3-Loop Heavy Flavor Corrections to DI Scattering

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DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

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Outline

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2. Status of OME calculations
3. Calculation of the 3-loop OMEs
4. Single mass contributions
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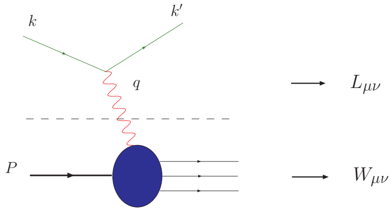
The deep–inelastic process



- Kinematic invariants:

$$Q^2 = -q^2, x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi)\right] |P\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Introduction

Why are Heavy Flavor Contributions important?

- They form a significant contribution to all structure functions particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(exp) \; {}^{+0.03}_{-0.02} \; (scale), \; {}^{+0.00}_{-0.07} \; (thy)\text{GeV} \qquad (\overline{\text{MS}}\text{-scheme})$$

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS and PS corrections are already final [J. Ablinger et al. (Nucl. Phys. B 844 (2011), B886 and B890 (2014))]

EIC: many more high precision data ahead for various detailed unpolarized and polarized precision measurements.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs). Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))] factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j.
For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Unpolarized and Polarized:

Leading Order:

1976 [Witten] and others later.

Next-to-Leading Order:

1992 full m^2/Q^2 dependence (numeric) [Laenen, van Neerven, Riemersma, Smith]

1996 analytic asymptotic [Buza, Matiounine, Smith, Migneron, van Neerven] , [Bierenbaum,Blümlein,Klein (2007)]

Next-to-Next-to-Leading Order:

2009 a series of moments: [Bierenbaum,Blümlein,Klein]

2010 the N_F terms

2014 the $A_{gq,Q}$, NS, PS,$A_{qg,Q}$, $A_{gg,Q}$ and all unpolarized log. terms; CC @$O(a_s^2)$

2015 A_{gg} nearly complete, HQ corrections to g_1, xF_3

2016 HQ $F_{L,2}^{W^+-W^-}$, HQ sum rules 2- and 3-loop, binomial topologies A_{Qg}

2017 all non-elliptic terms to A_{Qg} completed, unpolarized 2-mass terms: NS, $A_{gg,Q}$, PS

unp. 3-loop anom. dims $\propto T_F$

2018 2-mass $A_{gg,Q}$ unpol.

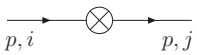
2019 2-mass A_{qq}^{PS} pol., pol. 3-loop anom. dims $\propto T_F$; complete analytic 2-loop PS pol+unpol WC

2020 2-mass $A_{gg,Q}$ pol., single mass A_{qq}^{PS} pol.

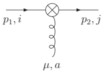
2021 $A_{gq,Q}$ pol., all pol. logarithmic contributions.

Calculation of the 3-loop operator matrix elements

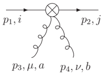
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



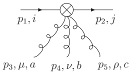
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



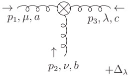
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] , \quad N \geq 3$$



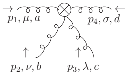
$$\begin{aligned} & g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\ & \Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & + (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & + (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & + (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \\ & + (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & + (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] , \\ & N \geq 4 \end{aligned}$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right] , \quad N \geq 2$$



$$\begin{aligned} & -ig \frac{1+(-1)^N}{2} f^{abc} \Bigg(\\ & \Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \Big] (\Delta \cdot p_1)^{N-2} \\ & + \Delta_\lambda \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \Big] \\ & \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\ & + \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2 \end{aligned}$$



$$\begin{aligned} & g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \\ & + f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) , \end{aligned}$$

$$\begin{aligned} & O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \\ & + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \\ & - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \\ & + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \\ & \times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\} \\ & - \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \ p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2 \end{aligned}$$

Method of calculation

- Resummation of operator insertion into propagator structure ($\Delta \cdot \Delta = 0$):

$$\sum_{N=0}^{\infty} t^N (\Delta \cdot k)^N = \frac{1}{1 - t\,\Delta \cdot k}$$

- Reduction to master integrals using IBP relations implemented in `Reduze2` [Manteuffel, Studerus (2012)]
.
- Solution of master integrals obtained by different methods:
  - direct integration using Mellin-Barnes and ${}_pF_q$-techniques
  - differential equations in resummation variable t
- method of arbitrary high moments, i.e. reconstructing all-N solution from a large number of fixed moments

