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A sum rule for tidal Love number

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Work in collaboration with Myungbo Shim

The main result

$$C_E = -\frac{1}{8G}\left[\sum_{m'>m}\frac{\Gamma_{m'\to mh}}{(\Delta m)^6} - \sum_{m'<m}\frac{\Gamma_{m\to m'h}}{(\Delta m)^6}\right]$$

A relation between

- LHS : tidal response (Love number)
- RHS : single graviton emission rates

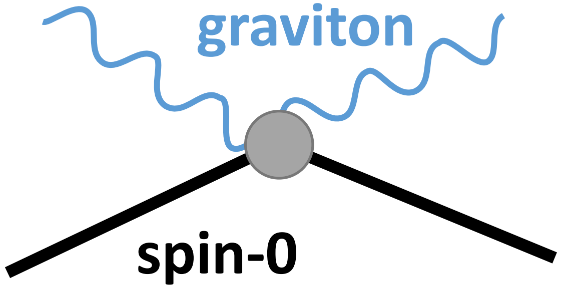
$\boxed{\hbar = c = 1}$

The main result

$$\boxed{C_E} = -\frac{1}{8G}\left[\sum_{m'>m}\frac{\Gamma_{m'\to mh}}{(\Delta m)^6} - \sum_{m'<m}\frac{\Gamma_{m\to m'h}}{(\Delta m)^6}\right]$$

tidal response (Love number)

$$\langle Q_{\mu\nu}\rangle = 2C_E\langle E_{\mu\nu}\rangle$$



$$S = \int(-m + C_E E_{\mu\nu}E^{\mu\nu} + C_B B_{\mu\nu}B^{\mu\nu})d\tau$$

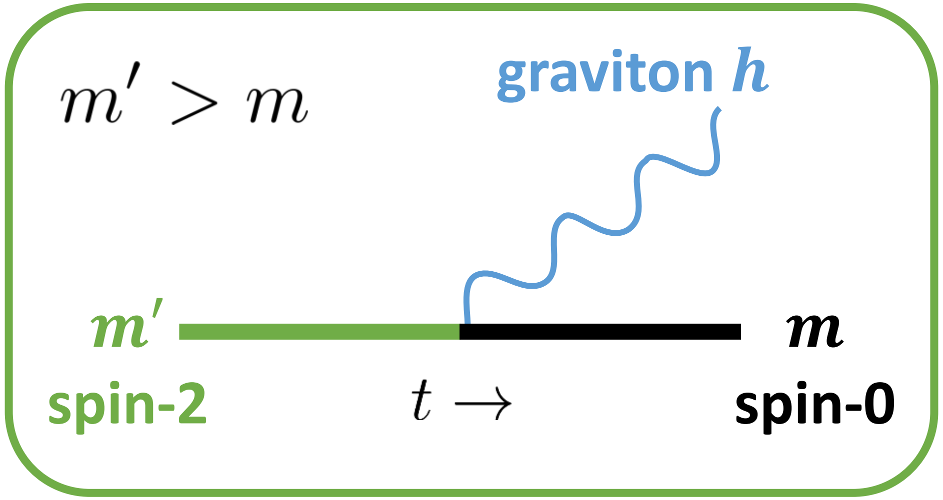
$$E_{\mu\nu} = W_{\mu\alpha\nu\beta}u^\alpha u^\beta$$

