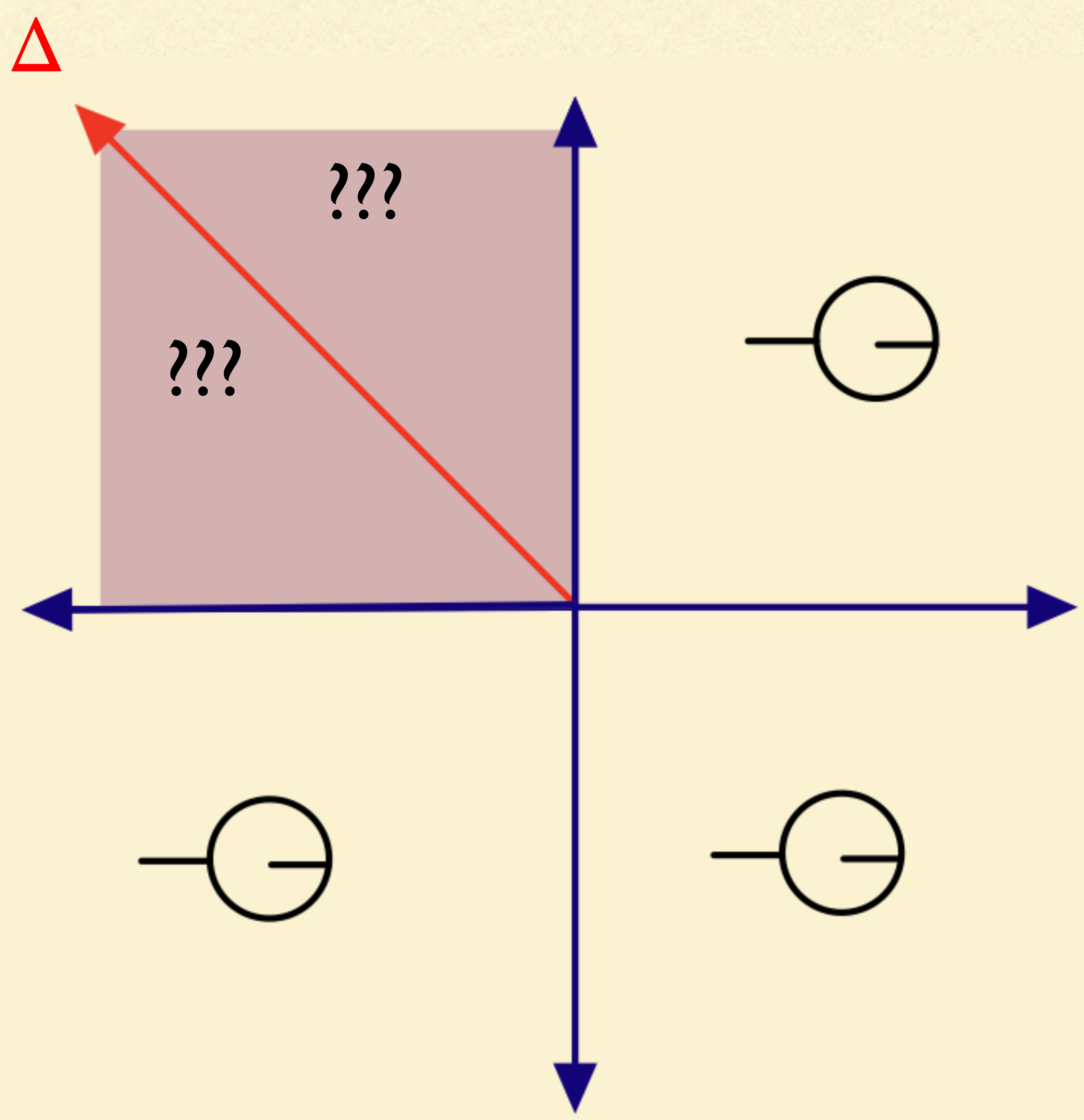

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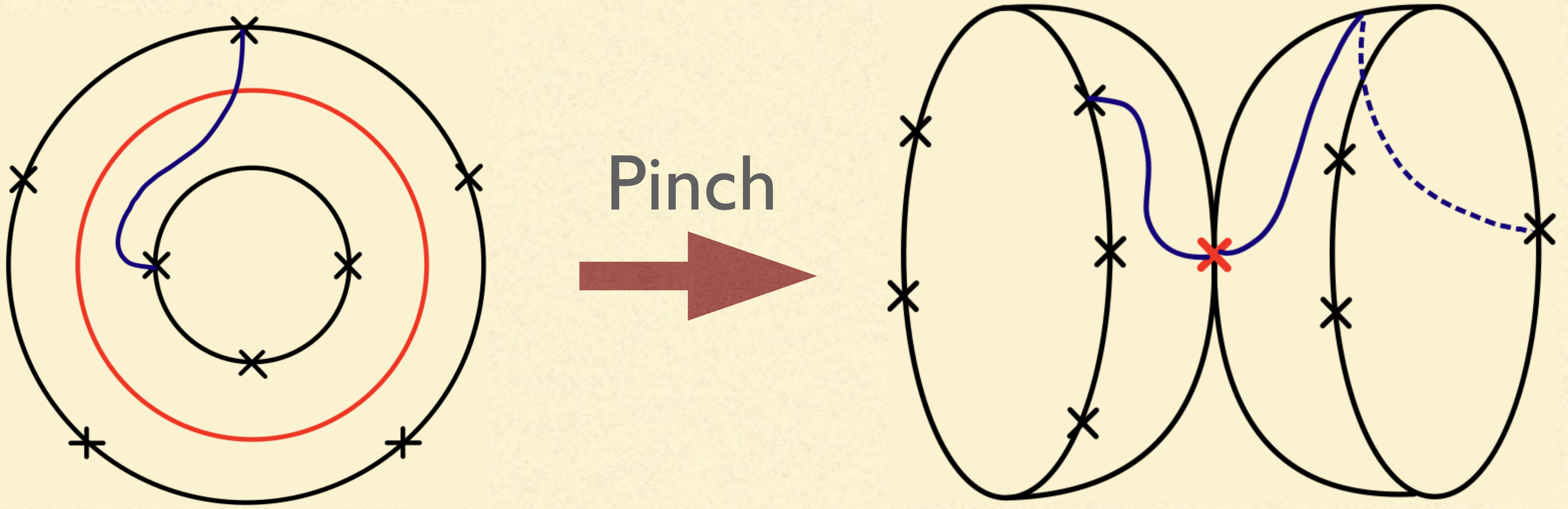
Complete the fan by completing the theory?

We gain some insight on the meaning of Ω_Δ by studying the geometry of Δ-cones.

The dual $\Delta -$facet follows an *entangled factorization*:



New, non cluster-algebraic recursive structure!

$\sim$ closed string

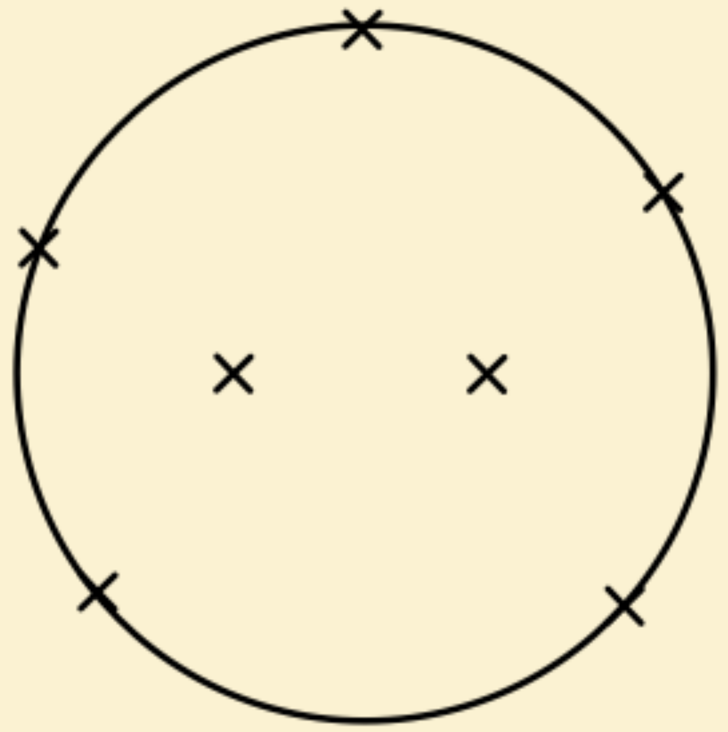
Furthermore, Δ is homologous to one component of ∂S so that $q_{[\Delta]} = \sum_i q_{[b_i]}$

$$\Omega_S = \left(M \times I_s + \frac{1}{\Delta} \underline{\Omega}_\Delta \right) dX$$

The pole and residue at Δ resembles of an extra *colorless* particle (graviton/dilaton) going on-shell

