Classical black hole scattering from a worldline QFT

based on [2010.02865] with Jan Plefka & Jan Steinhoff

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December 3, 2020

Graviton-dressed scalar propagator

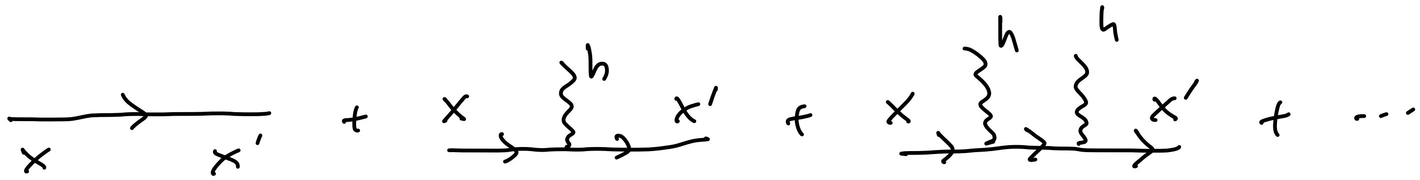
Massive complex scalar $\phi(x)$ interacting with gravity:

$$S = \int d^D x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi - \xi R \phi^\dagger \phi \right)$$

$$\left(\nabla_\mu \nabla^\mu + m^2 + \xi R \right) G(x,x') = \sqrt{-g}\, \delta^{(D)}(x-x')$$

Weak-field approximation

$G(x,x') =$



This gravitationally-dressed Green's function has a worldline path integral form [deWitt, Behenstein, Parker]:

$$G(x,x') \sim \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x \exp\left[-i \int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \underbrace{\left(\xi - \frac{1}{4}\right) R}_{} \right) \right]$$

↑ suppressed in $\hbar \to 0$ limit

Worldline action familiar from [Kälin, Porto].

This provides a link: QFT $\longleftrightarrow$ worldline.

From the S-matrix to the worldline

$$G(x,x') = \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}[x,a,b,c] \exp\left[-i\int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu}\left(\dot{x}^\mu \dot{x}^\nu + \underbrace{a^\mu a^\nu + b^\mu c^\nu}_{\text{ghosts}}\right) + \left(\xi - \frac{1}{4}\right) R\right)\right]$$

Non-propagating "Lee-Yang" ghosts a^μ, b^μ, c^μ needed to control $\langle \dot{x}^\mu(\tau)\dot{x}^\nu(\tau)\rangle \sim \delta(0)$ divergences.

$$G_i(x,x') = \int \mathcal{D}\phi_i\, \phi_i(x)\, \phi_i^\dagger(x')\, e^{iS_i}$$

$$S_{EH} = -2 m_{Pl}^{D-2} \int d^D x \sqrt{-g} R$$

Insert into a time-ordered correlator:

$$\begin{aligned} &\langle \Omega | T\{\phi_1(x_1)\phi_1^\dagger(x_1')\phi_2(x_2)\phi_2^\dagger(x_2')\} | \Omega \rangle \\ &= \int \mathcal{D}[h_{\mu\nu}, \phi_1, \phi_2]\, \phi_1(x_1)\phi_1^\dagger(x_1')\, \phi_2(x_2)\phi_2^\dagger(x_2')\, e^{i(S_{EH} + S_1 + S_2)} \\ &= \int \mathcal{D}h_{\mu\nu}\, G_1(x_1, x_1')\, G_2(x_2, x_2')\, e^{iS_{EH}} \end{aligned}$$

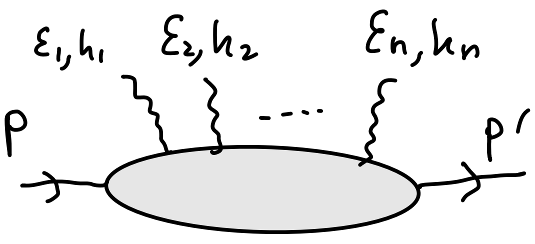
Neglect virtual scalar loops using classical $\hbar \to 0$ limit.