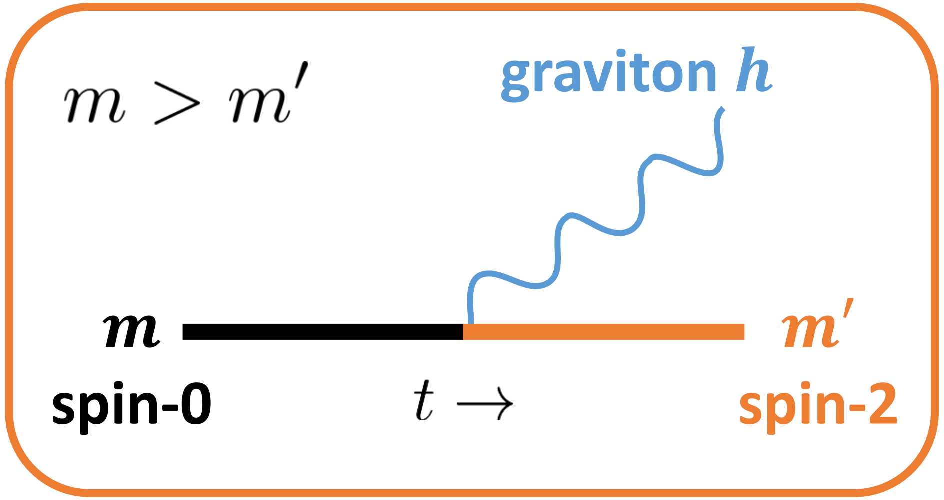
$$B_{\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}W^{\alpha\beta}{}_{\delta\nu}u^\gamma u^\delta$$

The main result

$$C_E = -\frac{1}{8G}\left[\sum_{m'>m}\frac{\overset{\text{transition to } m}{\boxed{\Gamma_{m'\to mh}}}}{(\Delta m)^6} - \sum_{m'<m}\frac{\overset{\text{transition from } m}{\boxed{\Gamma_{m\to m'h}}}}{(\Delta m)^6}\right]$$

$$\boxed{\Delta m = m' - m}$$





Understanding the relation : EM & QM

- A similar relation for electromagnetism

$$\alpha = -2\pi \left[\sum_{m'>m} \frac{\Gamma_{m'\to m\gamma}}{(\Delta m)^4} - \sum_{m'<m} \frac{\Gamma_{m\to m'\gamma}}{(\Delta m)^4} \right]$$

- Relation between
  - LHS : electric polarisability $\langle \vec{p} \rangle = \alpha \langle \vec{E} \rangle$
  - RHS : single photon emission rates

- Can be understood using (spinless) hydrogen atom

Example : Hydrogen atom in QM (1)

$$H = \frac{p^2}{2m_e} - \frac{e^2}{4\pi r}$$

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$$H = \frac{p^2}{2m_e} - \frac{e^2}{4\pi r} + \boxed{eEz}$$

perturbation

- Leading perturbation order wavefunction

$$|n^{\text{full}}\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \cdots \qquad |n^{(1)}\rangle = \sum_{m\neq n} \frac{|m\rangle\langle m|eEz|n\rangle}{E_n - E_m}$$

- Leading perturbation order dipole moment

$$\langle\vec{p}\rangle = -e\langle z\rangle \simeq -e\left[\langle n^{(0)}|z|n^{(1)}\rangle + \langle n^{(1)}|z|n^{(0)}\rangle\right] = \sum_{m\neq n} \frac{2|\langle m|ez|n\rangle|^2}{E_n - E_m} E$$

Example : Hydrogen atom in QM (2)

- Polarisability

$$\alpha = \frac{\partial \langle p \rangle}{\partial E} = \sum_{m \neq n} \frac{2|\langle m|ez|n\rangle|^2}{E_n - E_m}$$

- Spontaneous emission rate (dipole transition) $\omega = |E_n - E_m|$

$$\Gamma_{n \to m\gamma} = 2\pi |\langle m\gamma|H_{\rm int}|n\rangle|^2 = \frac{\omega^3}{\pi} |\langle m|ez|n\rangle|^2$$

Example : Hydrogen atom in QM (2)

- Polarisability

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$$\Gamma_{n\to m\gamma} = 2\pi |\langle m\gamma|H_{\text{int}}|n\rangle|^2 = \frac{\omega^3}{\pi}\boxed{|\langle m|ez|n\rangle|^2}$$

- Combine!