$\Rightarrow$ All these methods use the packages `Sigma`, `EvaluateMultiSums` [Schneider (2007-)] and `HarmonicSums` [Ablinger et al. (2010-)] which have been developed with these calculations.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\rm NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]\\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]\\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.\\
&\quad\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]\\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]\\
&\quad + a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]\\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]\\
&\quad + a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]\\
&\quad + a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.\\
&\quad\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

The variable flavor number scheme

- Matching conditions for parton distribution functions: important for LHC physics

$$
\begin{aligned}
f_k(N_F+2)+f_{\overline{k}}(N_F+2) &= A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)\\
&\quad+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,\\
f_Q(N_F+2)+f_{\overline{Q}}(N_F+2) &= A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,\\
\Sigma(N_F+2) &= \left[A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)\\
&\quad+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,\\
G(N_F+2) &= A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.
\end{aligned}
$$

$m_c/m_b \sim 3$. Decoupling of these two flavors at once; two mass OMEs @ 3 loops.

The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

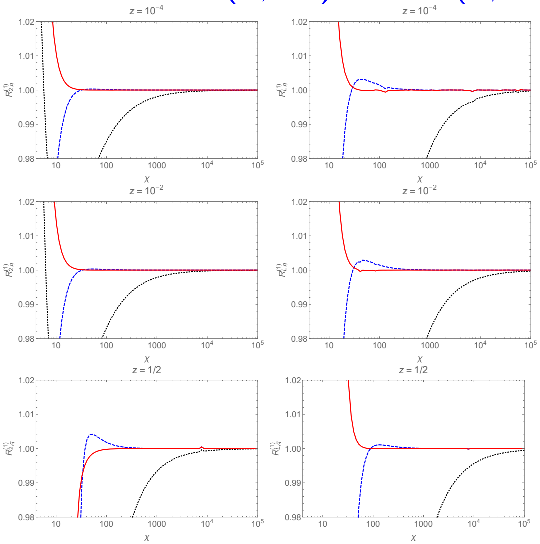


Figure 1: The ratios $R^{(1)}_{2,q}$ (left) and $R^{(1)}_{L,q}$ (right) as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines : $O((m^2/Q^2)^2)$ improved.

Polarized OMEs

- Polarized OMEs for heavy flavor production at 2-loop order have been calculated before.
  [Buza et al. (1996), Klein (2009), Hasselhuhn (2013)]
- Calculation of OMEs relied on tensor-decomposition in order to arrive at Larin scheme, i.e.

$$\gamma_5 \to \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

- We found out that a change of projector can accomplish the same:

$$\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left[\not p\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma G_q\right] \to \epsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma\mathrm{tr}\left[\not p\gamma^\mu\gamma^\nu G_q\right]$$

$\Rightarrow$ this allows to apply all technologies for the unpolarized OMEs directly to the polarized ones, i.e.

1. the terms $\sim T_F$ of the polarized 3-loop anomalous dimensions from a massive calculation
   [Behring et al. (Nucl. Phys. B948 (2020))]
2. $A_{qq,Q}^{(3),\mathrm{PS}}$, $A_{Qq}^{(3),\mathrm{PS}}$, $A_{gq,Q}^{(3)}$ (single and 2-mass contributions)
   [Ablinger et al. (Nucl. Phys. B952,953,955 (2020)), Behring et al. (Nucl. Phys. B964 (2021))]