LSZ Reduction

To analyze S-matrices switch to momentum space.
Insert a plane-wave basis:

$$h_{\mu\nu} = \sum_{\ell=1}^{N} \varepsilon_{\mu\nu}^{(\ell)} \, e^{i k_\ell \cdot x(\sigma_\ell)}$$

$$F(p,p' | \{\varepsilon^{(\ell)}, k_\ell\}) = (p^2 - m^2)(p'^2 - m^2)\, G(p,p' | \{\varepsilon^{(\ell)}, k_\ell\}) \Big|_{p^2 = p'^2 = m^2}$$

$=$ 

Surprisingly compact form factor describing n-graviton emission:

$$F(p,p' | \{\varepsilon^{(\ell)}, k_\ell\}) = \left(-\frac{i\kappa}{4}\right)^N \delta^{(D)}(P) \prod_{\ell=1}^{N} \left[\int_{-\infty}^{\infty} d\sigma_\ell \, \varepsilon^{(\ell),\mu\nu} \left(\partial_{\varepsilon_\ell^\mu} \partial_{\varepsilon_\ell^\nu} + \partial_{\alpha_\ell^\mu} \partial_{\alpha_\ell^\nu} + \partial_{\beta_\ell^\mu} \partial_{\gamma_\ell^\nu} \right) \right]$$
$$\delta\left(\sum_{\ell=1}^{N} \sigma_\ell \right) \exp\left[-(p+p') \cdot \sum_{\ell=1}^{N} (i k_\ell \sigma_\ell + \varepsilon_\ell) - i \sum_{\ell,\ell'=1}^{N} \left\{ \frac{|\sigma_\ell - \sigma_{\ell'}|}{2} k_\ell \cdot k_{\ell'} \right.\right.$$
$$\left.\left. - i \, \mathrm{sgn}(\sigma_\ell - \sigma_{\ell'}) \, \varepsilon_\ell \cdot k_{\ell'} + \delta(\sigma_\ell - \sigma_{\ell'}) (\varepsilon_\ell \cdot \varepsilon_{\ell'} + \alpha_\ell \cdot \alpha_{\ell'} - 4 \gamma_\ell \cdot \beta_{\ell'}) \right\} \right] \Bigg|_{\substack{\varepsilon_\ell = \alpha_\ell = \beta_\ell = \gamma_\ell = 0; \\ p^2 = p'^2 = m^2}}$$

Now we compare this with the worldline.

Worldline Quantum Field Theory (WQFT)

Use a path-integral representation of the WQFT:

$$\Xi(b,v) \equiv \int Dx \int D[a,b,c] \exp\left[-i \int_{-\infty}^{\infty} d\sigma \left(\frac{1}{4} g_{\mu\nu}\left(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right)\right)\right]$$

Background field expansion: $\quad x^\mu(\sigma) = b^\mu + v^\mu \sigma + z^\mu(\sigma)$

$$\left.\begin{aligned} \langle z^\mu(\sigma)\, z^\nu(\sigma')\rangle &= 2i\eta^{\mu\nu}\Delta(\sigma-\sigma') \\ \langle a^\mu(\sigma)\, a^\nu(\sigma')\rangle &= -2i\eta^{\mu\nu}\delta(\sigma-\sigma') \\ \langle b^\mu(\sigma)\, c^\nu(\sigma')\rangle &= 4i\eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned}\right\} \quad \partial_\sigma \partial_{\sigma'} \Delta(\sigma-\sigma') = -\delta(\sigma-\sigma')$$

MAIN RESULT

$$\Xi(b,v) = \delta(q\cdot v)\, e^{iq\cdot b}\, F(p,p')$$

$q = \sum_\ell k_\ell =$ sum of graviton momenta emitted along worldline

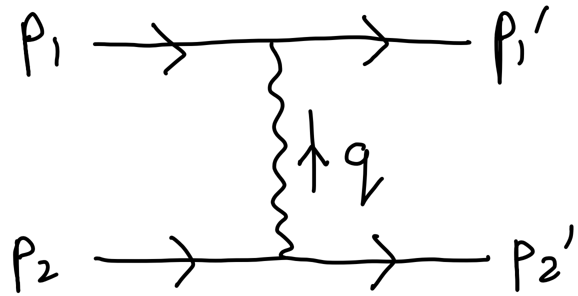
But only if we identify $\quad \Delta(\sigma) = \frac{|\sigma|}{2}, \quad \frac{p^\mu + p'^\mu}{2} = \hat{m} v^\mu$