$$\alpha = -2\pi \left[\sum_{m'>m} \frac{\Gamma_{m'\to m\gamma}}{(\Delta m)^4} - \sum_{m'<m} \frac{\Gamma_{m\to m'\gamma}}{(\Delta m)^4} \right]$$

Derivation from QED (1)

- The on-shell $m' \to m\gamma$ amplitudes[Arkani-Hamed,Huang&Huang `17]

$$\mathcal{A}^{+}_{m'm\gamma} = \frac{e\mathcal{C}}{m}[\mathbf{13}]^2\,, \qquad \mathcal{A}^{-}_{m'm\gamma} = \frac{e\mathcal{C}}{m}\langle\mathbf{13}\rangle^2$$

$\omega = \Delta m = m' - m$

Leading in $\Delta m/m$

- Transition rate

$$\Gamma_{m'\to m\gamma} = 4\pi \times \frac{\omega}{32\pi^2 m^2}\sum_{m',\gamma}|\mathcal{A}_{m'm\gamma}|^2 = \left(\frac{e\mathcal{C}}{m}\right)^2\frac{\omega^3}{\pi}$$

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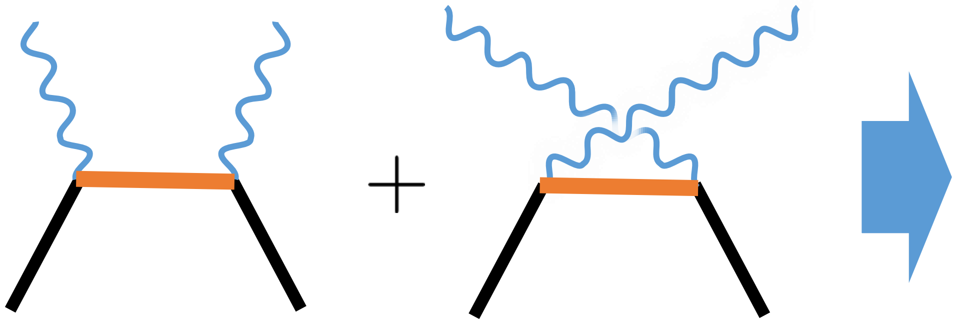
- Gluing construction for 4pt[Arkani-Hamed,Huang&Huang `17]



$$\mathcal{A}^{++}_{m\gamma\gamma m} = -\left(\frac{e\mathcal{C}m'}{m}\right)^2[23]^2\left[\frac{1}{s-m'^2}+\frac{1}{u-m'^2}\right]$$

$$\mathcal{A}^{+-}_{m\gamma\gamma m} = -\left(\frac{e\mathcal{C}}{m}\right)^2[2|p_1|3\rangle^2\left[\frac{1}{s-m'^2}+\frac{1}{u-m'^2}\right]$$

Derivation from QED (2)

$$E^\mu = F^{\mu\nu} u_\nu$$

$$B^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} u_\alpha F_{\beta\gamma}$$

- The effective point-particle worldline action

$$S = \int \left(-m - qA_\mu u^\mu - \frac{\alpha}{2} E_\mu E^\mu - \frac{\chi}{2} B_\mu B^\mu\right) d\tau$$

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- The effective point-particle worldline action

$$\begin{aligned}
S &= \int (-m - qA_\mu u^\mu - \frac{\alpha}{2} E_\mu E^\mu - \frac{\chi}{2} B_\mu B^\mu) d\tau \\
&\simeq \int (\frac{mv^2}{2} - q\phi + q\vec{v}\cdot\vec{A} + \frac{\alpha}{2}\vec{E}^2 + \frac{\chi}{2}\vec{B}^2) dt \\
&= \int (T - V) dt
\end{aligned}$$

$$\langle \vec{p} \rangle = -\left\langle \frac{\partial F}{\partial \vec{E}} \right\rangle \simeq -\left\langle \frac{\partial V}{\partial \vec{E}} \right\rangle = \alpha \langle \vec{E} \rangle \,, \; \langle \vec{\mu} \rangle = \chi \langle \vec{B} \rangle$$

