Exceptional field theories and the double copy based on 2010.08255 [hep-th] with Kwangeon Kim, Kanghoon Lee

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Motivation and context

Kaluza Klein theory unifyies gravity and electromagnetism into a single neat geometric package. The action and local symmetries of electromagnetism emerge from the reduction of a single geometric theory in one dimension higher. As such we are used to seeing vector fields as part of geometry and gravity in higher dimensions. With the advent of supergravities, theories with two-form potentials and indeed more generally p-form potentials have become common place. A natural question to ask is whether one can lift these theories with p-forms into a *geometric* theory in higher dimensions. With ordinary geometry one cannot.

So called, Double Field theory does this job. It is a lift of the NS-NS sector of supergravity to a purely *geometric* theory in higher dimensions. It is a sort of Kaluza Klein theory that gives ordinary gravity and 2-form gauge theory under reduction. Its local symmetries are a combination of diffeomorphisms with the gauge tranformations of the 2-form potentials and its action and equations of motion must contain the usual SUGRA ones once one removes dependences on any extra dimensions we have added.

Exceptional Field Theory does this for even higher degree forms such as the three form of eleven dimensional supergravity.

The classical double copy

The double copy relation gave a map between gravity and Yang-Mills. It was originally formulated for scattering amplitudes. The so called classical double copy makes this map between classical solutions. Central to this construction is the Kerr-Schild ansatz. We will show how to generalise the Kerr Schild ansatz and in doing so construct a double copy type relation for all the bosonic fields in supergravity.

Introduction to Double Field Theory

Novel formulation of string theory

- Bosonic NS-NS sector: $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ
- Makes $O(D, D; R)$ a manifest symmetry of the action
- Metric and B-field on equal footing - geometric unification

Double the dimension of space but require a global $O(D, D)$ structure

- $O(D, D)$ structure $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Geometric Framework

Doubling the dimension of space to $2D$

- Introduce new coordinates $\tilde{x}_\mu$
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Unification of two concepts

- Metric and B-field $\rightarrow$ generalized metric
- Diffeos and gauge transformations $\rightarrow$ generalized diffeos
- Generated by generalized Lie derivative

The Doubled Formalism

Generalized coordinates

- Combine x^μ and $\tilde{x}_\mu$ into

$$X^M = (x^\mu, \tilde{x}_\mu)$$

- $\mu = 1, \ldots, D$ and $M = 1, \ldots, 2D$

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Generalized metric

- Combine metric $g_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ into

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\sigma} B_{\sigma\nu} & g^{\mu\nu} \end{pmatrix}$$

- Rescale the dilaton $\quad e^{-2d} = \sqrt{g} e^{-2\phi}$

The DFT Action

The action integral

$$S = \int \mathrm{d}^{2D} X e^{-2d} R$$

The *generalized Ricci scalar*

$$\begin{aligned} R = &\frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_K \mathcal{H}_{NL} \\ &+ 4\mathcal{H}^{MN}\partial_M \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} \\ &- 4\mathcal{H}^{MN}\partial_M d \partial_N d + 4\partial_M \mathcal{H}^{MN}\partial_N d \end{aligned}$$

Extension to M-Theory

Extended theories

- Make U-duality manifest
- Include brane wrapping directions
- Geometrically unify metric and C-field(s)

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Example: $SL(5)$

- Duality group for M-theory in 4 dimensions x^{μ}
- Combine with 6 wrapping directions $y_{\mu\nu}$
- Wave in extended space gives M2-brane

The generalised tangent vector V^M in the $\mathbf{10}$ representation of $SL(5)$ is parametrised by a 4-vector v^μ and two-form $\lambda_{\mu\nu}$ in 4d as follows:

$$V^M = \begin{pmatrix} v^\mu \\ \lambda_{\mu\nu} \end{pmatrix} , \tag{4.1}$$

with $\mu, \nu (= 0, 1, 2, 3)$ being indices on the original M^4. On the other hand we may represent the $\mathbf{10}$ of $SL(5)$ $(M, N = 1 \cdots 10)$ as antisymmetric pairs of $\mathbf{5}$ indices, $[mn]$, where $m, n = 0, 1, 2, 3$ and 5, so that

$$V^{[mn]} = \begin{cases} V^{\mu 5} = -V^{5\mu} = v^\mu \\ V^{\mu\nu} = \tilde{\lambda}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\lambda_{\rho\sigma} \end{cases} \tag{4.2}$$