The eikonal phase

Define the exponentiated eikonal phase χ in the classical $\hbar \to 0$ limit:

$$e^{i\chi} \equiv \frac{1}{4m_1m_2}\int \frac{d^Dq}{(2\pi)^{D-2}}\,\delta(q\cdot v_1)\,\delta(q\cdot v_2)\,e^{iq\cdot b}\,\langle \phi_1\phi_2|S|\phi_1\phi_2\rangle$$

where $b^\mu = b_2^\mu - b_1^\mu$, $q^\mu = p_1'^\mu - p_1^\mu = p_2^\mu - p_2'^\mu$

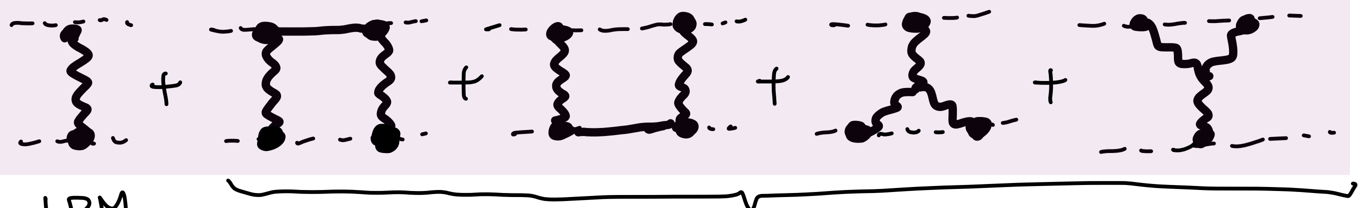


$$Z_{WQFT} = e^{i\chi} \quad \Longleftarrow$$

The free energy Z_{WQFT} is

$$Z_{WQFT} = \int Dh_{\mu\nu}\int D[x_i, a_i, b_i, c_i]\exp\Big[iS_{EH} - i\sum_{i=1}^{n}\frac{m_i}{2}\int d\tau_i\, g_{\mu\nu}\big(\dot{x}_i^\mu \dot{x}_i^\nu + a_i^\mu a_i^\nu + b_i^\mu c_i^\nu\big)\Big]$$

Eikonal phase given by sum of tree-level vacuum diagrams:

$i\chi =$  $+ \mathcal{O}(G^3)$

1PM

2PM

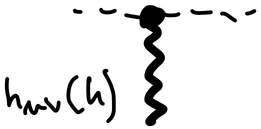
WQFT Feynman Rules

Promote $z_i^\mu(\tau)$ to a propagating degree of freedom:

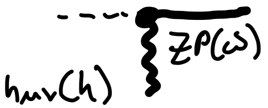
$$S = -2m_{Pl}^2 \int d^4x \sqrt{-g}\, R - \sum_i \frac{m_i}{2} \int d\tau_i\, g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu$$

$$\left.\begin{aligned} g_{\mu\nu}(x) &= \eta_{\mu\nu} + m_{Pl}^{-1} h_{\mu\nu}(x) \\ x_i^\mu(\tau_i) &= b_i^\mu + \tau_i v_i^\mu + z_i^\mu(\tau_i) \end{aligned}\right\}$$

Feynman rules from bulk action are standard; on the worldline w/ $z_1^\mu(\omega)$ only energy ω is conserved:



$$= -i \frac{m}{2m_{Pl}} e^{ik\cdot b} \delta(k\cdot v) v^\mu v^\nu \quad \Big\} \text{ source for } h_{\mu\nu}(k)$$



$$= \frac{m}{2m_{Pl}} e^{ik\cdot b} \delta(k\cdot v + \omega) \left(2\omega v^{(\mu} \delta_\rho^{\nu)} + v^\mu v^\nu k_\rho\right) \quad \Big\} \text{ mixing with } z^\rho(\omega)$$

Vertices couple linearly to $h_{\mu\nu}(k)$; known n-point expression:

$$V^{WL,\mu\nu}_{\rho_1\rho_2\cdots\rho_n}(k;\omega_1,\cdots,\omega_n) = i^{n-1} \frac{m}{m_{Pl}} e^{ik\cdot b} \delta\Big(k\cdot v + \sum_i \omega_i\Big) \times$$
$$\left(\frac{1}{2}\Big(\prod_i k_{\rho_i}\Big) v^\mu v^\nu + \sum_{i=1}^n \omega_i \Big(\prod_{j\neq i}^n k_{\rho_j}\Big) v^{(\mu}\delta^{\nu)}_{\rho_i} + \sum_{i<j}^n \omega_i\omega_j \Big(\prod_{l\neq i,j}^n k_{\rho_l}\Big) \delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_j}\right)$$