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- The effective point-particle worldline action

$$S = \int \left(-m - qA_\mu u^\mu \boxed{- \frac{\alpha}{2} E_\mu E^\mu} \boxed{- \frac{\chi}{2} B_\mu B^\mu}\right) d\tau$$
$$\simeq \int \left(\frac{mv^2}{2} - q\phi + q\vec{v}\cdot\vec{A} + \boxed{\frac{\alpha}{2}\vec{E}^2} + \boxed{\frac{\chi}{2}\vec{B}^2}\right) dt$$
$$= \int (T - V) dt$$

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Derivation from QED (3)

- Amplitude from point-particle worldline action[Chung,Huang,JWK&Lee `18,`19]

$$S = \int \left(-m - qA_\mu u^\mu - \frac{\alpha}{2}E_\mu E^\mu - \frac{\chi}{2}B_\mu B^\mu\right)d\tau$$

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$$S=\int\left(-m\boxed{-\,qA_\mu u^\mu}\boxed{-\,\frac{\alpha}{2}E_\mu E^\mu-\frac{\chi}{2}B_\mu B^\mu}\right)d\tau$$

$$\boxed{\partial_\mu A_\nu\to -ik_\mu\varepsilon_\nu(k)\,,\ u^\mu\to p_1^\mu/m}$$

up to normalisation N

$$\boxed{\mathcal{A}^{\text{Eff. Act.}}_{mm\gamma}=-qN\varepsilon_\mu(p_1/m)^\mu}$$

$$\mathcal{A}^{++}_{m\gamma\gamma m}=(\alpha-\chi)\frac{N[23]^2}{4}$$

$$\mathcal{A}^{+-}_{m\gamma\gamma m}=(\alpha+\chi)\frac{N[2|p_1|3\rangle^2}{4m^2}$$

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$$\boxed{\partial_\mu A_\nu \to -ik_\mu \varepsilon_\nu(k)\,,\; u^\mu \to p_1^\mu/m}$$

$$\boxed{\mathcal{A}^{\text{Scalar QED}}_{\phi\bar{\phi}\gamma} = -q\varepsilon_\mu (p_2 - p_1)^\mu}$$

$$\boxed{N = -2m}$$

$$\boxed{\mathcal{A}^{\text{Eff. Act.}}_{mm\gamma} = -qN\varepsilon_\mu (p_1/m)^\mu}$$

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Derivation from QED (4)

3pt glued amplitude

$$\mathcal{A}^{++}_{m\gamma\gamma m} = -\left(\frac{e\mathcal{C}m'}{m}\right)^2 [23]^2 \left[\frac{1}{s-m'^2} + \frac{1}{u-m'^2}\right]$$

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Worldline action amplitude

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$$|t|, |s-m^2| \ll |m'^2 - m^2|\,,\ \Delta m/m \ll 1$$

$$\mathcal{A}^{++}_{m\gamma\gamma m} = \left(\frac{e\mathcal{C}}{m}\right)^2 \left[\frac{2}{\Delta m} + \frac{4}{m}\right] \frac{m[23]^2}{2}$$

$$\mathcal{A}^{+-}_{m\gamma\gamma m} = \left(\frac{e\mathcal{C}}{m}\right)^2 \frac{2}{\Delta m} \frac{[2|p_1|3\rangle^2}{2m}$$

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$$\alpha = -\left(\frac{e\mathcal{C}}{m}\right)^2 \left[\frac{2}{\Delta m} + \frac{2}{m}\right]$$

$$\chi = +\left(\frac{e\mathcal{C}}{m}\right)^2 \frac{2}{m}$$

Derivation from QED (5)

Electric susceptibility

$$\alpha = -\left(\frac{e\mathcal{C}}{m}\right)^2 \left[\frac{2}{\Delta m}\right]$$

