Classical Double Copies:

The view from twistor space

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SAGEX Scattering Amplitudes: from Geometry to Experiment

Introduction

- The classical double copy relates solutions in gauge and gravity theories.
- Closely related to the original double copy for scattering amplitudes (Bern, Carrasco, Johansson).
- In most approaches, one can relate classical solutions only order-by-order in perturbation theory.
- Two particular approaches, however, allow *exact* solutions to be related.

The Kerr-Schild double copy

- Defining the graviton field by

Minkowski metric

graviton

Full metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

there is a special class of *Kerr-Schild* gravitons

$$h_{\mu\nu} = \phi k_\mu k_\nu, \quad k^2 = 0, \quad (k \cdot \partial) k^\mu = 0.$$

scalar field

vector field

- Then can construct gauge and biadjoint fields (Monteiro, O'Connell, White):

constant colour vectors

$$A_\mu^a = c^a \phi k_\mu, \quad \Phi^{aa'} = c^a \tilde{c}^{a'} \phi.$$

- All of these turn out to linearise their respective field equations, so are exact solutions!

The spinorial formalism

- An alternative exact procedure is the *Weyl double copy* (Luna, Monteiro, Nicholson, O'Connell).
- It relies on a formalism in which field equations are written in terms of 2-component spinors π_A and their complex conjugates $\pi_{A'}$.
- Convert tensors $T_{\mu\nu\ldots\rho}$ using *Infeld-van-der-Waerden symbols* $\sigma^{\mu}_{AA'}$:

$$\sigma^0_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2_{AA'} \quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Raise / lower indices using Levi-Civita symbols $\epsilon_{AB},\ \epsilon_{A'B'}$.

- It turns out that all multi-index spinors can be reduced to sums of products of Levi-Civita symbols and fully symmetric spinors.

- The electromagnetic field strength tensor translates as:

$$F_{\alpha\beta} \to F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{AB}\epsilon_{A'B'}$$

anti-self-dual part self-dual part

- In gravity, we will be concerned with vacuum spacetimes, such that the Riemann tensor reduces to the *Weyl tensor*, with spinorial translation

“Weyl spinor”

$$C_{\alpha\beta\gamma\delta} \to \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$$

- The vacuum Maxwell and Einstein equations then reduce to the spin-1 and spin-2 cases of the *massless free field equation*

$$\nabla^{AA'}\phi_{AB\ldots C} = 0, \quad \nabla^{AA'}\bar{\phi}_{A'B'\ldots C'} = 0$$

spinorial covariant derivative 2*n* indices for a spin-*n* field

Principal spinors

- All fully symmetric spinors decompose into symmetrised products of 1-index *principal spinors*.

- Thus, the EM and Weyl spinors may be written as e.g.

brackets denote symmetrisation

$$\phi_{AB} = \alpha_{(A}\beta_{B)}, \quad \Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

- EM spinors are *(non-)null*, according to whether the principal spinors are proportional or not.

Degeneracy	Petrov type
{1,1,1,1}	I
{2,1,1}	II
{3,1}	III
{2,2}	D
{4}	N

- Similarly, we may classify Weyl spinors according to the pattern of degeneracy of their principal spinors (the *Petrov classification*).

The Weyl double copy

- The Weyl double copy can now be stated as follows.
- Given two electromagnetic spinors, one may generate a Weyl spinor according to the rule

$$\Psi_{ABCD} = \frac{1}{S}\phi_{(AB}\tilde{\phi}_{CD)}$$

- Here S is a scalar satisfying the linearised biadjoint field equation.
- Not necessarily known how to fix S *a priori*, but can be found in specific examples.
- Argued to hold for general vacuum type D spacetimes (Luna, Monteiro, Nicholson, O'Connell)…
- …and to be "equivalent" to the Kerr-Schild double copy, where they overlap.

Open questions

- Why is it possible to formulate a classical double copy in **position space**, when the BCJ procedure involves products in **momentum space**?
- How can one fix the scalar function S in the Weyl double copy?
- Given a scalar or biadjoint field, how does one perform the inverse zeroth copy to get a gauge field?
- Can we formulate exact classical double copies in curved backgrounds? (see e.g. Bahjat-Abbas, Luna, White; Carrillo-González, Penco, Trodden)
- Or for less algebraically special cases?

We can answer all these questions by going to twistor space!

A Twistor Primer

- Twistor space $\mathbb{T}$ is the set of solutions of the *twistor equation*

$$\nabla_{A'}^{(A}\Omega^{B)} = 0 \quad \Rightarrow \quad \Omega^A = \omega^A - ix^{AA'}\pi_{A'}$$

which we can associate with 4-component objects (“twistors”)

$$Z^\alpha = \left(\omega^A, \pi_{A'}\right) .$$

- The “location” of a twistor in spacetime is defined by the vanishing of Ω^A, leading to the *incidence relation*

$$\omega^A = ix^{AA'}\pi_{A'}$$

- This is invariant under rescalings, so that twistors obeying the incidence relation are points in *projective twistor space* $\mathbb{PT}$*.*

Fixed point *x* in Minkowski space ⟷ Riemann sphere in $\mathbb{PT}$

- Can also define *dual twistors*, and (conformally invariant) product

$$Z^\alpha W_\alpha .$$

The Penrose transform

- Solutions of the massless free field equation are related to holomorphic functions in $\mathbb{PT}$:

$$\phi_{A'B'\ldots C'}(x) = \frac{1}{2\pi i}\oint_{\Gamma} \pi_{E'}d\pi^{E'}\pi_{A'}\pi_{B'}\ldots\pi_{C'}[\rho_x f(Z^\alpha)]$$

contour on the celestial sphere of *x*

Restrict to twistors obeying the incidence relation

- The function *f* must have homogeneity (-*n*-2), if there are *n* indices on the LHS.