Politopality/Projectivity/Completeness $\Rightarrow$ *forces "gravity"*

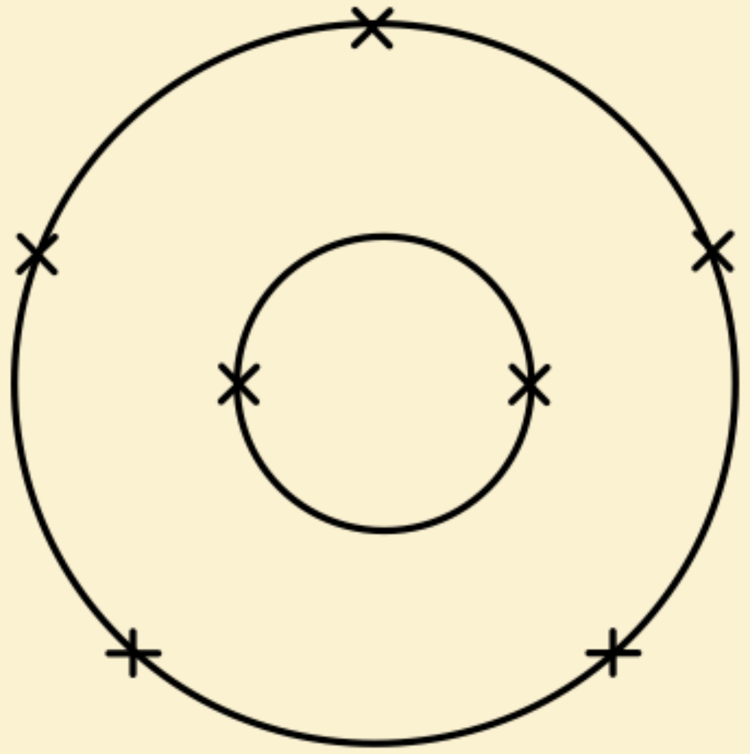
A plethora of checks: finitely many Δ's



Single trace 2-loop integrand

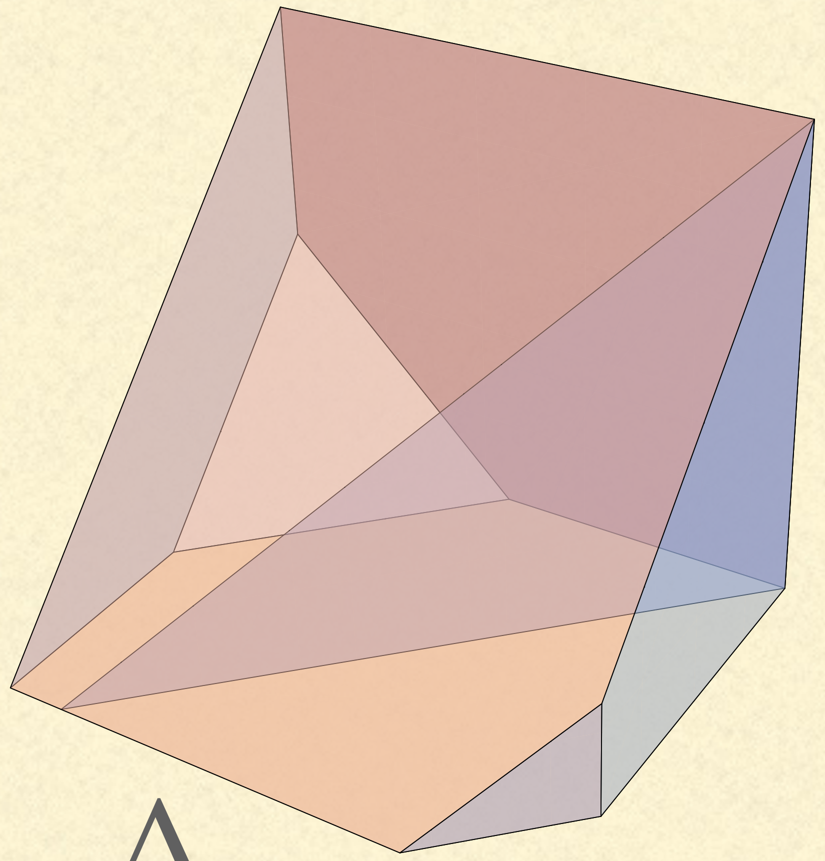
$\mathrm{MCG}(S) = \mathbb{Z}$

Cut

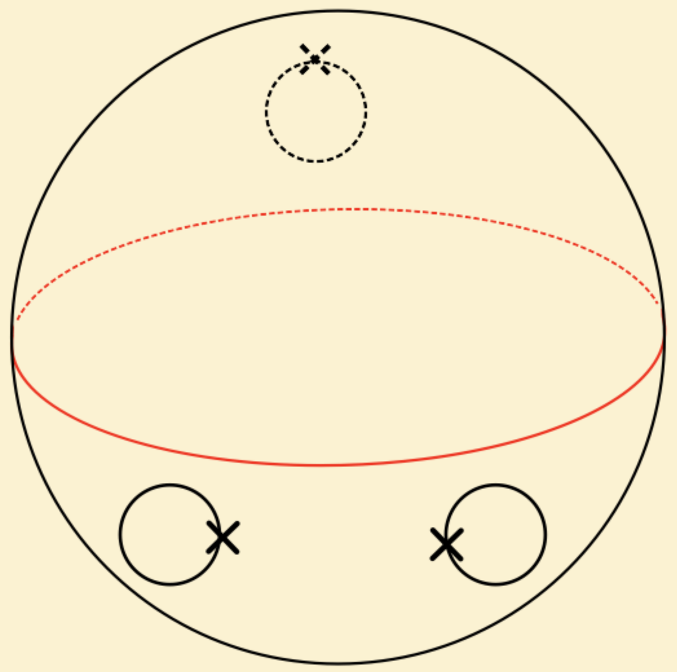


Double trace 1-loop integrand

$\mathrm{MCG}(S) = \mathbb{Z}$



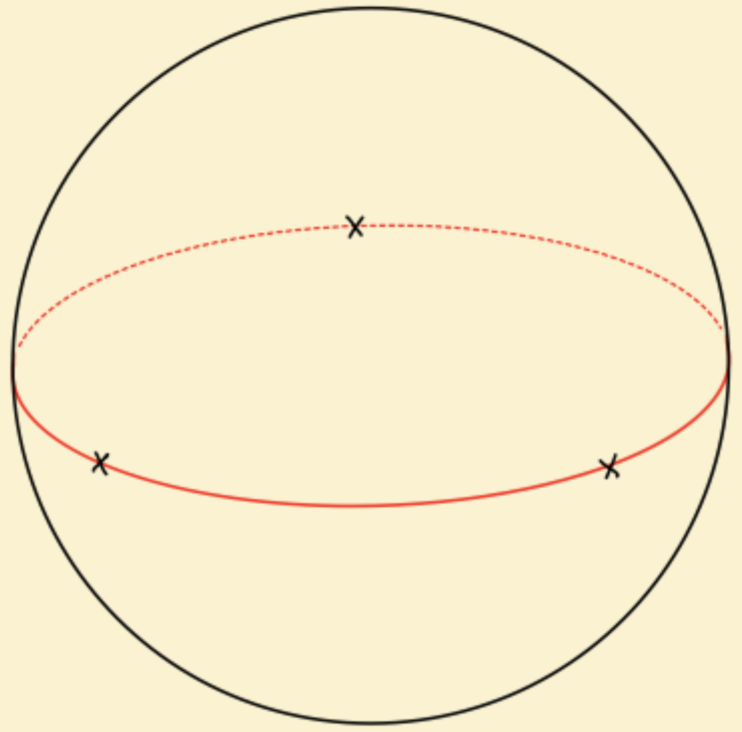
Δ



Triple trace 2-loop integrand

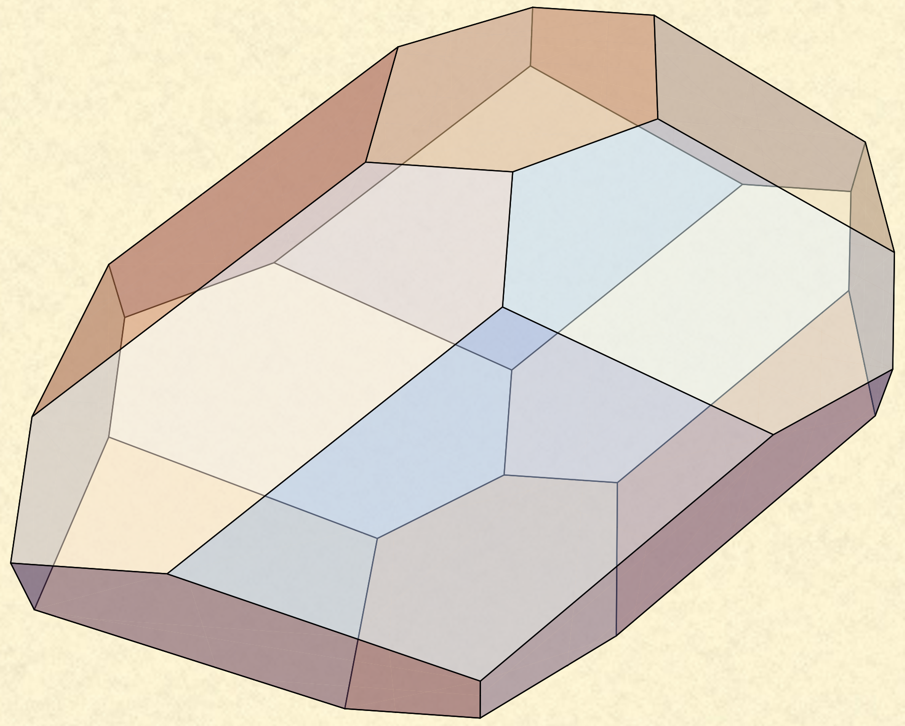