The $SL(5)$ generalized metric $\mathcal{M}_{MN}$ is then parametrized by the supergravity fields, $g_{\mu\nu}$ and $C_{\mu\nu\rho}$ as follows:

$$\mathcal{M}_{MN} = |g|^{\frac{1}{5}} \begin{pmatrix} g_{\mu\nu} + \frac{1}{4} C_{\mu\rho_1\rho_2} g^{\rho_1\rho_2,\sigma_1\sigma_2} C_{\sigma_1\sigma_2\nu} & \frac{1}{2} C_{\mu\rho_1\rho_2} g^{\rho_1\rho_2,\nu_1\nu_2} \\ \frac{1}{2} g^{\mu_1\mu_2,\rho_1\rho_2} C_{\rho_1\rho_2\nu} & g^{\mu_1\mu_2,\nu_1\nu_2} \end{pmatrix}, \qquad (4.3)$$

Here, we shall focus only on the (generalised) metric fields, G_{ij} and $\mathcal{M}_{MN}$, and ignore all the form fields A_i, B_{ij} and C_{ijk} for simplicity. We fix G_{ij} and treat it as a non-dynamical field.

$$S = \int_{\Sigma} \mathrm{d}^7 z\, \mathrm{d}^{10} X \sqrt{|G|} \left[R[G_{ij}] + \frac{1}{12} G^{ij} \partial_i \mathcal{M}_{MN} \partial_j \mathcal{M}^{MN} - V[\mathcal{M}, G] \right] \tag{4.4}$$

where $R[G_{ij}]$ is the Ricci scalar with respect to G_{ij}, and the scalar potential $V[\mathcal{M}, G]$ and $V[m, G]$ are given by

$$
\begin{aligned}
V[\mathcal{M},G] = & -\frac{1}{12}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_K\mathcal{M}_{LN} \\
& - \frac{1}{2}\partial_M\mathcal{M}^{MN}\partial_N \ln|G| \\
& - \frac{1}{4}\mathcal{M}^{MN}\left(\partial_M G_{ij}\partial_N G^{ij} + (\partial_M \ln|G|)(\partial_N \ln|G|)\ \right).
\end{aligned}
\tag{4.5}
$$

The variation of the action with respect to the generalized metric $\mathcal{M}$ gives

$$\delta_{\mathcal{M}} S = \int_{\Sigma} \mathrm{d}^7 z \, \mathrm{d}^{10} X \sqrt{|G|} \; \delta\mathcal{M}^{MN} \mathcal{K}_{MN} \,, \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{K}_{MN} = & -\frac{1}{6\sqrt{|G|}}\partial_P\big(\sqrt{|G|}\mathcal{M}^{PQ}\partial_Q\mathcal{M}_{MN}\big) + \frac{1}{\sqrt{|G|}}\partial_P\big(\sqrt{|G|}\mathcal{M}^{PQ}\partial_{(M}\\
& + \frac{1}{12}\partial_M\mathcal{M}_{PQ}\partial_N\mathcal{M}^{PQ} + \frac{1}{6}\mathcal{M}^{PQ}M^{RS}\partial_P\mathcal{M}_{RM}\partial_Q\mathcal{M}_{SN}\\
& - \frac{1}{2}\mathcal{M}^{PQ}\mathcal{M}^{RS}\partial_P\mathcal{M}_{RM}\partial_S\mathcal{M}_{QN} - \frac{1}{2}\partial_M\partial_N \ln|G| + \frac{1}{4}\partial_M G_{ij}\partial_N\\
& - \frac{1}{6\sqrt{|G|}}\partial_i\big(\sqrt{|G|}G^{ij}\partial_j\mathcal{M}_{MN}\big)\,.
\end{aligned} \tag{4.7}$$