Polarized anomalous dimensions

$$
\begin{aligned}
\gamma_{qg}^{(2)} = {} & C_A N_F^2 T_F^2 \Biggl\{ -\frac{5N^2+8N+10}{N(N+1)(N+2)} \frac{128}{9} S_{-2} - \frac{64 P_8}{9N(N+1)^2(N+2)^2} S_1^2 \\
& - \frac{64 P_9}{9N(N+1)^2(N+2)^2} S_2 + \frac{64 P_{25}}{27N(N+1)^3(N+2)^3} S_1 + \frac{16 P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4} \\
& + p_{qg}^{(0)}(N) \Biggl(\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{64}{9} S_3 + \frac{128}{3} S_{-3} + \frac{128}{3} S_{2,1} \Biggr) \Biggr\} \\
& + C_F N_F^2 T_F^2 \Biggl\{ \frac{5N^2+3N+2}{N^2(N+1)(N+2)} \frac{32}{3} S_2 + \frac{10N^3+13N^2+29N+6}{N^2(N+1)(N+2)} \frac{32}{9} S_1^2 \\
& - \frac{32 P_{12}}{27N^2(N+1)^2(N+2)} S_1 + \frac{4 P_{38}}{27(N-1)N^5(N+1)^5(N+2)^4} \\
& + p_{qg}^{(0)}(N) \Biggl(-\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{320}{9} S_3 \Biggr) \Biggr\} \\
& + C_A C_F N_F T_F \Biggl\{ -128 \frac{N^3-7N^2-6N+4}{N^2(N+1)^2(N+2)} S_{-2,1} + \frac{32 P_5}{N^2(N+1)^2(N+2)} S_{-3} \\
& + \frac{16 P_{18}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 - \frac{16 P_{24}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{8 P_{27}}{9(N-1)N^3(N+1)^3(N+2)^2} S_1^2 + \frac{8 P_{29}}{3(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{P_{37}}{27(N-1)N^5(N+1)^5(N+2)^4} + p_{qg}^{(0)}(N) \Biggl[\Biggl(\frac{640}{3} S_3 - 384 S_{2,1} \Biggr) S_1 + \frac{32}{3} S_1^4 \\
& + 160 S_1^2 S_2 - 64 S_2^2 + \bigl(192 S_1^2 + 64 S_2\bigr) S_{-2} + 96 S_{-2}^2 + 224 S_{-4} - 64 S_{2,-2} + 64 S_{3,1} \\
& + 192 S_{2,1,1} - 256 S_{-2,1,1} - 192 S_1 \zeta_3 \Biggr] - \frac{192 P_{17}}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& + \Biggl(\frac{16 P_{16}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 + \frac{16 P_{35}}{27(N-1)N^4(N+1)^4(N+2)^4} \Biggr) S_1 \\
& + \Biggl[-\frac{32 P_{15}}{N^3(N+1)^3(N+2)} + \frac{128\bigl(N^3-13N^2-14N-2\bigr)}{N^2(N+1)^2(N+2)} S_1 \Biggr] S_{-2} \\
& + \frac{96 N(N+1) p_{qg}^{(0)}(N)^2}{N-1} S_{2,1} \Biggr\} \\
& + C_A^2 N_F T_F \Biggl\{ -\frac{64 P_{11}}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2,1} - \frac{16 P_{20}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{32 P_{21}}{3(N-1)N^2(N+1)^2(N+2)^2} S_{-3} - \frac{8 P_{22}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 \\
& + \frac{16 P_{32}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_1^2 + \frac{16 P_{33}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_2 \\
& - \frac{8 P_{39}}{27(N-1)^2N^5(N+1)^5(N+2)^5} + p_{qg}^{(0)}(N) \Biggl[-\frac{32 P_{10}}{3(N-1)N(N+1)(N+2)} S_{2,1} \\
& + \Biggl(-\frac{704}{3} S_3 + 128 S_{2,1} + 512 S_{-2,1} \Biggr) S_1 - 512 S_{-3} S_1 - \frac{16}{3} S_1^4 - 160 S_1^2 S_2 - 16 S_2^2 - 32 S_4 \\
& + \Biggl(-192 S_1^2 + 320 S_2 \Biggr) S_{-2} - 96 S_{-2}^2 + 96 S_{-4} - 448 S_{2,-2} - 128 S_{3,1} + 512 S_{-3,1} \\
& - 768 S_{-2,1,1} + 192 S_1 \zeta_3 \Biggr] + \frac{96(N-2)(N+3) P_4}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& + \Biggl(\frac{8 P_{19}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 - \frac{8 P_{36}}{27(N-1)^2N^4(N+1)^4(N+2)^4} \Biggr) S_1 \\
& + \Biggl(-\frac{64 P_{13}}{(N-1)N^2(N+1)^2(N+2)^2} S_1 + \frac{32 P_{30}}{9(N-1)N^3(N+1)^3(N+2)^3} \Biggr) S_{-2} \Biggr\} \\
& + C_F^2 N_F T_F \Biggl\{ \frac{P_{31}}{N^5(N+1)^5(N+2)} - \frac{8 P_3}{3N^2(N+1)^2(N+2)} S_1^3 - \frac{16 P_6}{3N^2(N+1)^2(N+2)} S_3 \\
& + \frac{64 P_{14}}{N^3(N+1)^2(N+2)} S_{-2} - \frac{8 P_{23}}{N^3(N+1)^3(N+2)} S_1^2 + \frac{8 P_{26}}{N^3(N+1)^3(N+2)} S_2 \\
& + p_{qg}^{(0)}(N) \Biggl[\Biggl(-\frac{704}{3} S_3 + 256 S_{2,1} \Biggr) S_1 - 256 S_{-3} S_1 - \frac{16}{3} S_1^4 - 48 S_2^2 - 160 S_4 - 64 S_{-2}^2 \\
& - 192 S_{-4} - \frac{128}{N(N+1)} S_{2,1} - 128 S_{2,-2} + 64 S_{3,1} + 256 S_{-3,1} - 192 S_{2,1,1} \Biggr] \\
& + \frac{96(N-1)\bigl(3N^2+3N-2\bigr)}{N^2(N+1)^2} \zeta_3 - 256 \frac{2-N+N^2}{N^2(N+1)(N+2)} \left[S_{-2} S_1 - S_{-2,1} \right] \\
& + \Biggl(-\frac{8 P_{28}}{N^4(N+1)^4(N+2)} - \frac{8 P_7}{N^2(N+1)^2(N+2)} S_2 \Biggr) S_1 - \frac{128(N-1)}{(N+1)^2(N+2)} S_{-3} \Biggr\},
\end{aligned}
$$