$\mathrm{MCG}(S) = \mathbb{Z}^3$

Pinch



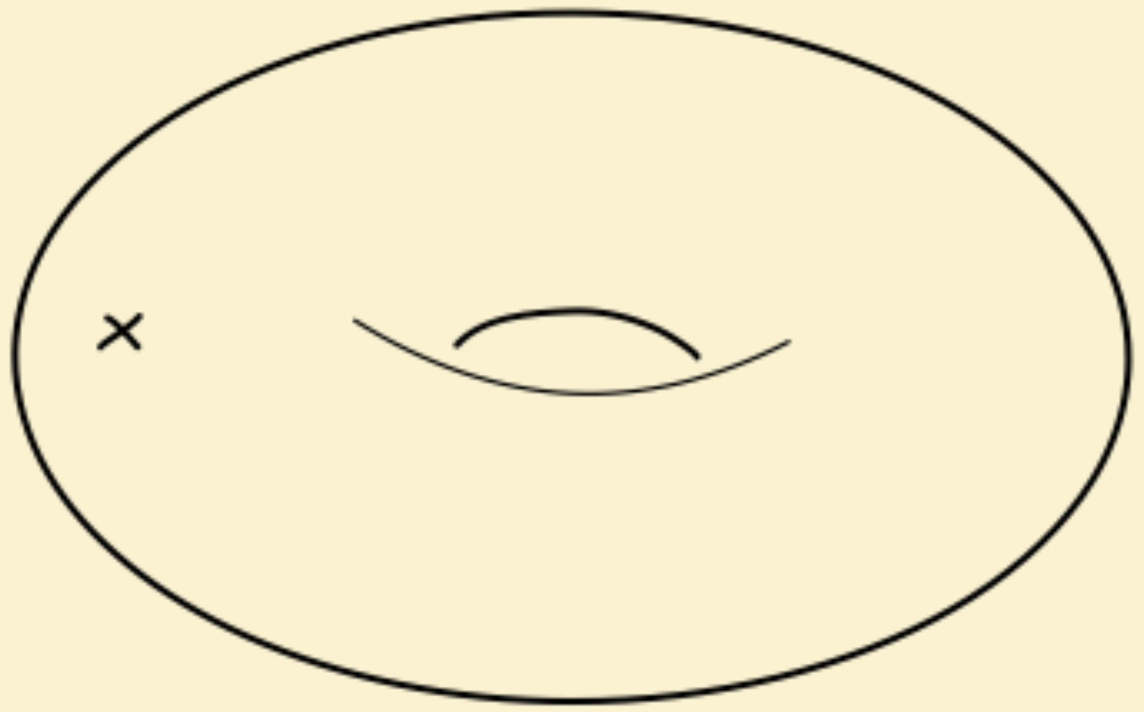
3-pts "gravity" amplitude

$\mathrm{MCG}(S) = S_3$

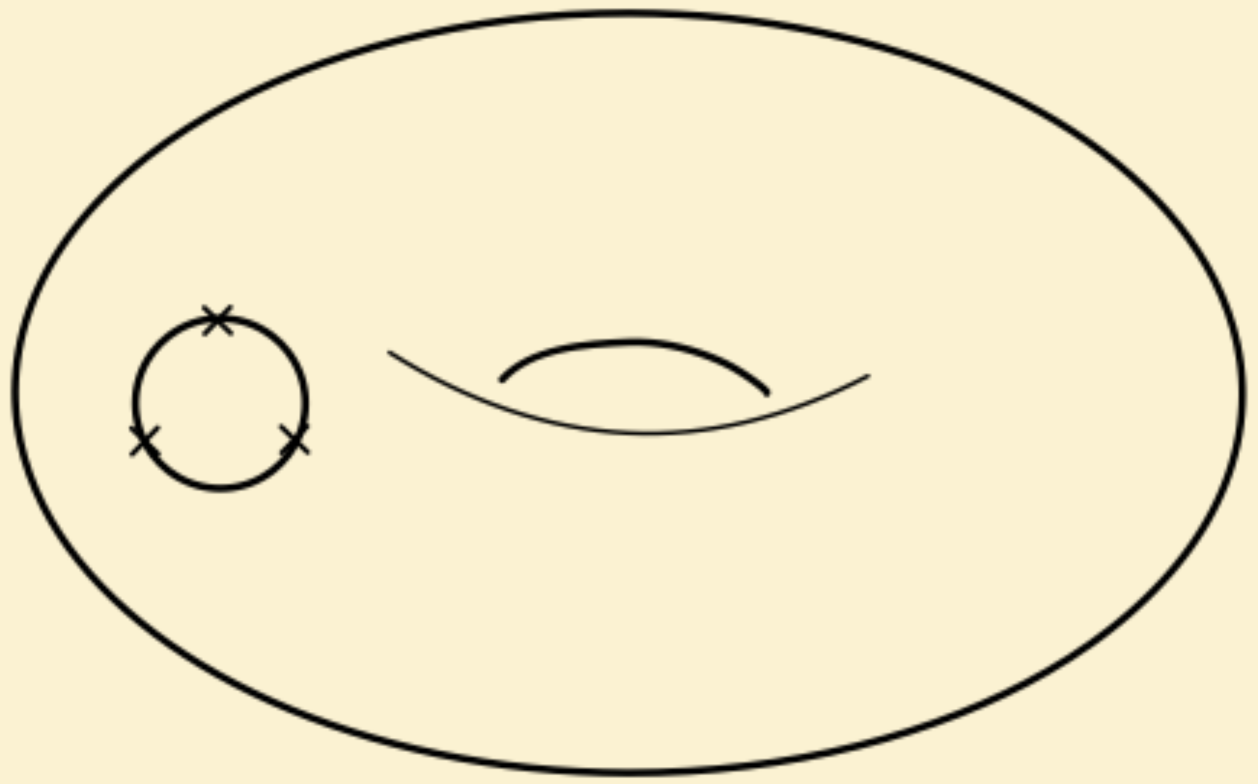


A plethora of checks: infinitely many Δ's

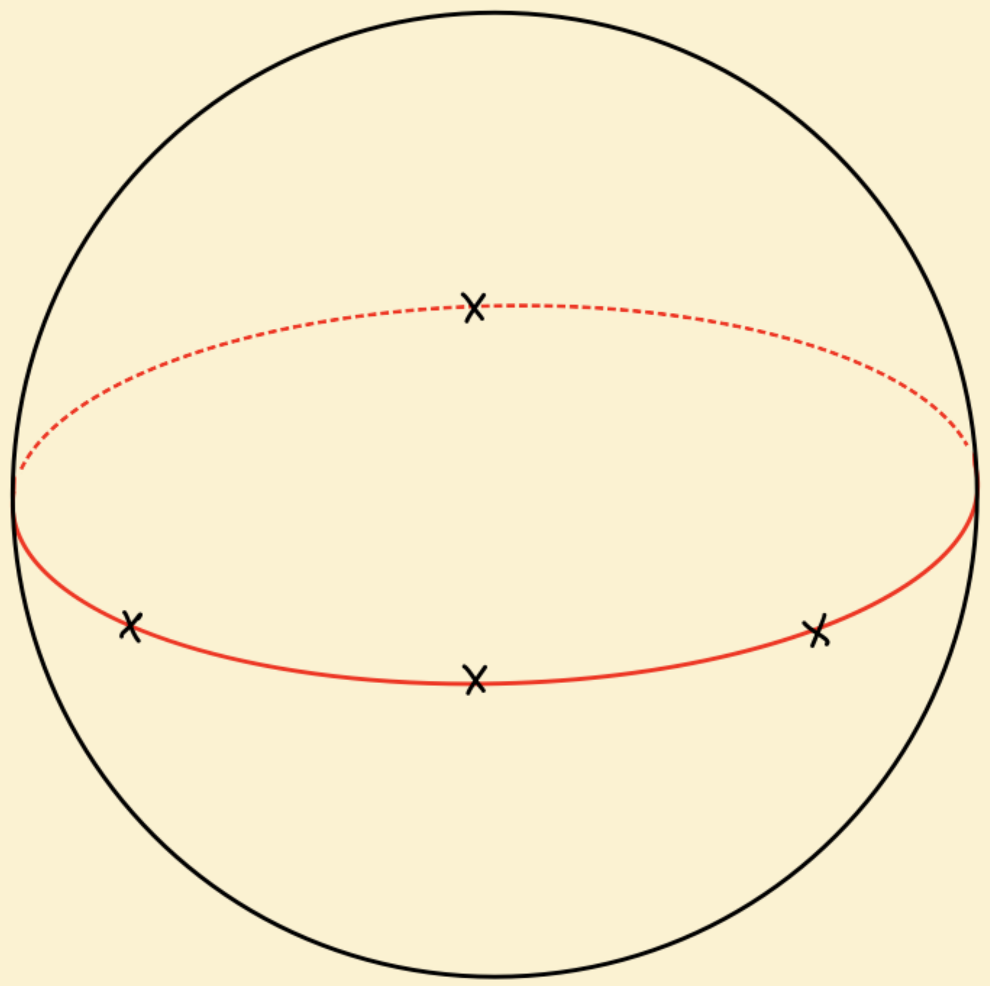
We have found complete fans for these non-Abelian MCG cases:



$\mathrm{MCG}(S) = \mathrm{SL}(2,\mathbb{Z})$



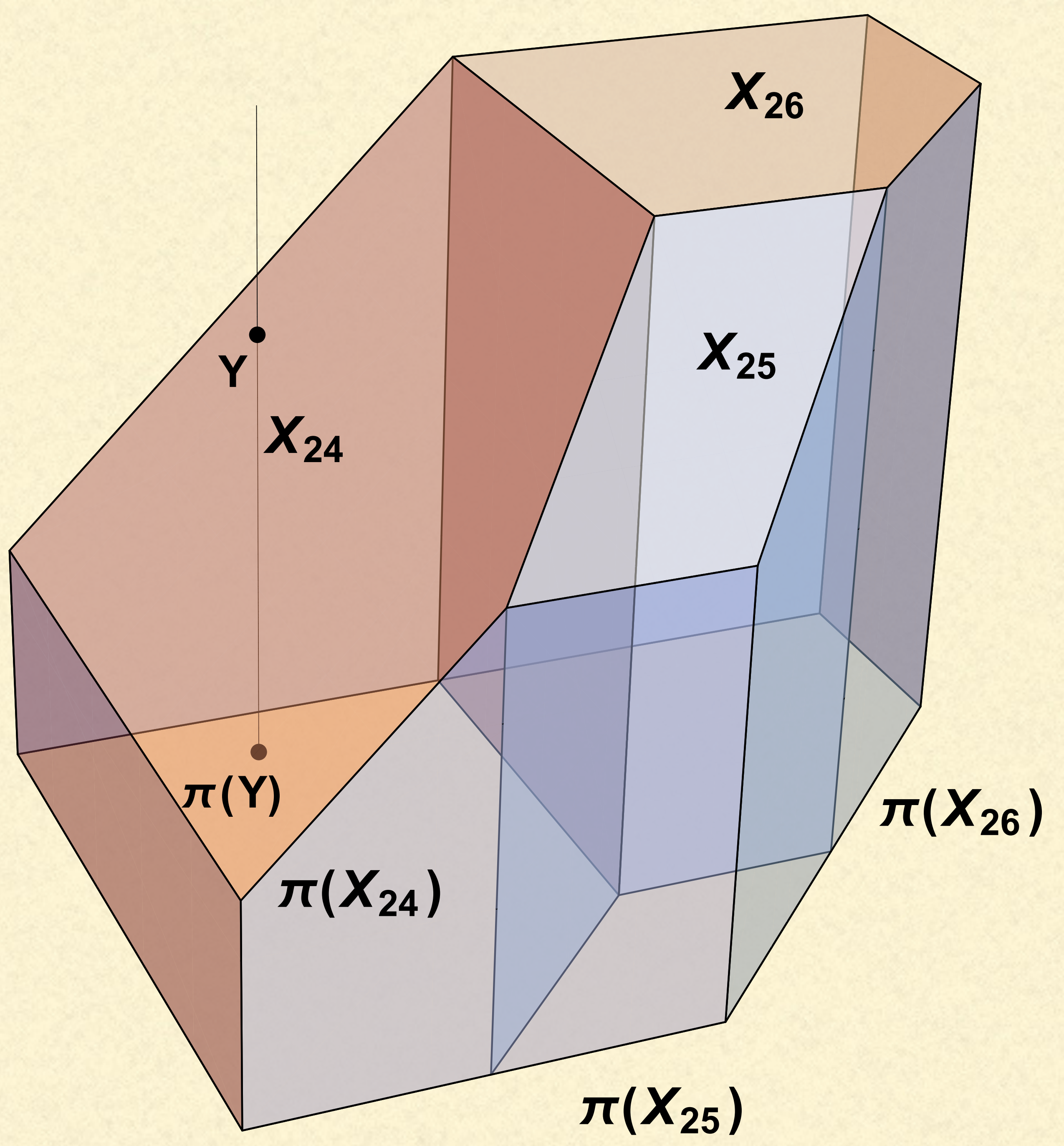
$\mathrm{MCG}(S) = \mathrm{SL}(2,\mathbb{Z}) \times \mathbb{Z}$

