

# Coaction for Feynman Integrals

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We conjecture three compatible coactions on Feynman integrals:

- **Local coaction** on multiple polylogarithms (MPLs), elliptic multiple polylogarithms, etc.

This one is well known. [\[Goncharov, Brown\]](#)

Applies in the Laurent expansion of Feynman integrals in the parameter  $\epsilon$  of dimensional regularization.

- **Global coaction** on generalized hypergeometric functions.

Applies to Feynman integrals in dimensional regularization without taking the Laurent expansion.

Conjectured to exist for arbitrary Feynman integrals; examples found with integer-based parameters; since proven for Lauricella functions [\[Brown, Dupont\]](#).

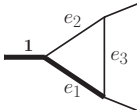
- **Diagrammatic coaction**

1st claim: the output from the other two coactions are compatible with each other and can be repackaged as Feynman integrals.

2nd claim: the coaction output can also be obtained by applying graphical operations before evaluating.

- Coaction is naturally compatible with discontinuities and differential operators, so we hope it can be applied to new computations
- Dimensional regularization is essential
- Formally, we distinguish motivic and de Rham MPLs, hypergeometric functions, etc. or use single-valued versions
- This talk: general principles and 1- and 2-loop examples

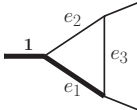
## Example: 3 equivalent expressions, 3 compatible coactions



$$= \frac{e^{\gamma_E \epsilon} \Gamma(1 + \epsilon)}{\epsilon(1 - \epsilon)} (m^2)^{-1 - \epsilon} {}_2F_1 \left( 1, 1 + \epsilon; 2 - \epsilon; \frac{p^2}{m^2} \right)$$

$$= \frac{1}{p^2} \left[ \frac{\log \left( \frac{m^2}{m^2 - p^2} \right)}{\epsilon} + \text{Li}_2 \left( \frac{p^2}{m^2} \right) + \log^2 \left( 1 - \frac{p^2}{m^2} \right) + \log(m^2) \log \left( 1 - \frac{p^2}{m^2} \right) \right] + \mathcal{O}(\epsilon)$$

## Example: 3 equivalent expressions, 3 compatible coactions

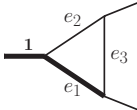


$$= \frac{e^{\gamma_E \epsilon} \Gamma(1 + \epsilon)}{\epsilon(1 - \epsilon)} (m^2)^{-1 - \epsilon} {}_2F_1 \left( 1, 1 + \epsilon; 2 - \epsilon; \frac{p^2}{m^2} \right)$$

$$= \frac{1}{p^2} \left[ \frac{\log \left( \frac{m^2}{m^2 - p^2} \right)}{\epsilon} + \text{Li}_2 \left( \frac{p^2}{m^2} \right) + \log^2 \left( 1 - \frac{p^2}{m^2} \right) + \log(m^2) \log \left( 1 - \frac{p^2}{m^2} \right) \right] + \mathcal{O}(\epsilon)$$

$$\begin{aligned} \Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\ \Delta(\log^2 z) &= 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1 \\ \Delta(\text{Li}_2(z)) &= 1 \otimes \text{Li}_2(z) + \text{Li}_2(z) \otimes 1 + \text{Li}_1(z) \otimes \log z \end{aligned}$$

## Example: 3 equivalent expressions, 3 compatible coactions



$$= \frac{e^{\gamma_E \epsilon} \Gamma(1 + \epsilon)}{\epsilon(1 - \epsilon)} (m^2)^{-1 - \epsilon} {}_2F_1 \left( 1, 1 + \epsilon; 2 - \epsilon; \frac{p^2}{m^2} \right)$$

$$= \frac{1}{p^2} \left[ \frac{\log \left( \frac{m^2}{m^2 - p^2} \right)}{\epsilon} + \text{Li}_2 \left( \frac{p^2}{m^2} \right) + \log^2 \left( 1 - \frac{p^2}{m^2} \right) + \log(m^2) \log \left( 1 - \frac{p^2}{m^2} \right) \right] + \mathcal{O}(\epsilon)$$

$$\Delta \left( {}_2F_1(\alpha, \beta; \gamma; x) \right) = {}_2F_1(1 + a\epsilon, b\epsilon; 1 + c\epsilon; x) \otimes {}_2F_1(\alpha, \beta; \gamma; x)$$

$$- \frac{b\epsilon}{1 + c\epsilon} {}_2F_1(1 + a\epsilon, 1 + b\epsilon; 2 + c\epsilon; x)$$

$$\otimes \frac{\Gamma(1 - \beta)\Gamma(\gamma)}{\Gamma(1 - \beta + \alpha)\Gamma(\gamma - \alpha)} x^{1 - \alpha} {}_2F_1 \left( \alpha, 1 + \alpha - \gamma; 1 - \beta + \alpha; \frac{1}{x} \right)$$