We need a projection operator ${P_{MN}}^{PQ}$ to get a consistent variation of $\mathcal{M}$ in order to have compatible with the coset structure of the generalised metric. Then the equations of motion become

$$\hat{\mathcal{R}}_{MN} = {P_{MN}}^{PQ}\mathcal{K}_{PQ} = 0\,, \tag{4.8}$$

with

$${P_{MN}}^{KL} = \frac{1}{\alpha}\left(\delta_M^{(K}\delta_N^{L)} - \omega\mathcal{M}_{MN}\mathcal{M}^{KL} - \mathcal{M}_{MQ}{Y^{Q(K}}_{RN}\mathcal{M}^{L)R}\right). \tag{4.9}$$

The constants α and ω depend on the theory and are as follows

G	H	α	ω
$GL(d)$	$SO(d)$	1	0
$O(D,D)$	$O(D)\times O(D)$	2	0
$SL(5)$	$SO(5)$	3	$-\frac{1}{5}$
$Spin(5,5)$	$Spin(5)\times Spin(5)$	4	$-\frac{1}{4}$
$E_{6(6)}$	$USp(8)$	6	$-\frac{1}{3}$
$E_{7(7)}$	$SU(8)$	12	$-\frac{1}{2}$
$E_{8(8)}$	$SO(16)$	60	-1

Table: List of duality groups G and their maximal compact subgroups H (Euclidean case) the generalised metric parameterises G/H

We now present the Kerr-Schild ansatz for the generalised metric

$$\begin{aligned} \mathcal{M}_{MN} &= \mathcal{M}_{0MN} + \kappa\varphi P_{0MN}{}^{PQ} K_P K_Q\,, \\ (\mathcal{M}^{-1})^{MN} &= (\mathcal{M}_0^{-1})^{MN} - \kappa\varphi P_0{}^{MN}{}_{PQ} K^P K^Q\,. \end{aligned} \tag{4.10}$$

$\mathcal{M}_{0MN}$ is the background generalised metric, this is often taken to be flat but it need not be. K_M is a null vector with respect to $\mathcal{M}_0$, which explicitly means:

$$K_M (\mathcal{M}_0^{-1})^{MN} K_N = 0\,. \tag{4.11}$$

Finally, φ is a scalar function.

Here all the indices are raised and lowered by the background metric $\mathcal{M}_0$, and ${P_{0MN}}^{PQ}$ is the projection operator constructed with the background generalised metric. It turns out that the null condition on K_M is not sufficient for the linear structure of the inverse metric. To make this so we need to impose an additional condition,

$$Q_{MP}Q^{PQ} = 0\,. \tag{4.12}$$

Where

$$Q_{MN} = \varphi {P_{0MN}}^{PQ} K_P K_Q \tag{4.13}$$

The simplest example is GR since the projection operator for GR is trivial,

$$\left(P^{\mathrm{GR}}\right)_{\mu\nu}{}^{\rho\sigma} = \delta_{\mu}{}^{(\rho}\delta_{\nu}{}^{\sigma)} \,. \tag{4.14}$$

According to the universal form (4.10), the KS ansatz for GR is given by

$$\begin{aligned} g_{\mu\nu} &= \tilde{g}_{\mu\nu} + \kappa\varphi K_{\mu}K_{\nu} \,, \\ \left(g^{-1}\right)^{\mu\nu} &= \left(\tilde{g}^{-1}\right)^{\mu\nu} - \kappa\varphi K^{\mu}K^{\nu} \,, \end{aligned} \tag{4.15}$$

where $\tilde{g}$ is a background metric and the vector field K^{μ} is null with respect to the background metric, that is it obeys:

$$K^{\mu}\tilde{g}_{\mu\nu}K^{\nu} = 0 \,. \tag{4.16}$$

One can then show that the additional nilpotency condition on Q is trivial, because the projection operator is trivial. With this ansatz for the metric, the Einstein equations reduce to Maxwell's equations for φK_{μ} and so we identify φK_{μ} as the single copy Maxwell field, A_{μ}.