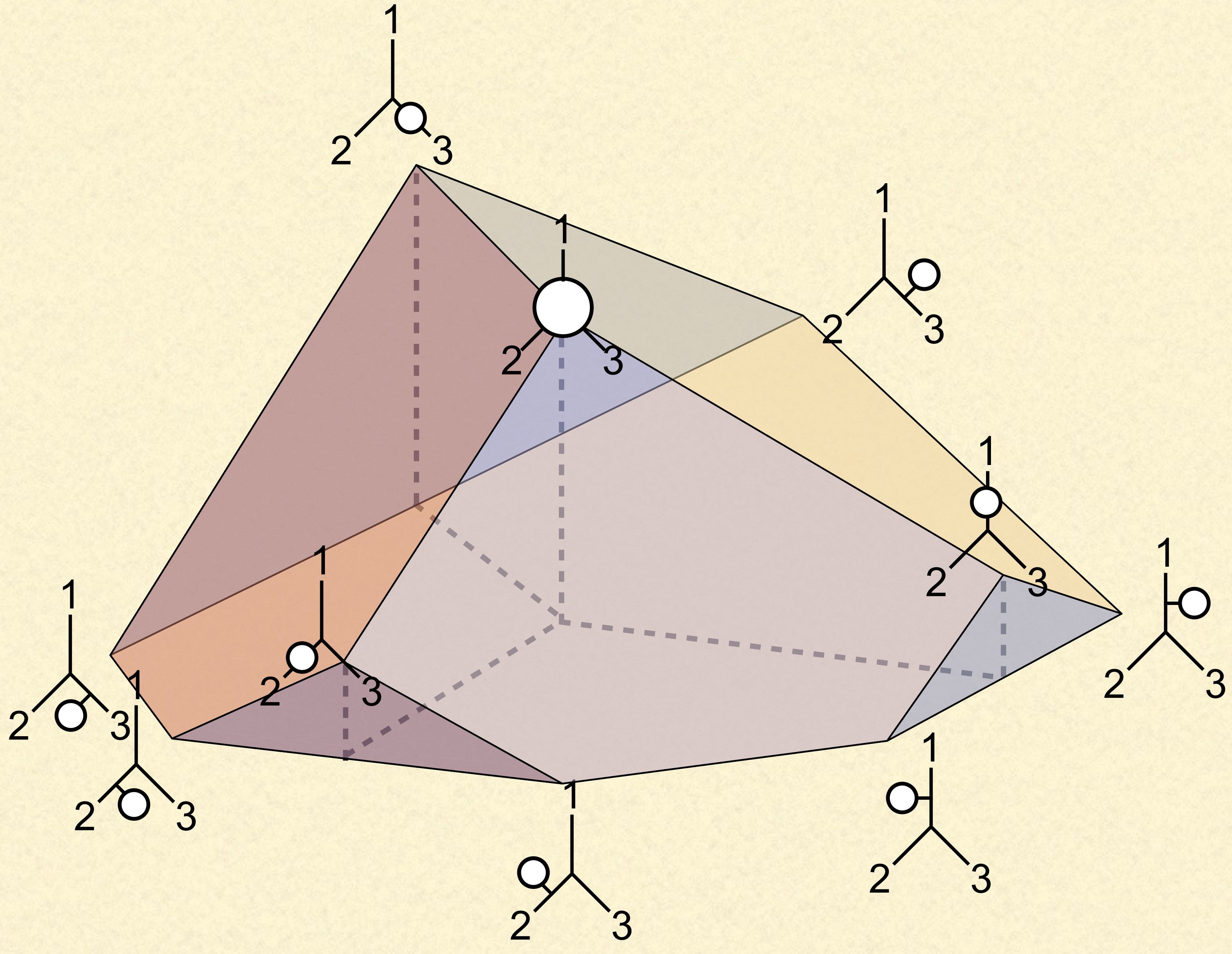


$\mathrm{MCG}(S) = \mathrm{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Recursive formulae from Projectivity

[GS, S. Stanojevic 1912.06125]
[Arkani-Hamed et al. 1703.04541]





$$\Omega_{0,0}^{(1,\ldots,n)} = \sum_{i=4}^{n} d\log\left(\frac{X_{2,i}}{X_{1,3}}\right) \wedge \hat{\Omega}_{0,0}^{(2,\ldots,i-1)} \wedge \hat{\Omega}_{0,0}^{(i,\ldots,1)}$$

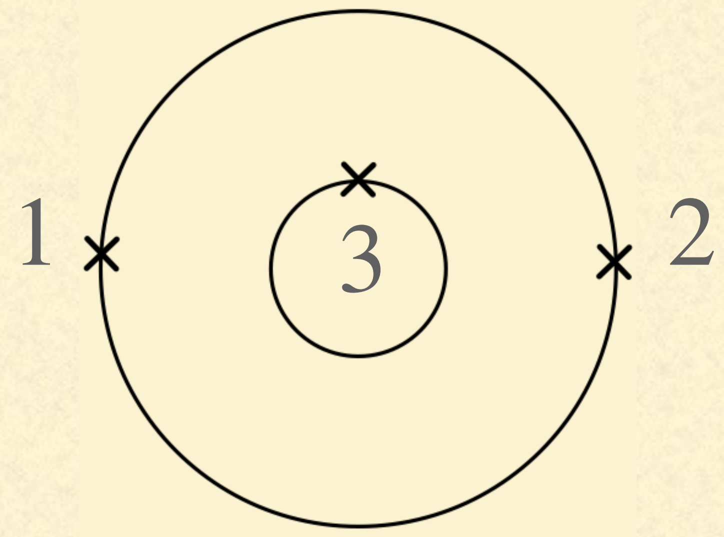
Trees from trees by *forgetting a particle*

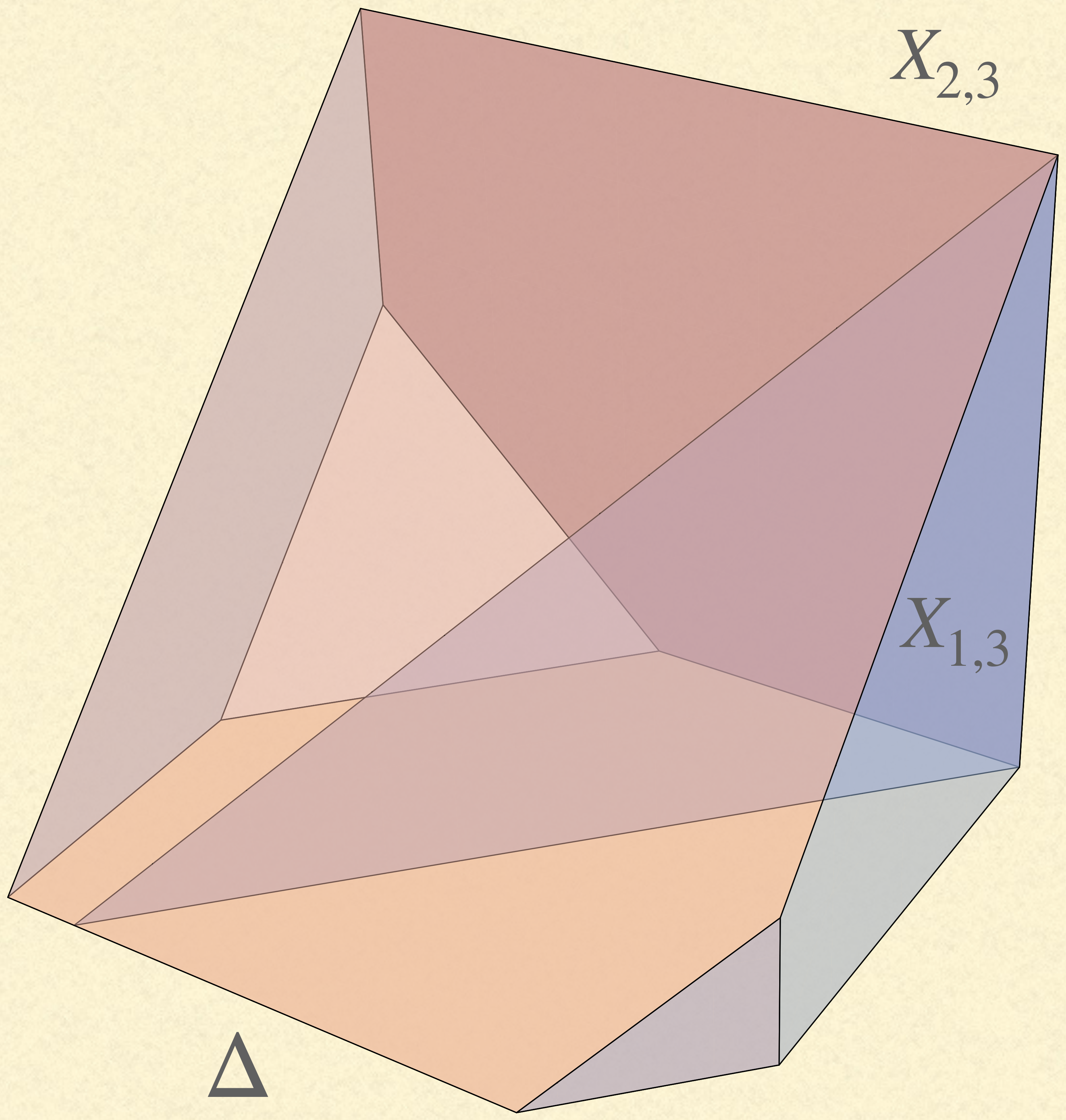
$$\Omega_{0,1}^{(1,\ldots,n)} = \sum_{i=1}^{n} d\log\left(\frac{X_i}{X_0}\right) \hat{\Omega}_{0,0}^{(\ldots,i,-,+,\ldots)}$$

1-loop from trees by *projecting on tadpoles*

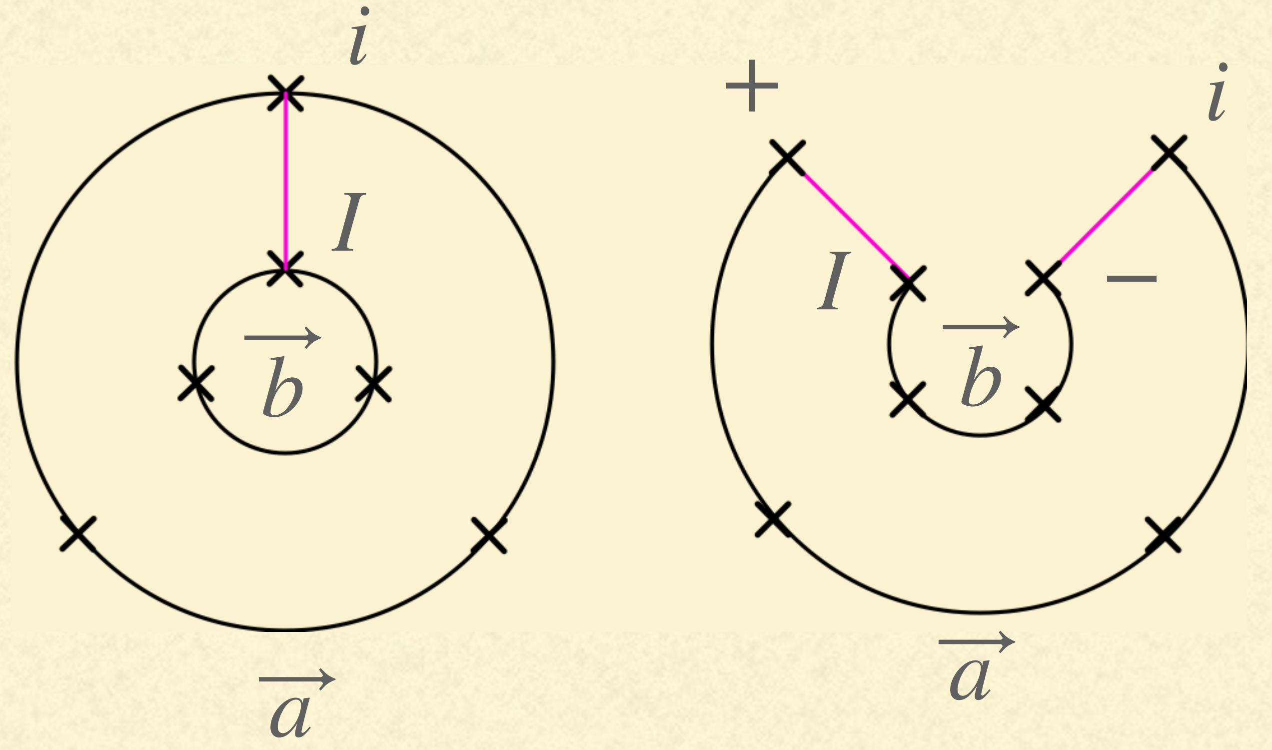
Recursive formulae from Projectivity

[GS, S. Stanojevic 1912.06125]
[Arkani-Hamed et al. 1703.04541]