Worldline propagator

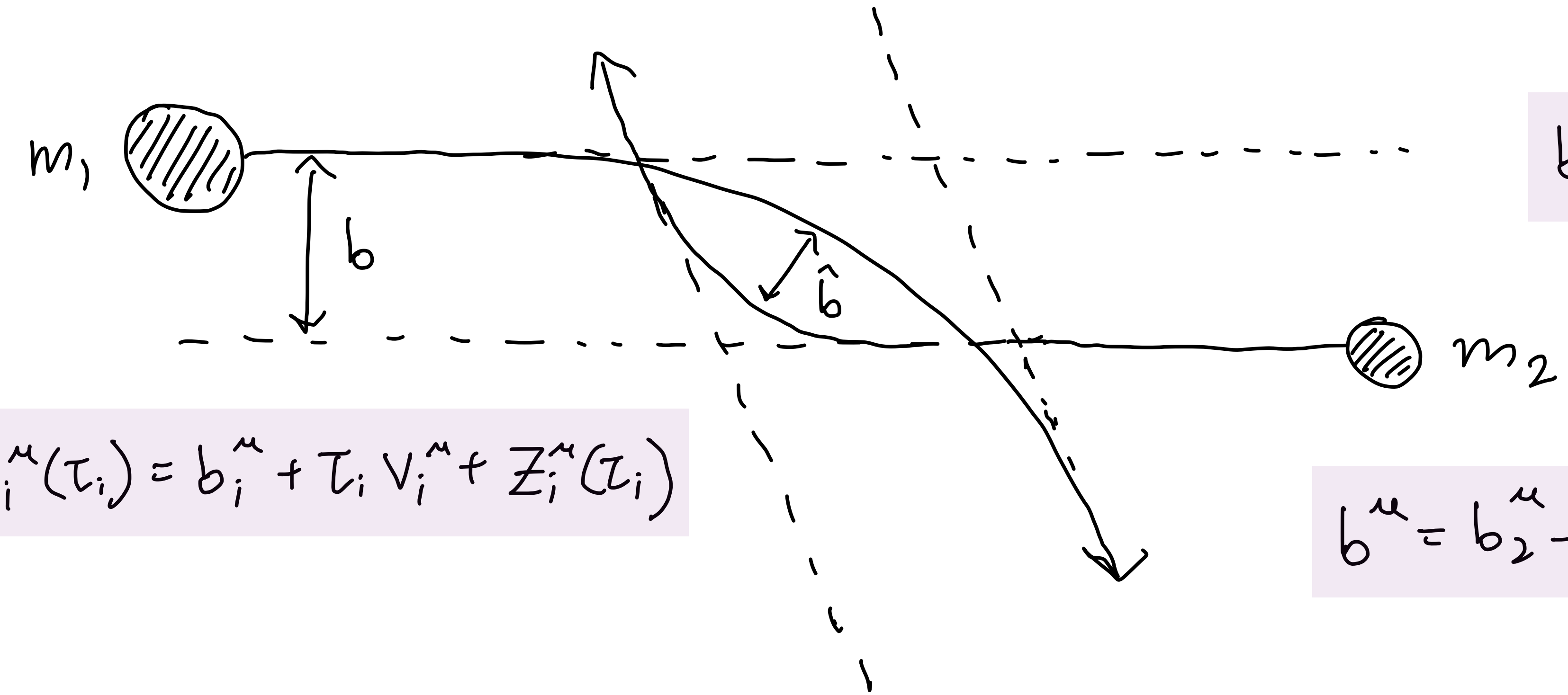
Pay attention to the $i\varepsilon$ prescription:

$\tau = \frac{\sigma}{2m}$

$$Z^\mu(\tau_1) \bullet\!\!-\!\!\bullet\, Z^\nu(\tau_2) = -i\,\frac{\eta^{\mu\nu}}{m}\int_\omega \frac{e^{-i\omega(\tau_1-\tau_2)}}{(\omega \pm i\varepsilon)^2} = \frac{i\eta^{\mu\nu}}{2m}\Big(|\tau_1-\tau_2| \pm (\tau_1-\tau_2)\Big)$$