## Example: 3 equivalent expressions, 3 compatible coactions

$$\begin{aligned}
 & \text{Diagram: A vertex with three external lines labeled } e_1, e_2, e_3. \text{ Line } e_1 \text{ is horizontal to the left, } e_2 \text{ goes up-right, } e_3 \text{ goes down-right.} \\
 &= \frac{e^{\gamma_E \epsilon} \Gamma(1+\epsilon)}{\epsilon(1-\epsilon)} (m^2)^{-1-\epsilon} {}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right) \\
 &= \frac{1}{p^2} \left[ \frac{\log\left(\frac{m^2}{m^2-p^2}\right)}{\epsilon} + \text{Li}_2\left(\frac{p^2}{m^2}\right) + \log^2\left(1 - \frac{p^2}{m^2}\right) + \log(m^2) \log\left(1 - \frac{p^2}{m^2}\right) \right] + \mathcal{O}(\epsilon)
 \end{aligned}$$

$$\begin{aligned}
 \Delta \left[ \text{Diagram: Same as above} \right] &= \text{Diagram: A circle with a vertical line through its center labeled } e_1 \text{ } \otimes \left( \begin{aligned} &\text{Diagram: Same as above, but line } e_1 \text{ is dashed red.} \\ &+ \frac{1}{2} \text{Diagram: Same as above, but lines } e_1 \text{ and } e_2 \text{ are dashed red.} \end{aligned} \right) \\
 &+ \text{Diagram: A bubble diagram with two horizontal external lines labeled } 1 \text{ and } 1, \text{ top arc labeled } e_1, \text{ bottom arc labeled } e_2 \text{ } \otimes \text{Diagram: Same as above, but lines } e_1 \text{ and } e_2 \text{ are dashed red.}
 \end{aligned}$$

## Preview of 2-loop example

Sunset with two massive propagators.

$$\Delta \left[ \text{Sunset}^{(1)} \right] = \text{Figure-eight} \otimes \left( \text{Sunset}^{(1)}_{\text{red}} + \text{Sunset}^{(1)}_{\text{orange}} + \text{Sunset}^{(1)}_{\text{blue}} + \text{Sunset}^{(1)}_{\text{green}} \right) \\ + \text{Sunset}^{(1)}_{\text{red}} \otimes \text{Sunset}^{(1)}_{\text{orange}} + \text{Sunset}^{(2)}_{\text{blue}} \otimes \text{Sunset}^{(1)}_{\text{blue}} + \text{Sunset}^{(3)}_{\text{green}} \otimes \text{Sunset}^{(1)}_{\text{green}},$$

General formula for coaction on integrals:

$$\Delta \left( \int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$



## General formula for coaction on integrals

$$\Delta \left( \int_{\gamma} \omega \right) = \sum_{i,j} c_{ij} \int_{\gamma} \omega_i \otimes \int_{\gamma_j} \omega$$

- All three coactions have this structure.
- Satisfies axioms of coaction.
- Claim of this formula: there exist sets  $\{\omega_i\}$ ,  $\{\gamma_j\}$ ,  $\{c_{ij}\}$  to make it true.
- $\{\omega_i\}$  generate cohomology
- $\{\gamma_j\}$  generate homology
- $\{c_{ij}\}$  are rational in  $\epsilon$ , algebraic in other parameters/kinematic variables; uniquely fixed by choices of  $\{\omega_i\}$ , and  $\{\gamma_j\}$

## General formula for coaction on integrals

$$\Delta \left( \int_{\gamma} \omega \right) = \sum_{i,j} c_{ij} \int_{\gamma} \omega_i \otimes \int_{\gamma_j} \omega$$

Choice of bases:

- $\{\omega_i\}$   
Left entries  $\int_{\gamma} \omega_i$  related to  $\int_{\gamma} \omega$  by standard **IBP reduction**. Choose them to be pure.
- $\{\gamma_j\}$   
Right entries  $\int_{\gamma_j} \omega$  related to  $\int_{\gamma} \omega$  by **change of contour**. Can start with all possible cuts, but there are relations among them.