$X_{i,I}$:



$\vec{a} \oplus_{i,I} \vec{b} = (a_1, \ldots, a_i, -, b_{I+1}, \ldots, b_I, +, a_{i+1}, \ldots)$

$$\Omega_{0,0}^{(\vec{a})(\vec{b})} = \sum_{i,I} d\log\left(\frac{X_{i,I}}{\Delta}\right) \hat{\Omega}_{0,0}^{(\vec{a}\oplus_{i,I}\vec{b})}$$

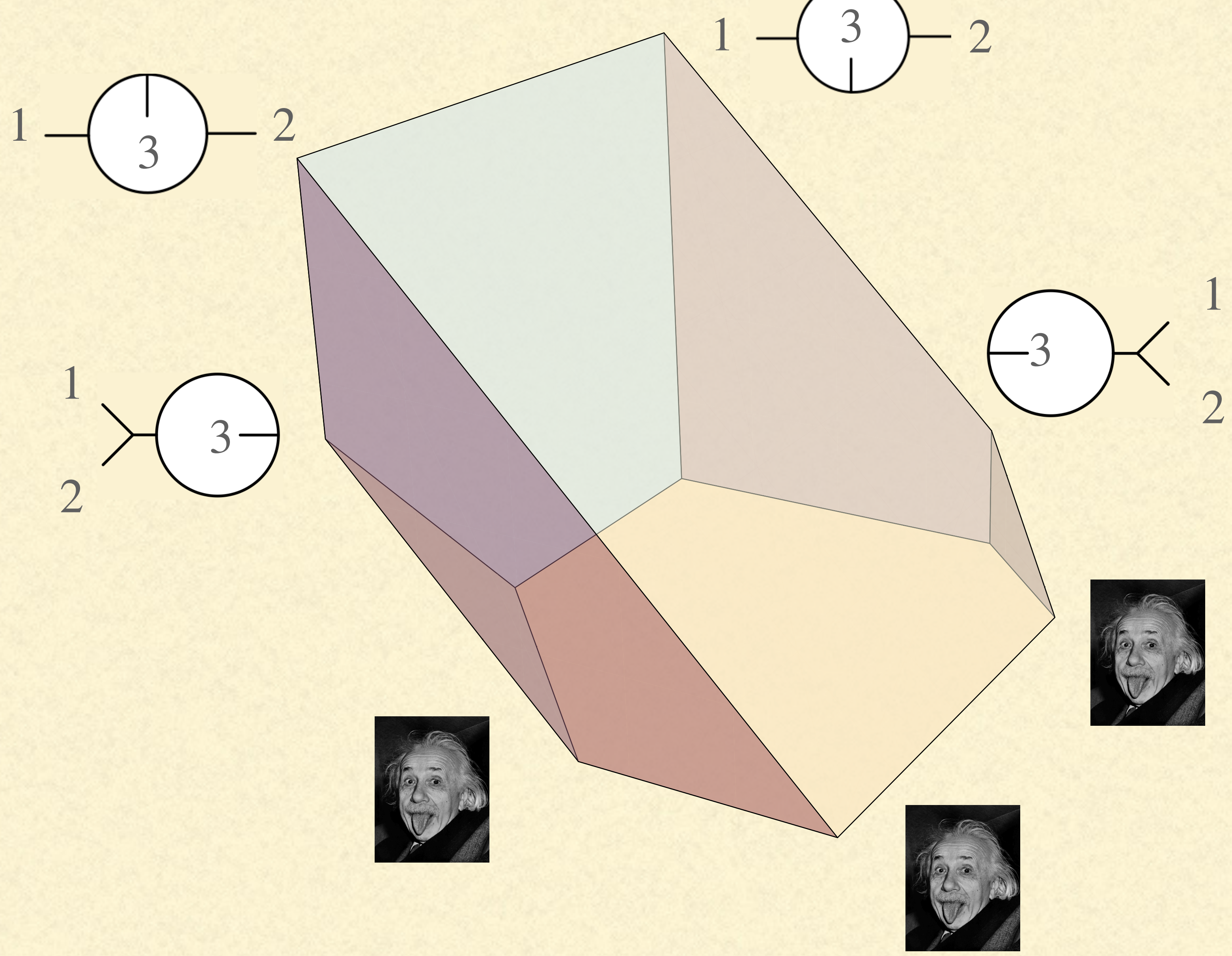
Double-trace 1-loop from trees by "*projecting*" on Δ

Outline

1. Integrands beyond the planar limit
2. Positive geometries...and “gravity”
3. **Conclusions**

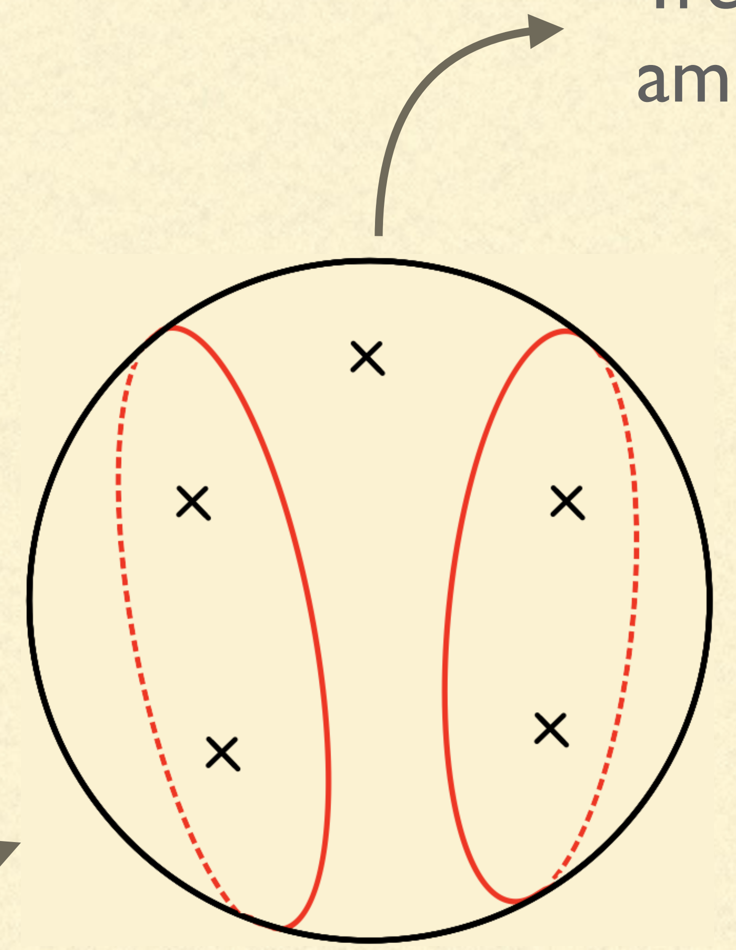
Summary

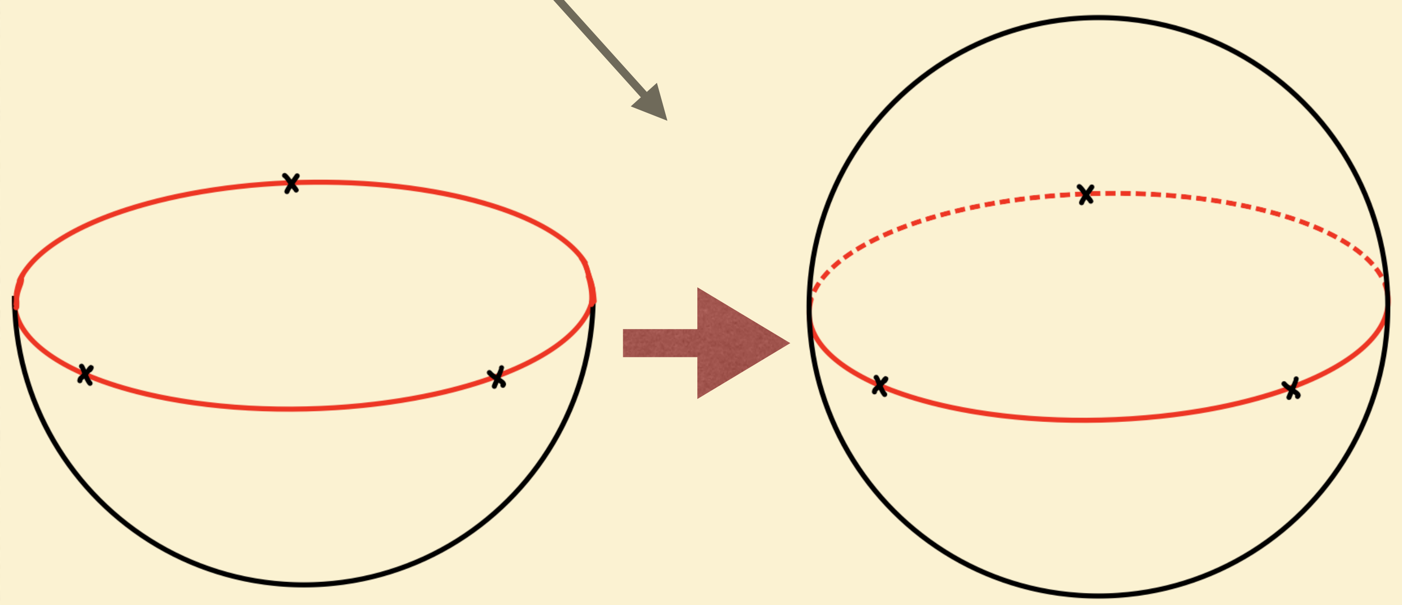
- Towards an all loop, all 1/N orders Amplituhedron for ϕ^3 theory (No SUSY, no Yangian, no planar limit)
- Meaningful definition of non-planar integrands
- Manifestly projectivity leads to new recursive formulae
- "QCD meets Gravity" : Non-planar positive geometries force on us "gravity"