We now review the KS ansatz for DFT. The generalised metric $\mathcal{H}_{MN}$ on the $2D$ dimensional doubled space is given by the coset $O(D,D)/O(1,D-1)\times O(1,D-1)$. In terms of the usual D dimensional metric g and the Kalb-Ramond two-form B it is written as follows:

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} & B_{\mu\rho}g^{\rho\nu} \\ -g^{\mu\rho}B_{\rho\nu} & g^{\mu\nu} \end{pmatrix} . \tag{4.17}$$

It is a symmetric $O(D,D)$ element and satisfies the $O(D,D)$ compatibility constraint, $\mathcal{H}_{MP}\mathcal{J}^{PQ}\mathcal{H}_{QN} = \mathcal{J}_{MN}$, where $\mathcal{J}_{MN}$ is the $O(D,D)$ metric which maybe chosen to be:

$$\mathcal{J}_{MN} = \begin{pmatrix} \mathbf{0} & \delta^{a}{}_{b} \\ \delta_{a}{}^{b} & \mathbf{0} \end{pmatrix} . \tag{4.18}$$

The projector uses the so called Y-tensor which is given in terms of the $O(D,D)$ metric as follows

$$Y^{PQ}{}_{MN} = \mathcal{J}^{PQ}\mathcal{J}_{MN} \,. \qquad (4.19)$$

The projection operator factorises as follows:

$$\left(P^{\rm DFT}\right)_{MN}{}^{PQ} = 2P_{(M}{}^{P}\bar{P}_{N)}{}^{Q} \,, \qquad (4.20)$$

where $P_M{}^N$ and $\bar{P}_M{}^N$ are projectors defined by the generalized metric of DFT, $\mathcal{H}_{MN}$, and the $O(D,D)$ metric $\mathcal{J}_{MN}$:

$$P_{MN} = \frac{1}{2}\left(\mathcal{J}_{MN}+\mathcal{H}_{MN}\right) , \qquad \bar{P}_{MN} = \frac{1}{2}\left(\mathcal{J}_{MN}-\mathcal{H}_{MN}\right) . \quad (4.21)$$

The factorisation of the projector, should not in fact be a surprise, it is in fact a consequence of the left and right decomposition of the closed string modes.

Applying to the general KS ansatz (4.10), we have the KS ansatz for $\mathcal{H}_{MN}$

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\varphi\big(P_0^{\mathrm{DFT}}\big)_{MN}{}^{PQ} K_P K_Q\,, \tag{4.22}$$

where K_M is a null vector, $K_M \mathcal{J}^{MN} K_N = 0$ and $\big(P_0^{\mathrm{DFT}}\big)_{MN}{}^{PQ}$ is a background projection operator. It is useful to write the projected null vectors as

$$L_M = P_{0M}{}^N K_N\,, \qquad \bar{L}_M = \bar{P}_{0M}{}^N K_N \tag{4.23}$$

and from the completeness relation, $\delta_M{}^N = P_M{}^N + \bar{P}_M{}^N$, K_M is decomposed to L_M and $\bar{L}_M$

The finite fluctuation part is written as

$$Q_{MN} = \varphi\big(L_M \bar{L}_N + L_N \bar{L}_M\big)\,. \tag{4.24}$$

At this stage, L and $\bar{L}$ do not have to be null. However, we need to require the nilpotency condition on Q_{MN} (4.12) for the linearity of the KS ansatz, which implies that L_M and $\bar{L}_M$ have to be a pair of mutually orthogonal null vectors with definite chirality,

$$L_M = P_{0M}{}^{N} L_N\,, \qquad \bar{L}_M = \bar{P}_{0M}{}^{N} \bar{L}_N \tag{4.25}$$

and

$$L_N L^N = 0\,, \qquad \bar{L}_N \bar{L}^N = 0\,. \tag{4.26}$$

Then the KS ansatz is rewritten as:

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\varphi\big(L_M \bar{L}_N + L_N \bar{L}_M\big)\,. \tag{4.27}$$

From this the equations of motion become that of two Maxwell fields from which the B-field and metric solutions may be reconstructed. This was done for DFT by Kanghoon Lee.

But what about exceptional field theory....?