An important relation for 1-loop integrals:

$$\sum_i C_i I_n + \sum_{i < j} C_{ij} I_n = -\epsilon I_n \mod i\pi$$

applies to subgraphs of multiloop groups.

Principle: no uncut loops needed in right entries. Use only “**genuine  $L$ -loop cuts.**”

Compact version of master formula:

$$\Delta \left( \int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

Look for bases such that

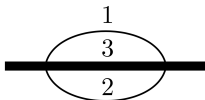
$$\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$$

# Principles of the diagrammatic coaction

$$\Delta \left( \int_{\gamma} \omega \right) = \sum_{i,j} c_{ij} \int_{\gamma} \omega_i \otimes \int_{\gamma_j} \omega$$

- Left entries related by IBP, only  $L$ -loop diagrams
- Right entries are **genuine  $L$ -loop cuts**
- Coefficients  $c_{ij}$  are explicit for  $L = 1$ , ad hoc for  $L > 1$  as of now.
- Can dualize bases either before or after initial choice
- Consistent with degenerate limits
- UV/IR pole cancellation
- $\gamma$  can be a cut contour. Gives coaction on cut integrals.

## First 2-loop example: 1-mass sunset



$$S(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5; D; p^2, m^2) = \left( \frac{e^{\gamma_E \epsilon}}{i\pi^{D/2}} \right)^2 \int d^D k d^D l \frac{[(k+l)^2]^{-\nu_4} [(l+p)^2]^{-\nu_5}}{[k^2]^{\nu_1} [l^2]^{\nu_2} [(k+l+p)^2 - m^2]^{\nu_3}}$$

2 master integrals, normalized to  $1 + \mathcal{O}(\epsilon)$ .

$$\begin{aligned} S^{(1)} &= \epsilon^2 (p^2 - m^2) S(1, 1, 1, 0, 0; 2 - 2\epsilon; p^2, m^2) \\ &= (m^2)^{-2\epsilon} \left( 1 - \frac{p^2}{m^2} \right) e^{2\gamma_E \epsilon} \Gamma(1 + 2\epsilon) \Gamma(1 - \epsilon) \Gamma(1 + \epsilon) {}_2F_1 \left( 1 + 2\epsilon, 1 + \epsilon; 1 - \epsilon; \frac{p^2}{m^2} \right) \end{aligned}$$

$$\begin{aligned} S^{(2)} &= -\epsilon^2 S(1, 1, 1, -1, 0; 2 - 2\epsilon; p^2, m^2) \\ &= (m^2)^{-2\epsilon} e^{2\gamma_E \epsilon} \Gamma(1 + 2\epsilon) \Gamma(1 - \epsilon) \Gamma(1 + \epsilon) {}_2F_1 \left( 2\epsilon, \epsilon; 1 - \epsilon; \frac{p^2}{m^2} \right) \end{aligned}$$

Hence only two independent integration contours, e.g. the maximal cuts.

## First 2-loop example: 1-mass sunset

Maximal cut integral:

$$\mathcal{C}_{123} S^{(1)} \sim \int dk_0 k_0^{-1-2\epsilon} \left( p^2 - m^2 + 2\sqrt{p^2} k_0 \right)^{-1-2\epsilon} \left( p^2 + 2\sqrt{p^2} k_0 \right)^{2\epsilon}$$

After taking three residues, there is one integration left, of the hypergeometric form  ${}_2F_1$ .

Can choose two independent contours:

$$\Gamma_{123}^{(1)} : k_0 \in \left[ -\frac{\sqrt{p^2}}{2}, 0 \right] \quad \Gamma_{123}^{(2)} : k_0 \in \left[ \frac{m^2 - p^2}{2\sqrt{p^2}}, 0 \right]$$

Results:

$$\int_{\Gamma_{123}^{(1)}} \omega^{(1)} = 2\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (p^2 - m^2)^{-2\epsilon} {}_2F_1 \left( -2\epsilon, 1+2\epsilon; 1-\epsilon; \frac{p^2}{p^2 - m^2} \right)$$

$$\int_{\Gamma_{123}^{(2)}} \omega^{(1)} = 4\epsilon e^{2\gamma_E \epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-4\epsilon)} (p^2)^{2\epsilon} (p^2 - m^2)^{-4\epsilon} {}_2F_1 \left( -2\epsilon, -\epsilon; -4\epsilon; 1 - \frac{m^2}{p^2} \right)$$

## First 2-loop example: 1-mass sunset

Can arrive at compact/dual form  $\Delta \left( \int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$  by requiring  $\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$ .