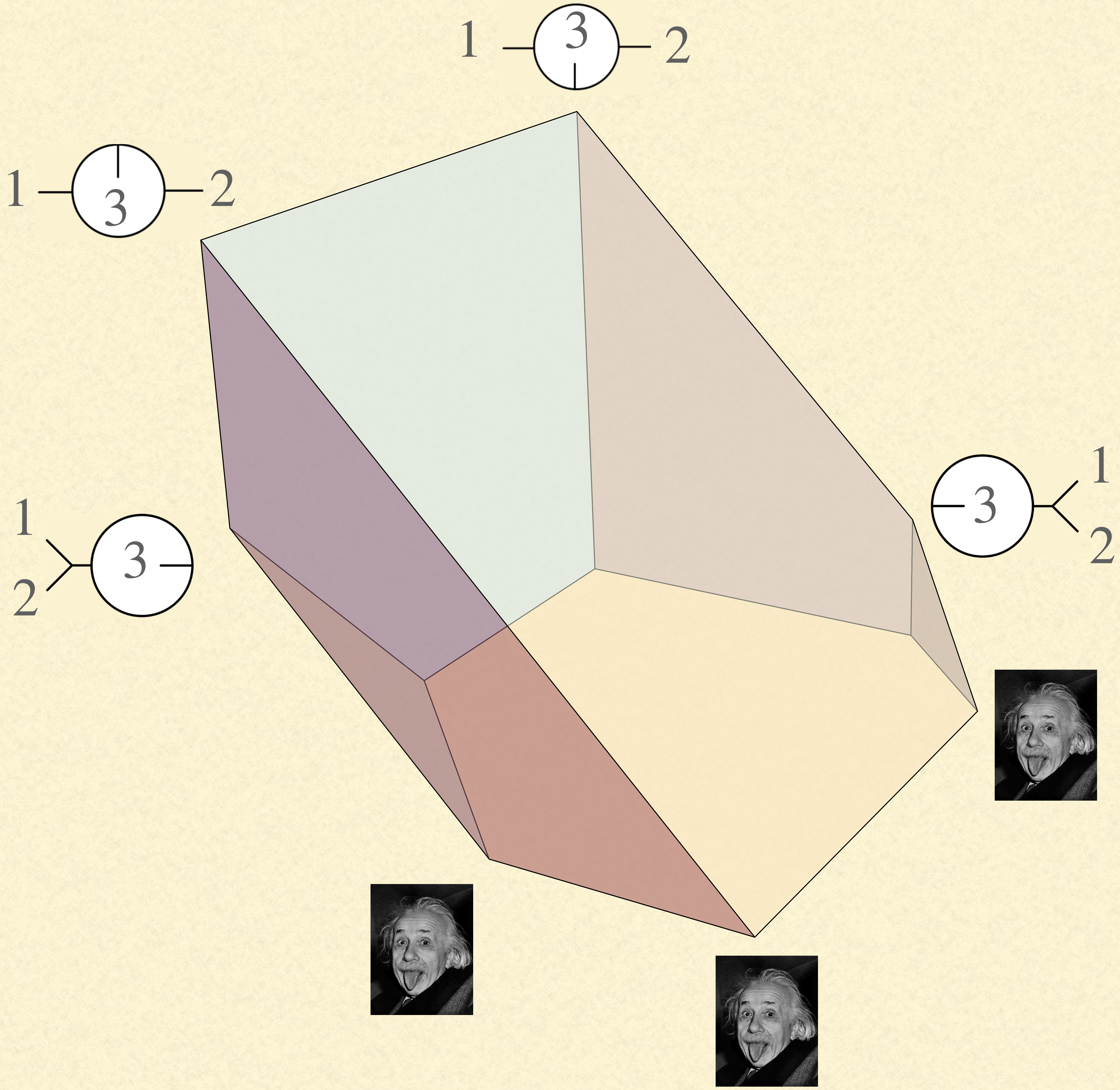


Future directions

- A complete fan for *every* surface
- Binary positive geometries/stringy canonical forms? [Arkani-Hamed, He, Lam]
- A pure gravitational geometry with factorizing properties in codimension 2
- Color kinematics duality/double copy? Doubling a bordered surface into a borderless surface
- Loop integration? (Dual polytope → Feynman Trick Polytope?)





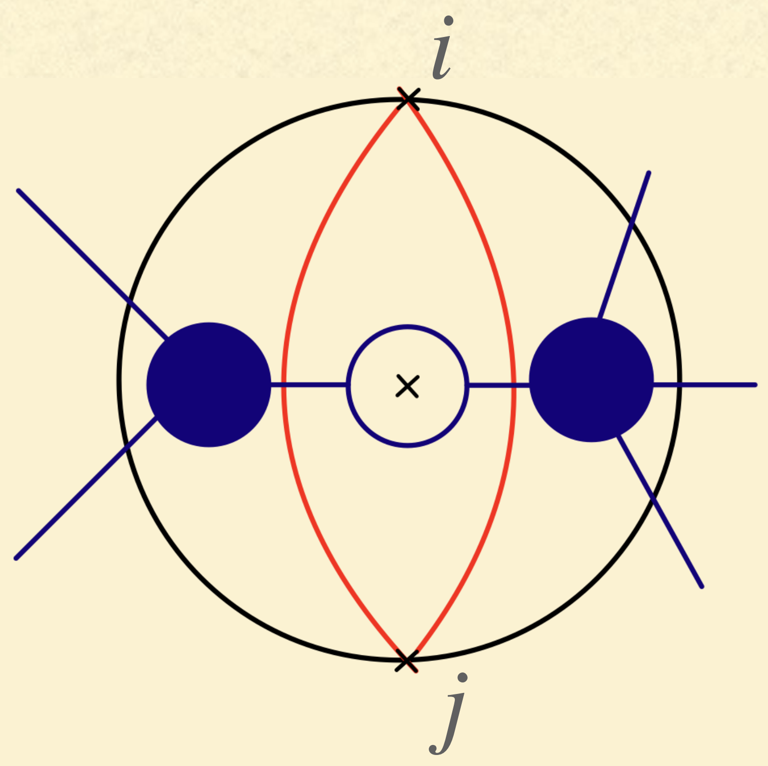


Thanks!

Extra slides

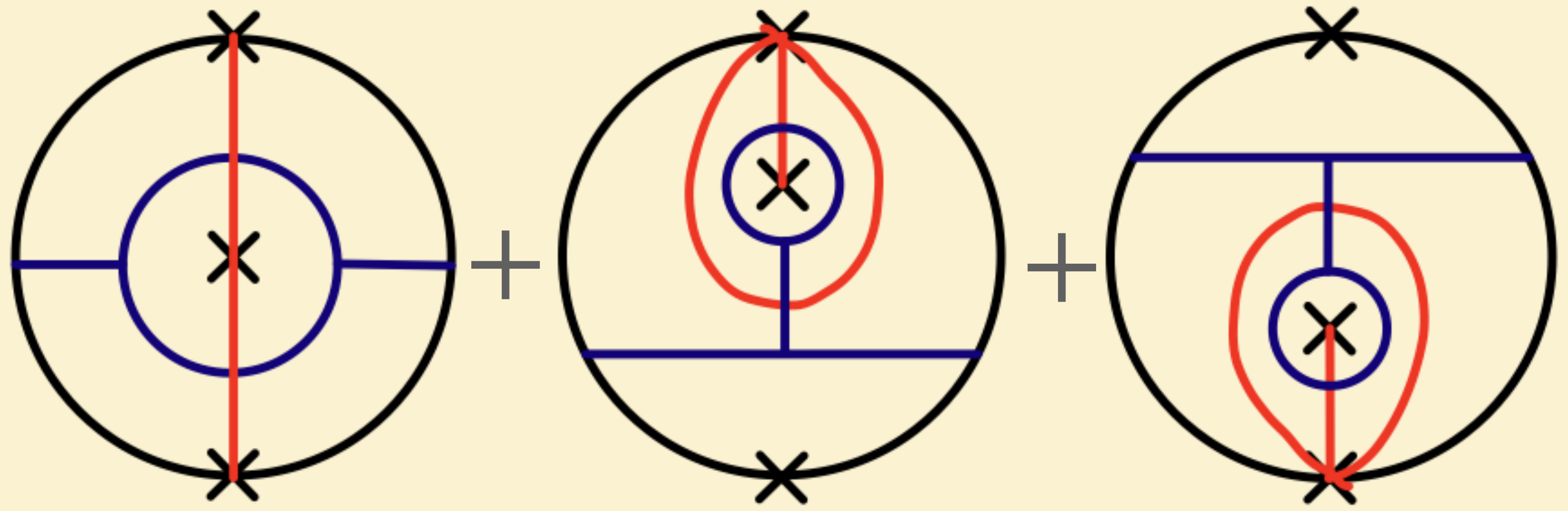
Double poles & Tadpoles

Double poles are lifted to pairs of simple poles in X variables



$$I_S(X) = \frac{1}{X_{ij}X_{ji}\ldots} + \ldots \to \frac{1}{(\ell^2 - m^2)^2\ldots} + \ldots$$

Tadpoles are killed by $X \to \infty$ limits: bi-adjoint from $\mathrm{Tr}(\phi^3)$



$$I_S(X) = \frac{1}{X_1X_2} + \frac{1}{X_1X_3} + \frac{1}{X_2X_4} \xrightarrow[X_{3,4}\to\infty]{} \frac{1}{X_1X_2} \to \frac{1}{\ell^2(\ell+p)^2}$$