also $\gamma_{qq}^{(2),\mathrm{PS}}$ complete. Agree with [Moch, Vermaseren, Vogt (2014)] + all other terms $\propto T_F$.

Results: $A_{Qg}^{(3)}$

- 2010: N_F terms
- 2017: calculation of all ζ-color factors except of the pure rational and ζ_3 terms
- 1000 even moments are available; 8000 moments for the T_F^2 terms
- Difference equations for the T_F^2 available:
  (d,o) = (1407;46) [$T_F^2 C_A$], (447;24) [$T_F^2 C_A \zeta_3$]; (654;27) [$T_F^2 C_F$], (283;15) [$T_F^2 C_F \zeta_3$].
- The 1st difference eq. is more voluminous than the biggest for the massless WCs
  [JB, Kauers, Klein, Schneider, (2008)]
- 3-loop massive form factor: 2 equal mass cases: (d,o) = (1324;55).
- The solution of the associated differential equations lead to an exponential singularity in $z \in [0,1]$ both for the rational and ζ_3 T_F^2 terms
- The unification of both difference eqs. is necessary to cancel this spurious singularity.
- New method to solve the associated differential eq.: several formal Laurent series to a given finite order around $z_0 \in [0,1]$ to map out the whole region with overlapping convergence radii.
  [J. Ablinger, J. Blümlein, C. Schneider, 2021]
- In this way a full representation, tuneable to any precision is obtained.
  Here it is irrelevant, if elliptic, hyper-elliptic or whatsoever structures are present.
- One should notice that even any HPL solution finally needs an efficient numerical solution.
- The generation of high number of moments for the non T_F^2 terms is underway.