Single photon emission rate

$$\Gamma_{m' \to m\gamma} = \left(\frac{e\mathcal{C}}{m}\right)^2 \frac{\omega^3}{\pi}$$

Derivation from QED (5)

Electric susceptibility

$$\alpha = -\boxed{\left(\frac{e\mathcal{C}}{m}\right)^2}\left[\frac{2}{\Delta m}\right]$$

Single photon emission rate

$$\Gamma_{m'\to m\gamma} = \boxed{\left(\frac{e\mathcal{C}}{m}\right)^2}\frac{\omega^3}{\pi}$$

$$\boxed{\omega = |\Delta m| = |m' - m|}$$

$$\boxed{\alpha = -2\pi\left[\sum_{m'>m}\frac{\Gamma_{m'\to m\gamma}}{(\Delta m)^4} - \sum_{m'<m}\frac{\Gamma_{m\to m'\gamma}}{(\Delta m)^4}\right]}$$

From QED to perturbative QG

QED

$$\mathcal{A}^{+}_{m'm\gamma} = \frac{e\mathcal{C}}{m}[\mathbf{13}]^2$$

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$$S \supset -1/2\int(\alpha E_\mu E^\mu + \mu B_\mu B^\mu)d\tau$$

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Perturbative QG

$$\mathcal{A}^{+}_{m'mh} = \frac{\kappa\tilde{\mathcal{C}}}{m^2}[\mathbf{13}]^4$$

$$\Gamma_{m'\to mh} = \left(\frac{\kappa\tilde{\mathcal{C}}}{m}\right)^2 \frac{4\omega^5}{\pi}$$

$$S \supset +\int(C_E E_{\mu\nu}E^{\mu\nu} + C_B B_{\mu\nu}B^{\mu\nu})d\tau$$

$$C_E = -\left(\frac{4\tilde{\mathcal{C}}}{m}\right)^2\left[\frac{1}{\Delta m}\right]$$

$$C_E = -\frac{1}{8G}\left[\sum_{m'>m}\frac{\Gamma_{m'\to mh}}{(\Delta m)^6} - \sum_{m'<m}\frac{\Gamma_{m\to m'h}}{(\Delta m)^6}\right]$$

From QED to perturbative QG

QED	Perturbative QG
$\mathcal{A}^{+}_{m'm\gamma} = \frac{e\mathcal{C}}{m}\boxed{[\mathbf{13}]^2}$	$\mathcal{A}^{+}_{m'mh} = \frac{\kappa\tilde{\mathcal{C}}}{m^2}\boxed{[\mathbf{13}]^4}$
$\Gamma_{m'\to m\gamma} = \left(\frac{e\mathcal{C}}{m}\right)^2 \frac{\boxed{\omega^3}}{\pi}$	$\Gamma_{m'\to mh} = \left(\frac{\kappa\tilde{\mathcal{C}}}{m}\right)^2 \frac{4\boxed{\omega^5}}{\pi}$
$S \supset -1/2\int(\alpha E_\mu E^\mu + \mu B_\mu B^\mu)d\tau$	$S \supset +\int(C_E E_{\mu\nu}E^{\mu\nu} + C_B B_{\mu\nu}B^{\mu\nu})d\tau$
$\alpha = -\left(\frac{e\mathcal{C}}{m}\right)^2\left[\frac{2}{\Delta m}\right]$	$C_E = -\left(\frac{4\tilde{\mathcal{C}}}{m}\right)^2\left[\frac{1}{\Delta m}\right]$
$\alpha = -2\pi\left[\sum_{m'>m}\frac{\Gamma_{m'\to m\gamma}}{\boxed{(\Delta m)^4}} - \sum_{m'<m}\frac{\Gamma_{m\to m'\gamma}}{\boxed{(\Delta m)^4}}\right]$	$C_E = -\frac{1}{8G}\left[\sum_{m'>m}\frac{\Gamma_{m'\to mh}}{\boxed{(\Delta m)^6}} - \sum_{m'<m}\frac{\Gamma_{m\to m'h}}{\boxed{(\Delta m)^6}}\right]$