Different signs give retarded/advanced propagators:

- retarded $\Rightarrow$ $\tau_1 > \tau_2$ $\Rightarrow$ identify b_i^μ, v_i^μ with incoming states
- advanced $\Rightarrow$ $\tau_1 < \tau_2$ $\Rightarrow$ " " " outgoing states
- time-symmetric $\Rightarrow$ $|\tau_1-\tau_2|$ $\Rightarrow$ " " " intermediate states



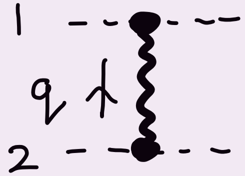
$$b = \hat{b}\cos\left(\frac{\theta}{2}\right)$$

$$X_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + Z_i^\mu(\tau_i)$$

$$b^\mu = b_2^\mu - b_1^\mu$$

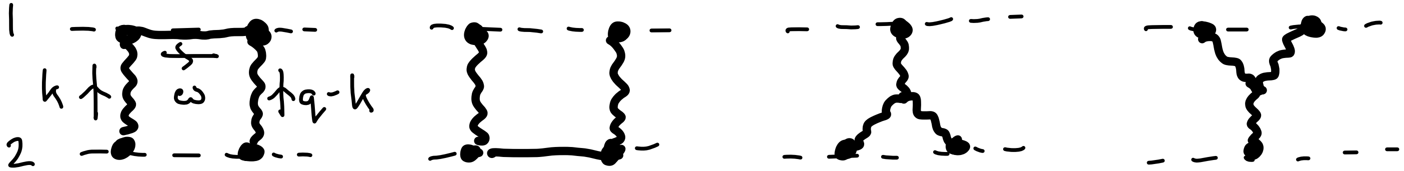
2PM Eikonal

Leading order described by a single diagram:



$$= \frac{m_1 m_2}{4 m_{Pl}^2} \int_q e^{iq\cdot b}\, \delta(q\cdot v_1)\, \delta(q\cdot v_2)\, v_1^\mu v_1^\nu \frac{P_{\mu\nu;\rho\sigma}}{q^2} v_2^\rho v_2^\sigma$$

Sub-leading $O(G^2)$ requires four diagrams:



E.g. $$\int_{k,q,\omega} \delta(k\cdot v_2)\, \delta((q-k)\cdot v_2)\, \delta(\omega - k\cdot v_1)\, \delta(\omega + (q-k)\cdot v_1) = \int_{k,q} \delta(q\cdot v_1)\, \delta(q\cdot v_2)\, \delta(k\cdot v_2)$$

Final integrated result, e.g. [Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove]

$$\chi = G m_1 m_2 \left(-\frac{2(2\gamma^2-1)}{\sqrt{\gamma^2-1}} \log b + \frac{3\pi}{4} \frac{(5\gamma^2-1)}{\sqrt{\gamma^2-1}} \frac{G(m_1+m_2)}{b} \right) + O(G^3)$$

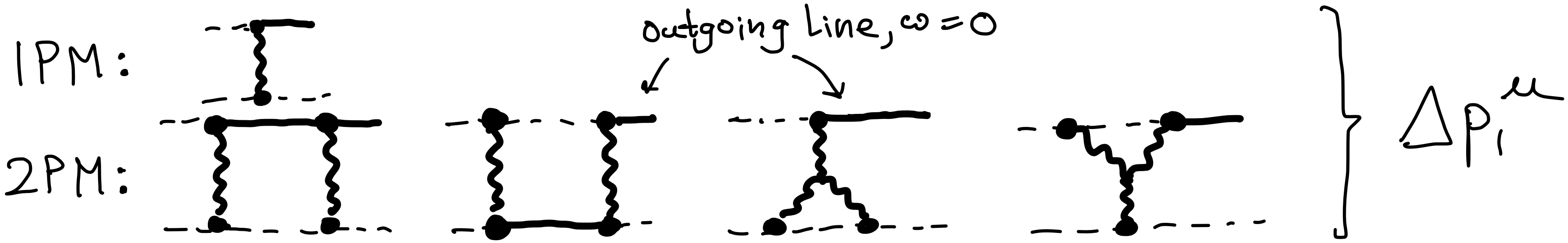
$\gamma = v_1 \cdot v_2$
$b = |b^\mu|$

Although all diagrams are tree level, we see "loop integrals" arising due to lack of momentum conservation at worldline vertices.