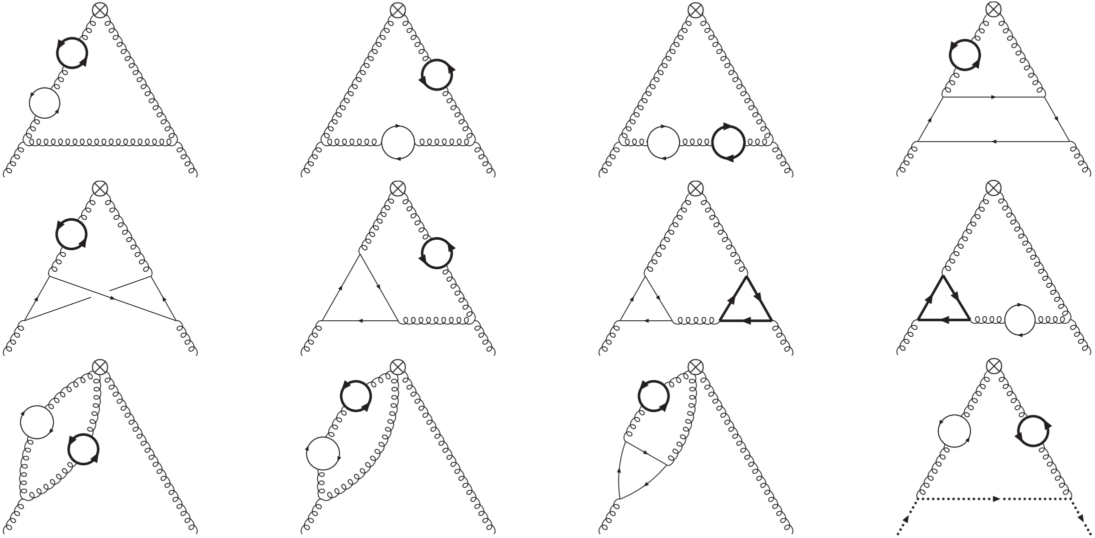
Single mass case: mathematical structures

- Before 1998:
  classical polylogarithms, Nielsen integrals [Lewin, 1958,1981] , [Kölbig, Mignaco, Remiddi, 1970; Kölbig, 1986]

- 1997: Shuffle algebras [Hoffman] , 2003 algebraic relations [Blümlein]
- 1998: Harmonic sums [Vermaseren;Blümlein & Kurth]
- 1998: generalized harmonic sums and polylogarithms [Goncharov] , 2001: [Moch,Uwer,Weinzierl]
- 1999: Harmonic polylogarithms [Remiddi,Vermaseren]
- 2004: infinite binomial and inverse binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2009: structural relations of harmonic sums [Bümlein]
- 2009: MZV data mine [Blümlein, Broadhurst, Vermaseren]
- 2011: cyclotomic harmonic sums and polylogarithms [Ablinger,Blümlein,Schneider]
- 2013: detailed theory generalized sums [Ablinger,Blümlein,Schneider]
- 2014: finite binomial and inverse binomial sums [Ablinger,Blümlein,Raab,Schneider]
- 2014– complete elliptic integrals in Feynman diagrams: [very many authors: Broadhurst et al., Remiddi et al., Weinzierl et al., Duhr et al., ...]
- 2017: method of arbitrary high moments [Blümlein, Schneider]
- 2021: Iterated integrals over letters induced by quadratic forms [Ablinger,Blümlein,Schneider]

List far from being complete.

2-mass contributions



2-mass contributions: mathematical functions

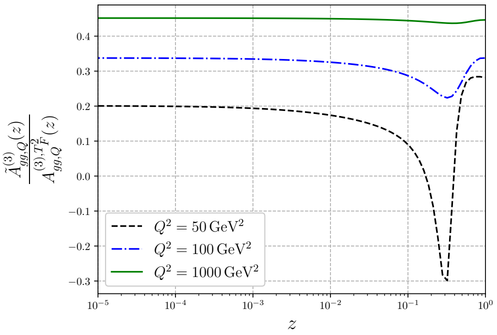
$A_{qq,Q}^{(3),\mathrm{NS}}, A_{gq,Q}^{(3)}$	Harmonic Sums [Vermaseren '98; Blümlein, Kurth '98] $\sum_{i=1}^{N} \frac{1}{i^3} \sum_{j=1}^{i} \frac{1}{j}$	HPLs [Remiddi, Vermaseren '99] $\int_0^x \frac{d\tau_1}{1+\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{1-\tau_2}$
$A_{gg,Q}^{(3)}$	Generalized harmonic and binomial sums [Ablinger, Blümlein, Schneider '13] [Ablinger, Blümlein, Raab, Schneider '14] $\sum_{i=1}^{N} \frac{4^i (1-\eta)^{-i}}{i \binom{2i}{i}} \sum_{j=1}^{i} \frac{(1-\eta)^j}{j^2}$	Iterated integrals over root and η valued letters [Ablinger, Blümlein, Raab, Schneider '14] $\int_0^x d\tau_1 \frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2}$
$A_{Qq}^{(3),\mathrm{PS}}$	–	Iterated integrals over root valued letters with restricted support $\theta(x-\eta_+) \int_0^{x(1-x)/\eta} d\tau \frac{\sqrt{1-4\tau}}{\tau}$