For simplicity we will now assume a flat background, $g_{\mu\nu} = \eta_{\mu\nu}$ and $C_{\mu\nu\rho} = 0$, so the background generalised metric $\mathcal{M}_0$ is:

$$\mathcal{M}_{0MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta^{\mu\mu',\nu\nu'} \end{pmatrix} , \quad \text{and} \quad \mathcal{M}_{0mm',nn'} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -\eta_{\mu\mu',\nu\nu'} \end{pmatrix} , \tag{4.28}$$

and the associated null vector K_M is parametrized as

$$K_M = \begin{pmatrix} l_\mu \\ k^{\nu\nu'} \end{pmatrix} , \qquad K^M = \begin{pmatrix} l^\mu \\ k_{\nu\nu'} \end{pmatrix} , \tag{4.29}$$

and

$$K_{mm'} = \begin{cases} K_{\mu 5} = l_\mu \\ K_{\nu\nu'} = \tilde{k}_{\nu\nu'} \end{cases} \qquad K^{mm'} = \begin{cases} K^{\mu 5} = l^\mu \\ K^{\nu\nu'} = -\tilde{k}^{\nu\nu'} \end{cases} . \tag{4.30}$$

Here the Greek indices are raised and lowered by the flat background metric $\eta_{\mu\nu}$.

$\tilde{k}_{\mu\nu}$ and $\tilde{k}^{\mu\nu}$ are related with $k_{\mu\mu'}$ and $k^{\mu\mu'}$ as

$$\tilde{k}_{\mu\mu'} = \frac{1}{2}\epsilon_{\mu\mu'\nu\nu'}k^{\nu\nu'}\,, \qquad \tilde{k}^{\mu\mu'} = \frac{1}{2}\eta^{\mu\mu',\nu\nu'}\tilde{k}_{\nu\nu'} = -\frac{1}{2}\epsilon^{\mu\mu'\nu\nu'}k_{\nu\nu'}\,, \tag{4.31}$$

and their inner product satisfies

$$k_{\mu\mu'}k^{\mu\mu'} = -\tilde{k}_{\mu\mu'}\tilde{k}^{\mu\mu'}\,. \tag{4.32}$$

Then the null condition for K is then

$$K_M K^M = K_{mm'}K^{mm'} = l_\mu l^\mu + \frac{1}{2}k_{\nu\nu'}k^{\nu\nu'} = l_\mu l^\mu - \frac{1}{2}\tilde{k}_{\nu\nu'}\tilde{k}^{\nu\nu'} = 0\,. \tag{4.33}$$

The projection necessary operator ${P_{MN}}^{PQ}$ for the $SL(5)$ EFT is given by:

$$ {P_{MN}}^{PQ} = \frac{1}{3}\Big(\delta_{(M}{}^{P}\delta_{N)}{}^{Q} + \frac{1}{5}\mathcal{M}_{MN}\mathcal{M}^{PQ} - \mathcal{M}_{MR}Y^{R(P}{}_{SN}\mathcal{M}^{Q)S}\Big)\,, \tag{4.34} $$

and the Y-tensor is

$$ Y^{MN}{}_{PQ} = Y^{mm'nn'}{}_{pp'qq'} = \epsilon^{mm'nn'r}\epsilon_{pp'qq'r}\,. \tag{4.35} $$

Combining the above results allows us to write the following Kerr-Schild ansatz for $\mathcal{M}_{MN}$

$$ \begin{aligned} \mathcal{M}_{MN} &= \mathcal{M}_{0MN} + \kappa\varphi {P_{0MN}}^{PQ} K_P K_Q\,,\\ (\mathcal{M}^{-1})^{MN} &= {\mathcal{M}_0}^{MN} - \kappa\varphi {P_0}^{MN}{}_{PQ} K^P K^Q\,, \end{aligned} \tag{4.36} $$

where κ is some constant parameter and the indices are raised and lowered by the background generalized metric $\mathcal{M}_{0MN}$ and $(\mathcal{M}_0^{-1})^{MN}$ in (4.28).