Solution:

$$\gamma_{123}^{(1)} = \frac{1}{4\epsilon} \Gamma_{123}^{(2)}, \quad \gamma_{123}^{(2)} = \frac{1}{2\epsilon} \left( \Gamma_{123}^{(1)} - \frac{1}{2} \Gamma_{123}^{(2)} \right).$$

$$\Delta S^{(1)} = S^{(1)} \otimes C_{123}^{(1)} S^{(1)} + S^{(2)} \otimes C_{123}^{(2)} S^{(1)}$$

$$\Delta S^{(2)} = S^{(1)} \otimes C_{123}^{(1)} S^{(2)} + S^{(2)} \otimes C_{123}^{(2)} S^{(2)}$$

$$\Delta \left[ \text{diagram with red label (1)} \right] = \text{diagram with red label (1)} \otimes \text{diagram with red dashed line and label (1)} + \text{diagram with blue label (2)} \otimes \text{diagram with blue dashed line and label (1)}$$

$$\Delta \left[ \text{diagram with blue label (2)} \right] = \text{diagram with red label (1)} \otimes \text{diagram with red dashed line and label (2)} + \text{diagram with blue label (2)} \otimes \text{diagram with blue dashed line and label (2)}$$

Check agreement with  $\Delta[{}_2F_1]$ .

# 1-mass sunset: comments on cuts

- $\gamma_1$  and  $\gamma_2$  generate full homology, including uncut contour

$$\int_{\Gamma_0} \omega^{(i)} = a \int_{\gamma_1} \omega^{(i)} + b \int_{\gamma_2} \omega^{(i)} \mod i\pi$$

$$\text{Diagram with purple cut} = a \text{Diagram with red cut} + b \text{Diagram with blue cut}$$

- Discontinuities can be recovered

$$\text{Disc}_{m^2} S^{(i)} \sim 2\epsilon \left( C_{123}^{(1)} S^{(i)} - C_{123}^{(2)} S^{(i)} \right)$$

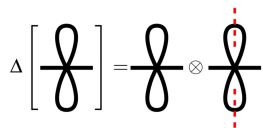
$$\text{Disc}_{p^2} S^{(i)} \sim -4\epsilon C_{123}^{(1)} S^{(i)}$$

- Coaction takes the same form on cut integrals

$$\Delta \left[ \text{Diagram with purple cut} \right] = \text{Diagram with purple cut} \otimes \text{Diagram with red cut} + \text{Diagram with purple cut} \otimes \text{Diagram with blue cut}$$



## Diagrammatic coaction at two loops

$$\Delta \left[ \text{diagram} \right] = \text{diagram} \otimes \text{diagram}$$
The diagram shows the coaction Δ applied to a two-loop Feynman diagram. The first diagram is a two-loop diagram with two vertices and two internal lines forming two loops. The second diagram is the same two-loop diagram, but with a vertical dashed red line passing through the center, representing a cut or a specific operation. The third diagram is the tensor product of the first and second diagrams, indicated by the ⊗ symbol.

$$\Delta(fg) = (\Delta f)(\Delta g)$$



$$S^{(1)} = -\epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon} k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon} l}{i\pi^{1-\epsilon}} \frac{\sqrt{\lambda(p^2, m_1^2, m_2^2)}}{(k^2 - m_1^2)(l^2 - m_2^2)(k + l + p)^2}$$

$$S^{(2)} = \epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon} k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon} l}{i\pi^{1-\epsilon}} \frac{m_2^2 - (k + p)^2}{(k^2 - m_1^2)(l^2 - m_2^2)(k + l + p)^2}$$

$$S^{(3)} = \epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon} k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon} l}{i\pi^{1-\epsilon}} \frac{m_1^2 - (l + p)^2}{(k^2 - m_1^2)(l^2 - m_2^2)(k + l + p)^2}$$

4th master integral is the double tadpole.

Expressions involve Appell  $F_4$ .

Basis of cuts: 3 maximal cuts  $\Gamma_{123}^{(i)}$ , and one 2-line cut  $\Gamma_{12}$  that is the max. cut of the double tadpole.