Results: $A_{gg,Q}^{(3)}$

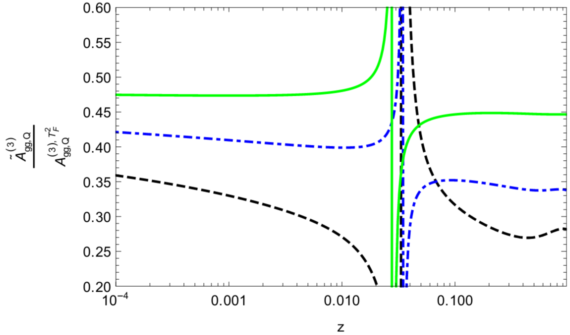
$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N) \quad = \quad & \frac{1}{2}\left(1+(-1)^N\right)\Biggl\{T_F^3\Biggl\{\frac{32}{3}\left(L_1^3+L_2^3\right)+\frac{64}{3}L_1L_2\left(L_1+L_2\right)+32\zeta_2\left(L_1+L_2\right)+\frac{128}{9}\zeta_3\Biggr\} \\
+ \quad & C_F T_F^2\Biggl\{\cdots+32\left(\mathrm{H}_0^2(\eta)-\frac{1}{3}S_2\right)S_1+\frac{128}{3}S_{2,1}-\frac{64}{3}S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right) \\
& -\frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)}\left(\frac{\eta}{1-\eta}\right)^N\Biggl[\mathrm{H}_0^2(\eta) \\
& -2\mathrm{H}_0(\eta)S_1\left(\frac{\eta-1}{\eta},N\right)-2S_2\left(\frac{\eta-1}{\eta},N\right)+2S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\Biggr]+\ldots\Biggr\} \\
+ \quad & C_A T_F^2\Biggl\{\cdots+\Biggl[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} \\
& +\frac{8P_{37}\mathrm{H}_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)}+\frac{2P_{23}\mathrm{H}_0^2(\eta)}{9\eta(N-1)N(N+1)^2}+\frac{32}{27}\mathrm{H}_0^3(\eta)+\frac{128}{9}\mathrm{H}_{0,0,1}(\eta) \\
& +\frac{64}{9}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{128}{9}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)\Biggr]S_1 \\
& +\frac{2^{-1-2N}P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}\Biggl\{\frac{1}{2}\mathrm{H}_0^2(\eta) \\
& S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)\Biggr\}+\ldots\Biggr\}, \qquad \eta=\frac{m_c^2}{m_b^2}.
\end{aligned}
$$

Results: $A_{gg,Q}^{(3)}$

The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$:



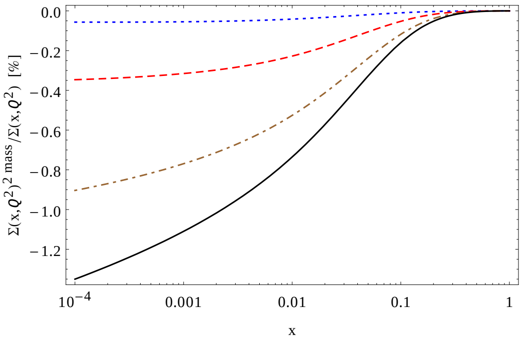
unpolarized case



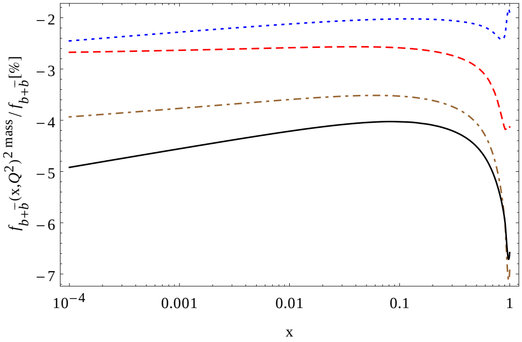
‘ polarized case

The 2-mass variable flavor number scheme at NLO

$\Sigma(x, Q^2)^{\text{2-mass}} / \Sigma(x, Q^2)$



$f_{b+\bar{b}}(x, Q^2)^{\text{2-mass}} / f_{b+\bar{b}}(x, Q^2)$



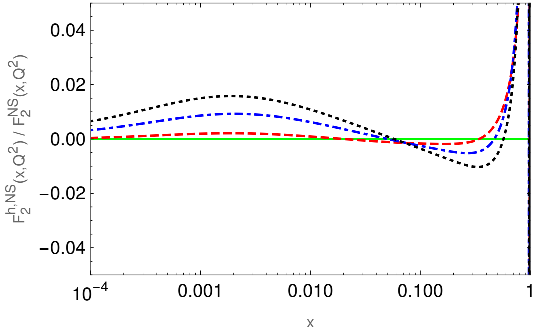
- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.
- For the PDFs the `NNLO` variant of ABMP16 with $N_f = 3$ flavors was used.
  Alekhin et al., Phys. Rev. D 96 (2017) 1

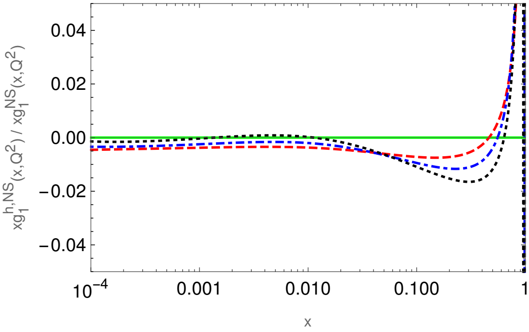
HQ contributions in N^{3}LO QCD analyses

[Blümlein & Saragnese, 2021]

$$F_2(x,Q^2)^{\rm NS}(N) = E_{\rm NS}(Q^2,Q_0^2;N) F_2(x,Q_0^2)^{\rm NS}(N)$$

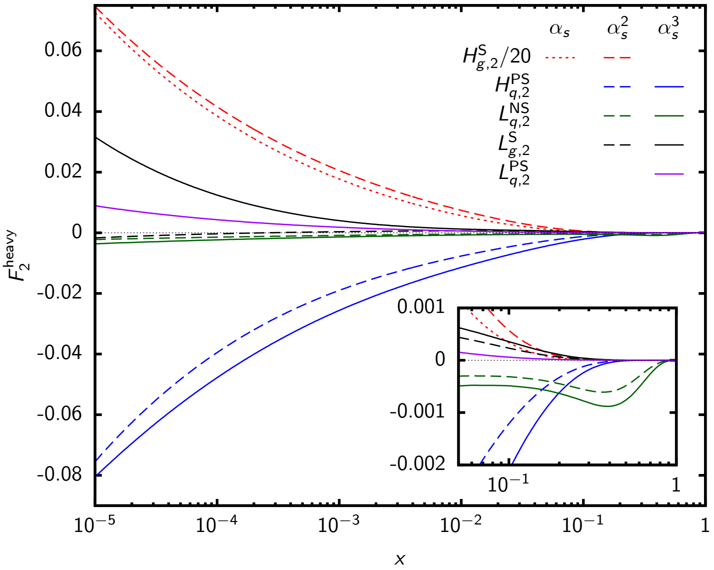
evolution operator: $E_{\rm NS}(Q^2,Q_0^2;N)$
$F_2(x,Q_0^2)^{\rm NS}(z)$ is measured. $\alpha_s(M_Z)$ from a one-parameter fit (knowing m_c and m_b).





⟹ Future high luminosity analyses at the EIC or LHeC.

Heavy Flavor contribution to F_2



$Q^2 = 100\ \text{GeV}^2$

Conclusions and Outlook

- Most of the massive 3–loop OMEs (VFNS) and asymptotic Wilson coefficients have been calculated in the unpolarized and polarized case (for single- and two-mass).
- $A_{gg,Q}^{(3)}$ nearly completed; $A_{Qg}^{(3)}$ going to come.
- Hugh difference equations can be solved either fully analytically or in representations which can be tune to high precision.
- Various new computing technologies were developed for massive Feynman diagram calculations
- The properties of the contributing mathematical function spaces were worked out.
- In the 2-loop case progress has been made in the analytic calculation of power corrections, relevant for the region of smaller virtualities.
- (T_F contributions to the) 3–loop anomalous dimensions appear as by-product,
  [see also the talks by P. Marquard and S. Moch.]
- applications also to QED corrections in $e^+e^- \to Z^*/\gamma^*$ [K. Schönwald's talk.]