$X_1 \to \ell^2 - m^2$

$X_2 \to (\ell + p)^2 - m^2$

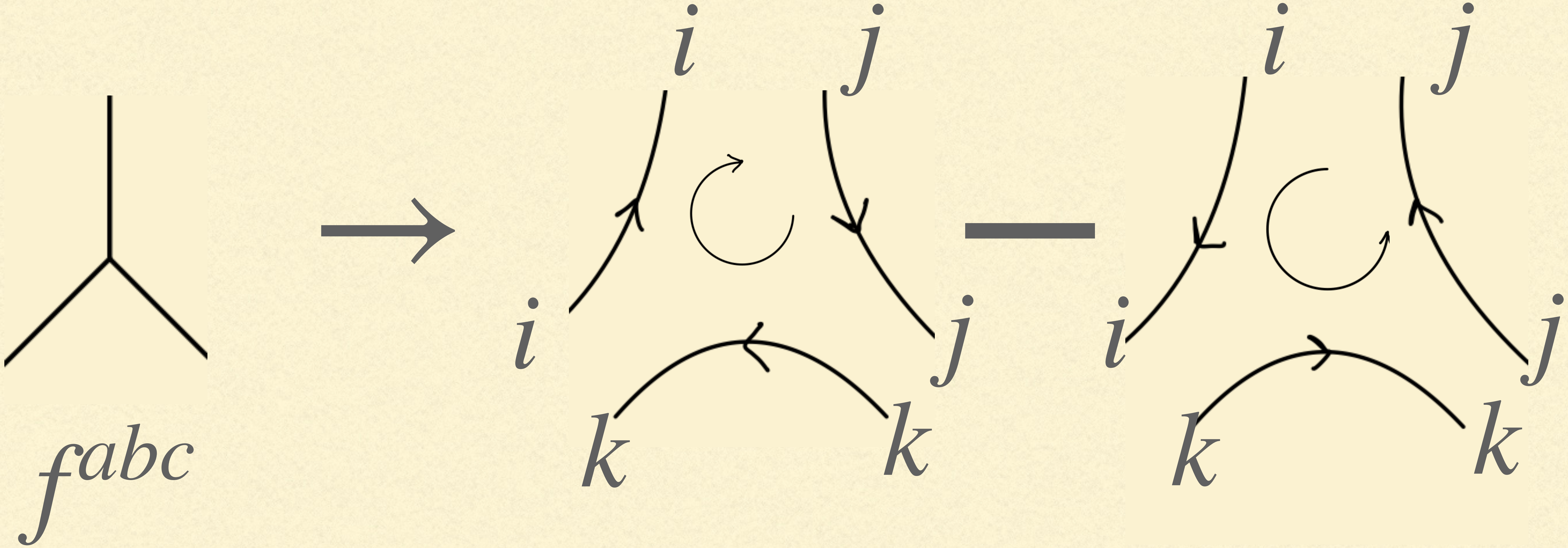
$X_{3,4} \to -m^2$

$$\frac{1}{\ell^2 - m^2}\frac{1}{(\ell+p)^2 - m^2} +$$
$$+\frac{1}{-m^2}\left(\frac{1}{\ell^2 - m^2} + \frac{1}{(\ell+p)^2 - m^2}\right)$$

ϕ^3 bi-adjoint

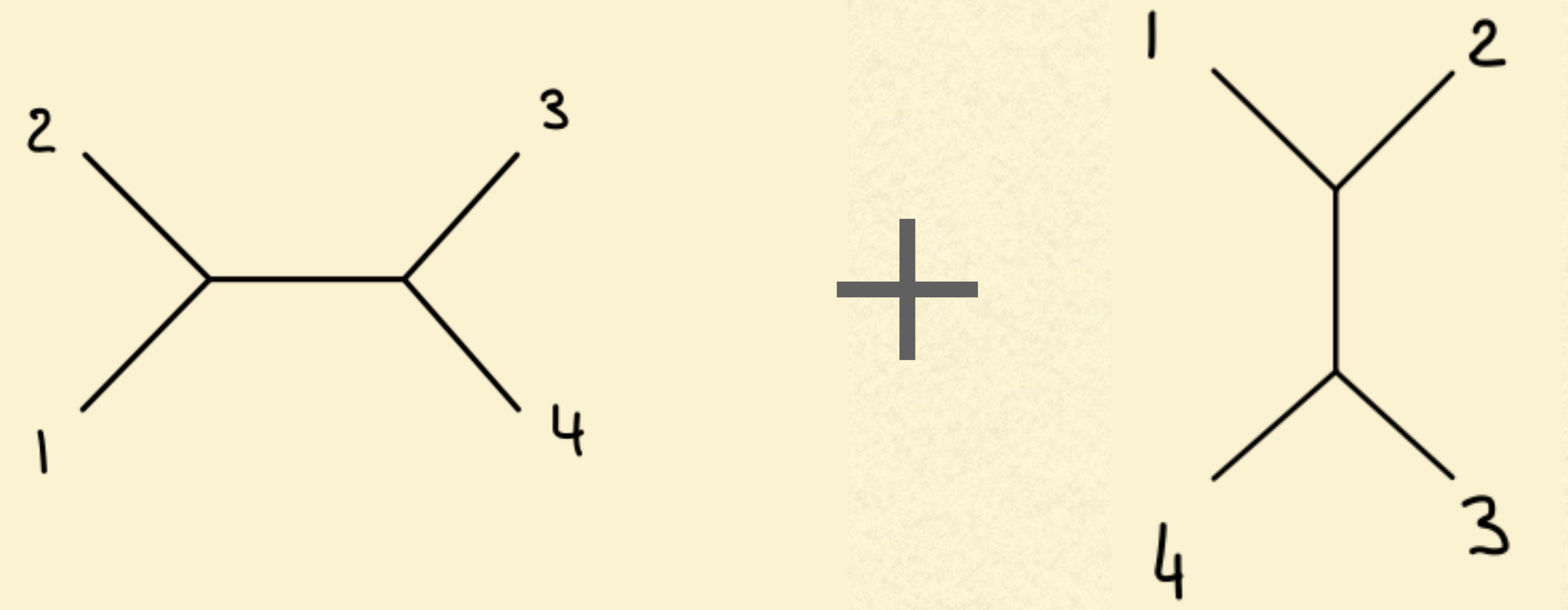
$$\mathscr{L} = \mathscr{L}_{free} + g\, f^{abc} f^{\overline{abc}} \phi_{a\bar{a}} \phi_{b\bar{b}} \phi_{c\bar{c}}$$