# Two-mass sunset

$$\begin{aligned}
 \Delta \left[ \text{Sunset}^{(1)} \right] &= \text{Diagram 1} \otimes \left( \text{Diagram 2}^{(1)} + \text{Diagram 3}^{(1)} + \text{Diagram 4}^{(1)} + \text{Diagram 5}^{(1)} \right) \\
 &+ \text{Diagram 6}^{(1)} \otimes \text{Diagram 7}^{(1)} + \text{Diagram 8}^{(2)} \otimes \text{Diagram 9}^{(1)} + \text{Diagram 10}^{(3)} \otimes \text{Diagram 11}^{(1)} \\
 \\
 \Delta \left[ \text{Sunset}^{(2)} \right] &= \text{Diagram 1} \otimes \left( \text{Diagram 2}^{(2)} + \text{Diagram 3}^{(2)} + \text{Diagram 4}^{(2)} + \text{Diagram 5}^{(2)} \right) \\
 &+ \text{Diagram 6}^{(1)} \otimes \text{Diagram 7}^{(2)} + \text{Diagram 8}^{(2)} \otimes \text{Diagram 9}^{(2)} + \text{Diagram 10}^{(3)} \otimes \text{Diagram 11}^{(2)} \\
 \\
 \Delta \left[ \text{Sunset}^{(3)} \right] &= \text{Diagram 1} \otimes \left( \text{Diagram 2}^{(3)} + \text{Diagram 3}^{(3)} + \text{Diagram 4}^{(3)} + \text{Diagram 5}^{(3)} \right) \\
 &+ \text{Diagram 6}^{(1)} \otimes \text{Diagram 7}^{(3)} + \text{Diagram 8}^{(2)} \otimes \text{Diagram 9}^{(3)} + \text{Diagram 10}^{(3)} \otimes \text{Diagram 11}^{(3)}
 \end{aligned}$$

# Double-edged triangle

4 master integrals:

2 in top topology, Appell  $F_4$

2 0-mass sunsets in  $p_1^2, p_2^2$

$$\begin{aligned}
 \Delta \left[ \text{triangle}(p_3, p_1, p_2) \right] &= p_1 \text{--} \text{sunset}(p_1) \text{--} p_1 \otimes \left( \text{triangle}(p_3, p_1, p_2) + \text{triangle}(p_3, p_1, p_2) \right) \\
 &+ p_2 \text{--} \text{sunset}(p_2) \text{--} p_2 \otimes \left( \text{triangle}(p_3, p_1, p_2) + \text{triangle}(p_3, p_1, p_2) \right) \\
 &+ \text{triangle}(p_3, p_1, p_2) \otimes \text{triangle}(p_3, p_1, p_2),
 \end{aligned}$$

The diagrams represent the following:

- Triangle:** A triangle with vertices  $p_3$ ,  $p_1$ , and  $p_2$ . Internal lines are labeled 1, 2, 3, 4. A green (1) is above the top line.
- Sunset:** A sunset diagram with two external lines and two internal lines forming a loop. Internal lines are labeled 1, 2, 3, 4. A green (1) is above the top line.
- Tensor Products:** Represented by  $\otimes$ .
- Summation:** Represented by  $+$ .
- Color Coding:** Red dashed lines indicate a sunset in  $p_1^2$ . Blue dashed lines indicate a sunset in  $p_2^2$ . Green dashed lines indicate a sunset in  $p_3^2$ .

## Examples of degenerate limits

Take  $p_2^2 \rightarrow 0$ . Basis of master integrals collapses to 2: 1 in top topology, 1 sunset.

Need to construct new dual integration contours.

$$\begin{aligned}
 \text{Diagram 1} &= \left( \text{Diagram 1a} - 6 \text{Diagram 1b} \right) \Big|_{p_2^2=0} \\
 \text{Diagram 2} &= \left( \text{Diagram 2a} - 2 \text{Diagram 2b} \right) \Big|_{p_2^2=0}
 \end{aligned}$$

Obtain coaction by taking the limit, or directly.

$$\Delta \left[ \text{Diagram 1} \right] = \text{Diagram 1a} \otimes \text{Diagram 1b} + \text{Diagram 1c} \otimes \text{Diagram 1d}$$

# Examples of degenerate limits

In the limit  $p_3^2 \rightarrow 0$ , the integral is reducible to the sunsets. 4-line cuts vanish.