One may now express the usual supergravity fields in terms of the components of K^M as follows:

$$\begin{aligned} |g| &= \left|1 - \frac{\kappa\varphi}{3} l \cdot l\right|^{-\frac{5}{3}}, \\ g_{\mu\nu} &= \left|1 - \frac{\kappa\varphi}{3} l \cdot l\right|^{-\frac{2}{3}} \left(\eta_{\mu\nu} + \frac{\kappa\varphi}{3}\left(l_\mu l_\nu - \tilde{k}_{\mu\rho}\tilde{k}_\nu{}^\rho\right)\right), \\ C_{\mu\nu\rho} &= \frac{2\kappa\varphi}{3\left|1 - \frac{\kappa\varphi}{3} l \cdot l\right|} l_{[\mu} k_{\nu\rho]} . \end{aligned} \tag{4.37}$$

where $l \cdot l = l_\mu \eta^{\mu\nu} l_\nu$. Remarkably, the KS ansatz for the generalized metric is linear in κ, but component fields are highly nonlinear. If we set $k_{\mu\nu} = 0$, then l^μ becomes a null vector and the KS ansatz reduces to the conventional KS ansatz in GR, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\varphi l_\mu l_\nu$ and $C_{\mu\nu\rho} = 0$.

We insert this into the equations of motion and discover:

$$\begin{aligned} &\partial^\sigma \partial_\sigma(\varphi l_\mu) - \partial^\sigma \partial_\mu(\varphi l_\sigma) = 0\,, \\ &\partial^\rho \partial_{[\rho}(\varphi k_{\mu\nu]}) = 0\,. \end{aligned} \tag{4.38}$$

From which we identify:

$$A_\mu = \varphi l_\mu\,, \qquad B_{\mu\nu} = \varphi k_{\mu\nu}\,. \tag{4.39}$$

as the single copy fields. Giving the Maxwell field and two form potential as the single copy for gravity and three form potential.

We will now construct the membrane solution in terms of the universal Kerr-Schild Ansatz and interpret these solutions from the point of view of the single copy fields.
Let's assume the worldvolume directions of the M2-brane are t, x^1, x^2 and denote them as $x^\alpha = \{t, x^1, x^2\}$ and choose the M-theory circle direction as x^2. The transverse directions are $\vec{x}_8 = \{x^3, z^i\}$, where $i, j, \cdots$ are 7-dimensional extra directions.
The M2-brane geometry is given by

$$\begin{aligned} \mathrm{d}s_{11}^2 &= H^{-\frac{2}{3}}\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + H^{\frac{1}{3}}\mathrm{d}x^3\mathrm{d}x^3 + H^{\frac{1}{3}}\delta_{ij}\mathrm{d}z^i\mathrm{d}z^j \,, \\ C_{t12} &= -\left(1 - H^{-1}\right), \qquad H = 1 + \frac{h}{|\vec{x}_8|^6} \,. \end{aligned} \tag{4.40}$$

To embed the 11-dimensional supergravity into the $SL(5)$ ExFT, we need to use the following Kaluza-Klein ansatz for the 11-dimensional metric $\hat{g}$

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} |g|^{-\frac{1}{5}} G_{ij} + A_i^\rho A_j^\sigma g_{\rho\sigma} & A_i^\rho g_{\rho\nu} \\ g_{\mu k} A_j^k & g_{\mu\nu} \end{pmatrix} , \tag{4.41}$$

The internal metric and C-field are

$$g_{\mu\nu} = H^{-\frac{2}{3}} \left(\eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta + H \mathrm{d}x^3 \mathrm{d}x^3 \right) , \qquad C_{012} = H^{-1} - 1 \, . \tag{4.42}$$

The single copy of this solution is then given by:

$$A_0 = \varphi \, , \qquad B_{12} = \varphi \, . \tag{4.43}$$

A quite remarkable result...

Thus the single copy of the membrane is described by a two dimensional plane of electric charges. That is the Maxwell field is given by solving the Harmonic equation for an electric source smeared over two spatial dimensions just like the classical electrostatic problem for a plane of charge. The two form solution is like that of a magnetic string that is smeared over one additional dimension to give a plane of string charge.

Both the Maxwell field and the two form field are given in terms of the same single harmonic function φ. This is due to the BPS nature of these solutions. If there were no C field and then the membrane was purely a gravitational object then its solution would be described by the usual gravitational single copy relation by the Maxwell field. The two form encodes the C field contribution. The fact being BPS sets the charge equal to the mass (or tension) of the solution means that what would be independent Harmonic functions for the Maxwell and two form fields are fixed to be equal. We thus conjecture that the non-BPS solutions are described by the same Ansatz but with independent Harmonic functions for A and B fields. We leave this for future research.