Side remarks

- Classical version of the relations[Chakrabarti,Delsate&Steinhoff `13]

$$\tilde{F}_l = \sum_n \frac{I_{nl}^2}{\omega_{nl}^2 - \omega^2}.$$

ω : AC frequency

- Missed opportunities[Rothstein `14/Brustein&Sherf `20]

$$Re F(\omega) = P \sum_m \frac{|\langle \Omega \mid Q_{ab} \mid m \rangle|^2}{E_\Omega - E_m - \omega} \qquad k_2 = -\frac{3}{4R^5} \sum_{n_r} \frac{|\langle \Psi_0 | \widehat{Q} | n_r, 2, 0 \rangle|^2}{|\Delta E_{1,n_r}|}$$

- Corrections from resummed propagators?

$$\frac{-i}{-q^2 + M^2} \to \frac{-i}{-q^2 + M^2 - iM\Gamma}$$

“What can we do with it?” (1)

$$C_E = -\frac{1}{8G}\left[\sum_{m'>m} \frac{\Gamma_{m'\to mh}}{(\Delta m)^6} - \sum_{m'<m} \frac{\Gamma_{m\to m'h}}{(\Delta m)^6}\right]$$

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- Consider black holes
  - LHS : Vanishes classically (GR)
  - RHS 1 : Quasi-normal modes (horizon excitations)
  - RHS 2 : Hawking radiation

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- Consider black holes
  - LHS : Vanishes classically (GR)
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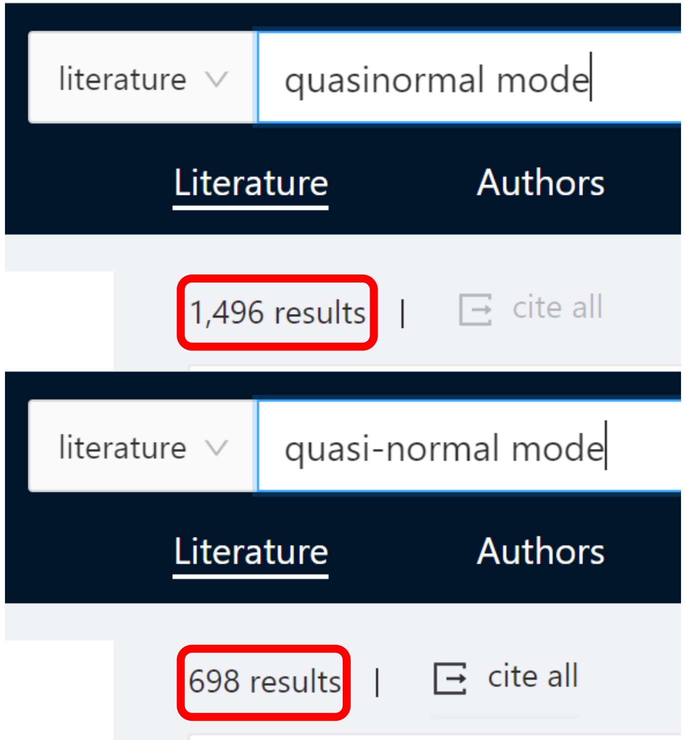
- Quasi-normal modes are well studied
  - Their contributions can be computed!