$$\Delta \left[ \text{triangle with internal lines 1, 2, 4 and external lines } p_3, p_2, p_1 \right] = p_2 \text{ (bubble)} \otimes p_3 \text{ (triangle with cut 1)} + p_1 \text{ (bubble)} \otimes p_3 \text{ (triangle with cut 4)}$$

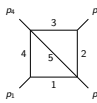
# Adjacent Triangles

$$e^{2\gamma_E \epsilon} \epsilon \left\{ \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(2-2\epsilon)} \frac{p_1^2 - p_2^2}{p_2^2} (-p_1^2)^{-2\epsilon} {}_2F_1 \left( 1-\epsilon, 1-2\epsilon; 2-2\epsilon; 1 - \frac{p_1^2}{p_2^2} \right) \right. \\ \left. - \frac{\Gamma(1+2\epsilon)\Gamma^3(1-\epsilon)}{2(1-2\epsilon)\Gamma(1-3\epsilon)} \left[ \frac{p_1^2 - p_2^2}{p_1^2} (-p_2^2)^{-2\epsilon} {}_3F_2 \left( 1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1 - \frac{p_2^2}{p_1^2} \right) \right. \right. \\ \left. \left. + \frac{p_1^2 - p_2^2}{p_2^2} (-p_1^2)^{-2\epsilon} {}_3F_2 \left( 1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1 - \frac{p_1^2}{p_2^2} \right) \right] \right\}$$

6 master integrals.

$$\Delta \left[ \text{Triangle}(p_1, p_2, p_3) \right] = \text{Term 1} \otimes \text{Triangle}_1 + \text{Term 2} \otimes \text{Triangle}_2 + \text{Term 3} \otimes \text{Triangle}_3 + \text{Term 4} \otimes \text{Triangle}_4 + \text{Term 5} \otimes \text{Triangle}_5 + \text{Term 6} \otimes \text{Triangle}_6$$

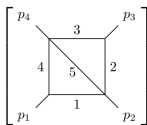
# 4-point integral: Diagonal Box



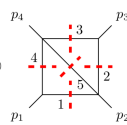
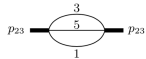
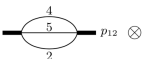
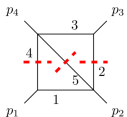
$$= -e^{2\gamma_E \epsilon} \frac{\epsilon(s+t)}{2(1-2\epsilon)} \frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}$$

$$\left[ \frac{t^{-2\epsilon}}{s} {}_2F_1\left(1-2\epsilon, 1-2\epsilon; 2-2\epsilon; 1+\frac{t}{s}\right) + \frac{s^{-2\epsilon}}{t} {}_2F_1\left(1-2\epsilon, 1-2\epsilon; 2-2\epsilon; 1+\frac{s}{t}\right) \right]$$

3 master integrals, 3 natural cuts.



$$\Delta \left[ \begin{array}{c} p_4 \\ 3 \\ p_3 \\ 4 \\ 5 \\ 2 \\ p_1 \\ 1 \\ p_2 \end{array} \right] = \begin{array}{c} p_4 \\ 3 \\ p_3 \\ 4 \\ 5 \\ 2 \\ p_1 \\ 1 \\ p_2 \end{array} \otimes \begin{array}{c} p_4 \\ 3 \\ p_3 \\ 4 \\ 5 \\ 2 \\ p_1 \\ 1 \\ p_2 \end{array} + p_{23} \text{---} \begin{array}{c} 3 \\ 5 \\ 1 \end{array} \text{---} p_{23} \otimes \begin{array}{c} p_4 \\ 3 \\ p_3 \\ 4 \\ 5 \\ 2 \\ p_1 \\ 1 \\ p_2 \end{array}$$

$$+ p_{12} \text{---} \begin{array}{c} 4 \\ 5 \\ 2 \end{array} \text{---} p_{12} \otimes \begin{array}{c} p_4 \\ 3 \\ p_3 \\ 4 \\ 5 \\ 2 \\ p_1 \\ 1 \\ p_2 \end{array}$$







- Diagrammatic coaction is conjectured to exist, compatible with
  - ▶ local coaction on MPLs, eMPLs, ...
  - ▶ global coaction on hypergeometric functions

Explicitly known at 1-loop. Beyond 1-loop, we find representations for various examples but lack a precise prediction.

- Coaction of  $L$ -loop graph has
  - ▶  $L$ -loop master integrals in left entries
  - ▶ genuine  $L$ -loop cuts in right entriesand exhibits a pairing between these objects.