- For scalar fields (*n* = 0), the spacetime field instead obeys the conformally invariant free field equation

$$\left(\Box + \frac{R}{6}\right)\phi = 0$$

- We can use these facts to obtain the Weyl double copy…

Weyl double copy from twistor space

- Consider the following product of functions in twistor space:

$$\text{homogeneity -6} \longrightarrow f_{\mathrm{grav.}} = \frac{f^{(1)}_{\mathrm{EM}} f^{(2)}_{\mathrm{EM}} \longleftarrow \text{homogeneity -4}}{f_{\mathrm{scal.}} \longleftarrow \text{homogeneity -2}}$$

- For certain choices, the corresponding spacetime fields from the Penrose transform turn out to obey

$$\phi_{A'B'C'D'} = \frac{\phi^{(1)}_{(A'B'}\phi^{(2)}_{C'D')}}{\phi},$$

which is precisely the Weyl double copy!

Type D solutions

- A suitable set of functions is:

here have set $\pi_{A'} = (1, \xi)$

$$f_m = \frac{\left[Q_{\alpha\beta} Z^\alpha Z^\beta\right]^{-m}}{m!} \equiv \frac{1}{m!}\left[\frac{N^{-1}(x)}{(\xi - \xi_1(x))(\xi - \xi_2(x))}\right]^m$$

m=(1,2,3) for (scalar, EM, grav)

constant

- Carrying out the Penrose transform yields spacetime fields

$$\phi = \frac{N(x)}{\xi_1 - \xi_2}, \quad \phi_{A'B'} = -\frac{N^2(x)}{(\xi_1 - \xi_2)^3}\alpha_{(A'}\beta_{B')}, \quad \phi_{A'B'C'D'} = \frac{N^3(x)}{(\xi_1 - \xi_2)^5}\alpha_{(A'}\beta_{B'}\alpha_{C'}\beta_{D')}$$

which clearly obey the type D Weyl double copy, with

$$\alpha = (1, \xi_1), \quad \beta = (1, \xi_2)$$

Example: Schwarzschild

- A concrete example is given by the (self-dual) Schwarzschild (Taub-NUT) solution, for which one has:

$$Q_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \xi_{1,2} = \frac{-z \pm r}{(x+iy)} = \frac{x-iy}{z \pm r}, \quad N(x) = \frac{i\sqrt{2}}{(x+iy)}$$

- The scalar function is found to be

$$\phi = \frac{i}{r\sqrt{2}},$$

agreeing with that found by Luna, Monteiro, Nicholson & O'Connell.

- Furthermore, the principal spinors give rise to the known possible choices of the null vector k_μ in the Kerr-Schild double copy.
- Different choices of $Q_{\alpha\beta}$ give all type D vacuum spacetimes (Haslehurst, Penrose).

What is this good for?

- The twistor space picture explains **where** the Weyl double copy comes from, and **why** it exists in position space.
- The scalar function that appears is no longer mysterious, but **fixed**.
- Furthermore, the scalar function already "knows" about the null directions appearing in the gauge and gravity solutions (i.e. through the locations of the poles in twistor space).
- Thus, the inverse zeroth copy makes sense for the first time!
- Conformal invariance of the twistor picture means the Weyl double copy should immediately extend to conformally flat spacetimes, generalising previous results (Bahjat-Abbas, Luna, White; Carrillo-González, Penco, Trodden).

General Petrov types

- Consider the Penrose transform pair (Bouwmeester, de Klerk, Dalhuisen, Swearngin, Thompson, van der Veen, Wickes)

$$\frac{1}{(A_\alpha Z^\alpha)^{1+a}(B_\alpha Z^\alpha)^{1+b}} \to \left(\frac{2}{\Omega|x-y|^2}\right)^{a+b+1} \mathcal{A}_{(A'_1}\dots\mathcal{A}_{A'_b}\mathcal{B}_{A'_{b+1}}\dots\mathcal{B}_{A'_{2h})},$$

where $\{\Omega,\ y,\ \mathcal{A}_A,\ \mathcal{B}_A\}$ are constants depending on *A* and *B.*

- The choice (*a*,*b*)=(0,0), (1,1) and (0,2) lead to the scalar and EM spinors:

$$\phi = \frac{2}{\Omega|x-y|^2}, \quad \phi^{(1,1)}_{A'B'} = \left(\frac{2}{\Omega|x-y|^2}\right)^3 \mathcal{A}_{A'}\mathcal{B}_{B'}, \quad \phi^{(0,2)}_{A'B'} = \left(\frac{2}{\Omega|x-y|^2}\right)^3 \mathcal{A}_{A'}\mathcal{A}_{B'}.$$

- Can multiply the twistor functions to make different gravity solutions, one of which is type III !

$$\boxed{\phi^{(1,1)\times(0,2)}_{A'B'C'D'} = \left(\frac{2}{\Omega|x-y|^2}\right)^5 \mathcal{A}_{(A'}\mathcal{A}_{B'}\mathcal{A}_{C'}\mathcal{B}_{D')}}$$

Conclusions

- The Weyl double copy is an exact relationship between classical solutions in biadjoint, gauge and gravity theories.
- Twistor theory resolves a number of open puzzles:
    1. Where the Weyl double copy comes from, and why it works in position space.
    2. How to fix the scalar function that appears.
    3. How the gauge field inherits information from the scalar theory (the "inverse zeroth copy").
    4. Why the double copy should work in general conformally flat spacetimes.
    5. Why it should also work for less algebraically special cases (i.e. not just Petrov type D).
- There is clearly much more that can be done.
- Discussion channel: #weyl_from_twistor_space