[INSPIRE, 04Dec`20]

“What can we do with it?” (2)

- A quantitative test for (*s*-wave) Hawking radiation as single quantum emission

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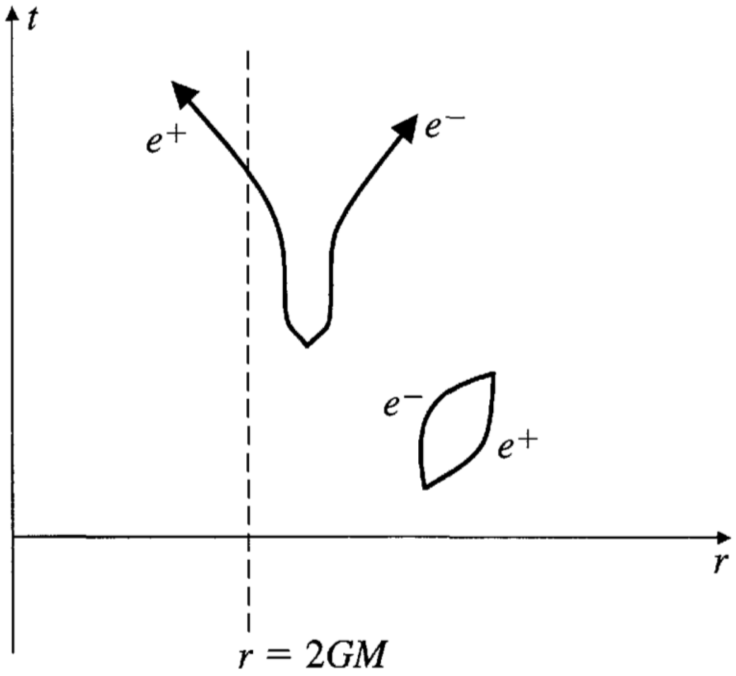
- A quantitative test for Hawking radiation as single quantum emission (*s*-wave)

- Usual understanding of HR
  - Pair creation near horizon
  - One falls in, the other goes out ⇒ HR
    - Single quantum emission!
  - Maximal entanglement btw BH and HR

Image taken from :
Sean Carroll, *Spacetime and Geometry*



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- “BH information paradoxes”
  - Assumes maximal entanglement

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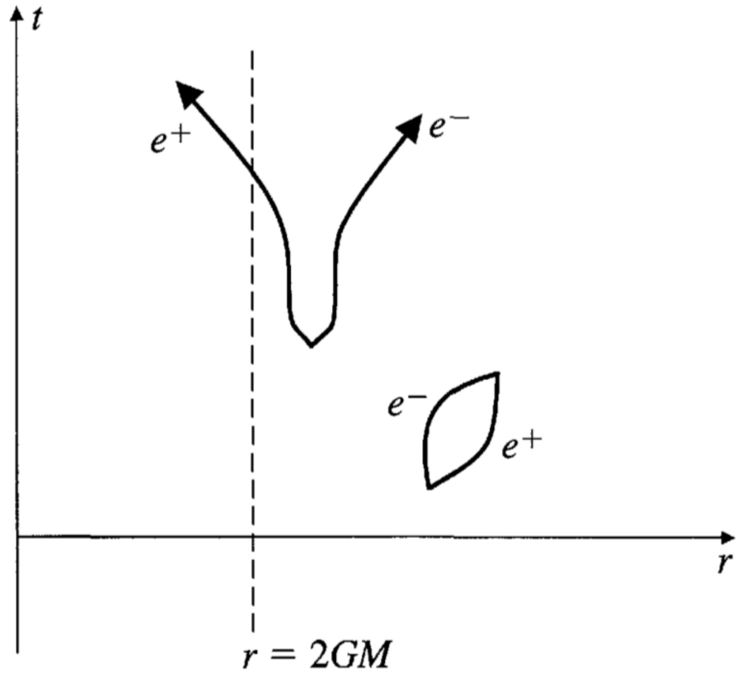

Conclusions and future directions

- A sum rule for Love
  - Relates tidal response to single graviton emission rates
  - A quantitative test for HR as successive single graviton emissions
    - May have implications to “BH information paradoxes”
  - The test can be generalised to photon emissions

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- Future directions
  - Nonlinear response / higher multipoles / spinning cases / $D > 4$ / $\Lambda \neq 0$
  - Loops and resummed propagator corrections
  - Running the test!