Deflections

$$\Delta p_i^\mu = -m_i\,\omega^2 \langle Z_i^\mu(\omega)\rangle_{\mathrm{WQFT}}\Big|_{\omega=0}$$

} Impulse given by graphs with a single outgoing line.



Diagrams closely related to the eikonal:

$$V^{\mathrm{WL},\mu\nu}_{\rho_1\cdots\rho_{n+1}}(k;\omega_1,\cdots,\omega_n,0) = \frac{\partial}{\partial b^{\rho_{n+1}}} V^{\mathrm{WL},\mu\nu}_{\rho_1\cdots\rho_n}(k;\omega_1,\cdots,\omega_n) \quad\Rightarrow\quad \boxed{\Delta p_{1,\mu} = -\frac{\partial \chi}{\partial b_1^\mu}}$$

Integration gives, c.f. [Kälin, Porto]:

$$\Delta p_1^\mu = \frac{G m_1 m_2 b^\mu}{b^2}\left(\frac{2(2\gamma^2-1)}{\sqrt{\gamma^2-1}} + \frac{3\pi}{4}\frac{(5\gamma^2-1)}{\sqrt{\gamma^2-1}}\frac{G(m_1+m_2)}{b}\right) + \mathcal{O}(G^3) \qquad \gamma = v_1\cdot v_2$$

Conclusions

- We've identified a link between QFT scattering amplitudes and observables in the WQFT.

- WQFT path integrals include $z_i^\mu(\tau_i)$ and $h_{\mu\nu}(x)$. Feynman rules on the worldline conserve only energy, so allow "loop integrals" from tree-level.

- Besides physical insight, WQFT offers an efficient path to PM integrands.

- Clear interpretation of the eikonal phase: $Z_{\text{WQFT}} = e^{i\chi}$

- We've also obtained 2PM 2-body / 3PM 3-body radiation $k^2\langle h^{\mu\nu}(k)\rangle_{k^2=0}$, former described by 5-point amplitude c.f. [Luna, Nicholson, O'Connell, White]

- Next steps:

  - push eikonal χ, impulse Δp_i^μ to 3PM / 4PM.
  - incorporate spin, radiation, finite-size effects.

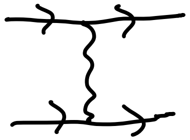
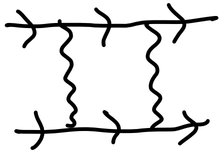
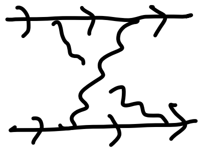
Thanks for listening!

Amplitudes-based PM methods

Use scalars as QFT avatars of non-spinning black holes: $g_{\mu\nu}(x) = \eta_{\mu\nu} + m_{pl}^{-1} h_{\mu\nu}(x)$

$$S = \int d^4x \sqrt{-g}\left(-2m_{pl}^2 R + \sum_i \left(g^{\mu\nu}\partial_\mu \phi_i^\dagger \partial_\nu \phi_i - m_i^2 \phi_i^\dagger \phi_i\right)\right) \qquad c = \hbar = 1,\ m_{pl} = \frac{1}{\sqrt{32\pi G}}$$

Calculate scalar-graviton amplitudes,

$\mathcal{M}^{(0)} =$  $\quad \mathcal{M}^{(1)} =$  $+$  $+ \dots$

$$\mathcal{M} = \underbrace{\mathcal{M}^{(0)}}_{\text{1PM}} + \underbrace{G\mathcal{M}^{(1)}}_{\text{2PM}} + \underbrace{O(G^2)}_{\text{3PM}}$$

To get physical observables (deflection, scattering angle, etc) different methods:

- EFT matching → [Cheung, Rothstein, Solon]
- Lippmann-Schwinger eq$^{\text{n}}$ → [Bjerrum-Bohr, Cristoffli, Damgaard, Vanhove]
- Eikonal phase ($\mathcal{N}=8$) → [di Vecchia, Heissenberg, Russo, Veneziano]
- " → [Parra-Martinez, Ruth, Zeng]
- Operator expectation values → [Kosower, Maybee, O'Connell, Vines]

3PM results: [Bern, Cheung, Roiban, Sawyer, Shen, Solon]
agreement with [Kälin, Porto] PM worldline EFT.