We take as gauge group U(N) and consider the large N expansion

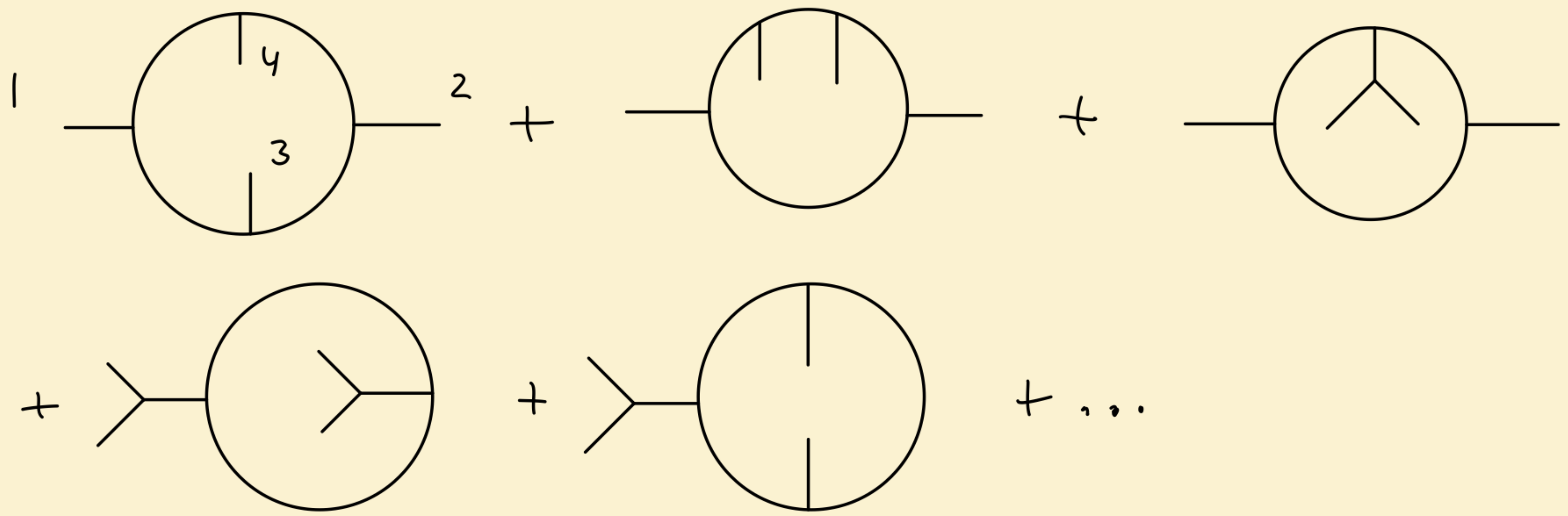


Some examples

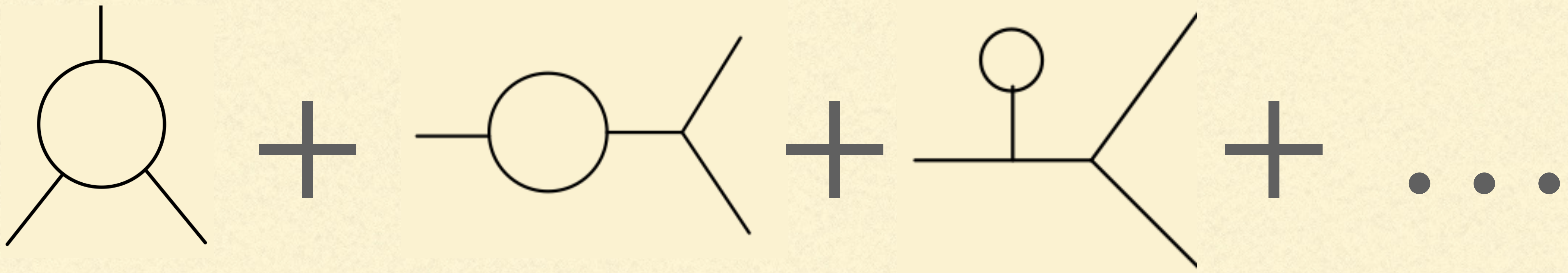
$$A^{(1234)}_{0,0} =$$



$$A^{(12)(34)}_{0,0} =$$

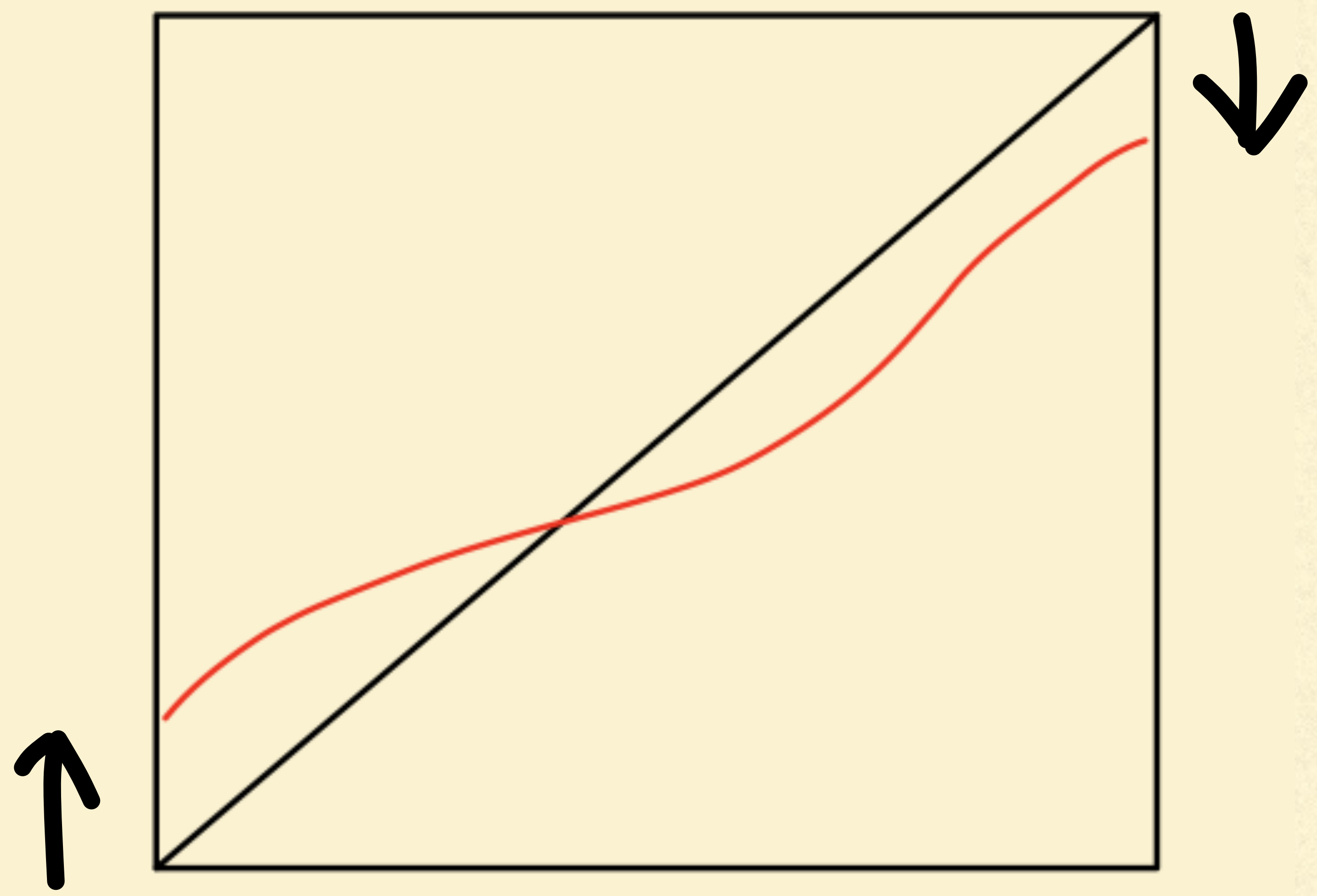


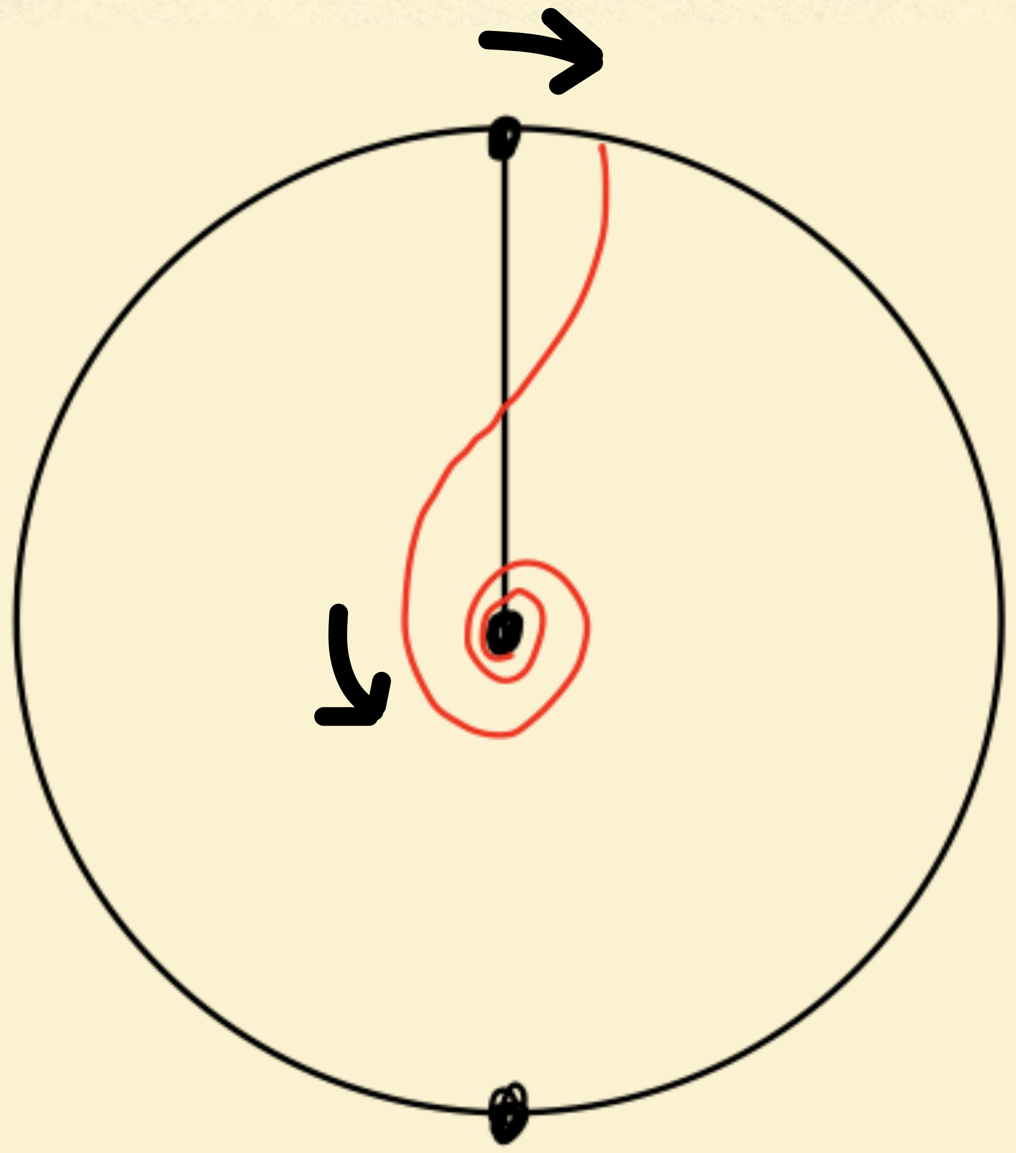
$$A^{(123)}_{0,1} =$$



Tropical Teichmüller via Laminations

$$\mathrm{Trop}(T) = \{X_a + X_b = \max(X_c + X_d, X_e + X_f)\}$$

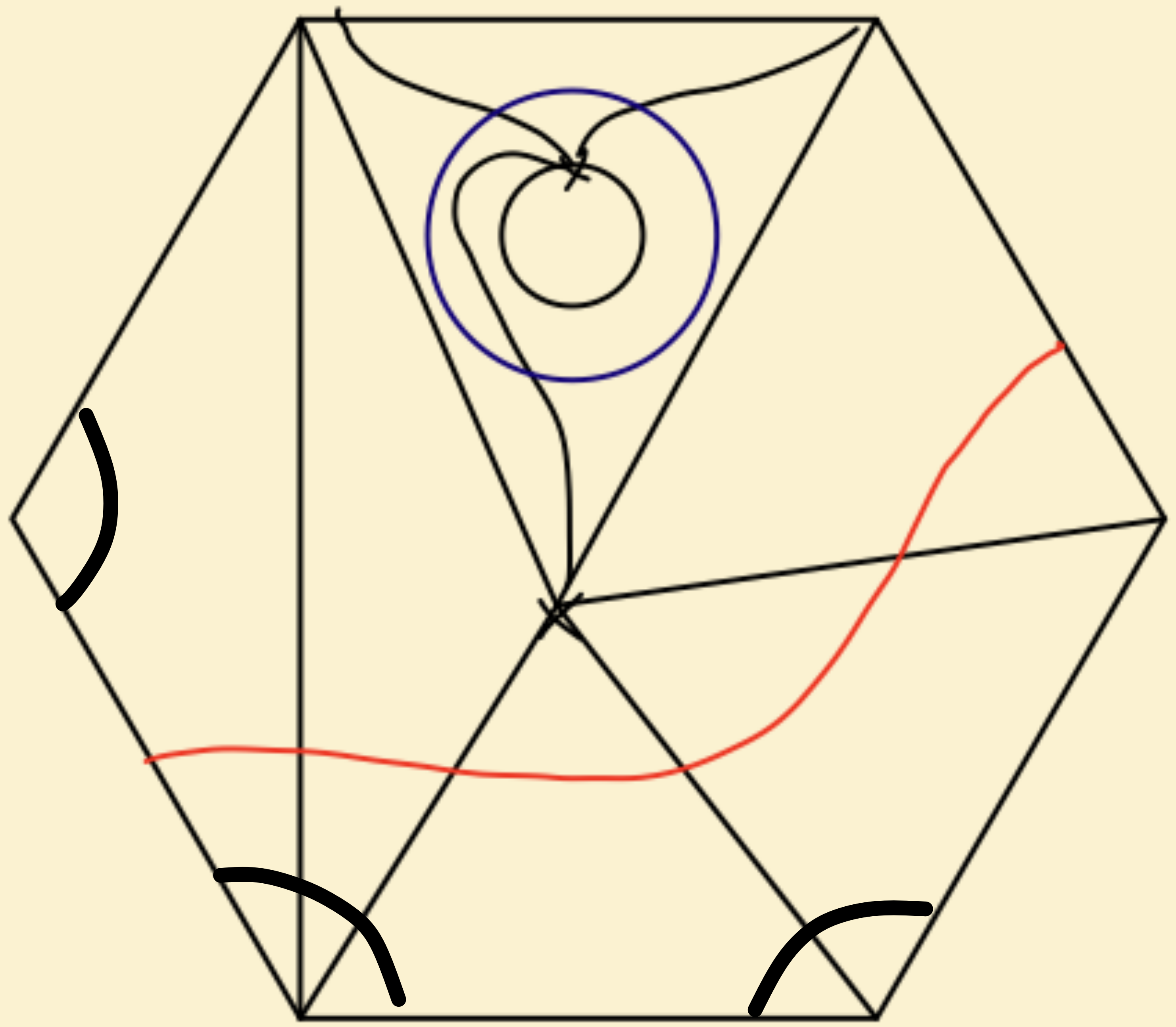




Laminations are in 1:1 correspondence with arcs of a triangulation

Tropical Teichmüller via Laminations

$$\mathrm{Trop}(T) = \{X_a + X_b = \max(X_c + X_d, X_e + X_f)\}$$



The intersection number of a lamination with respect to a triangulation solves the Tropical mutation relation

Projecting through the *boundary laminations* produces an interesting fan