Coaction for Feynman Integrals

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with Samuel Abreu, Claude Duhr, Einan Gardi, and James Matthew

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We conjecture three compatible coactions on Feynman integrals:

- **Local coaction** on multiple polylogarithms (MPLs), elliptic multiple polylogarithms, etc.

  This one is well known. [Goncharov, Brown]

  Applies in the Laurent expansion of Feynman integrals in the parameter ϵ of dimensional regularization.

- **Global coaction** on generalized hypergeometric functions.

  Applies to Feynman integrals in dimensional regularization without taking the Laurent expansion.

  Conjectured to exist for arbitrary Feynman integrals; examples found with integer-based parameters; since proven for Lauricella functions [Brown, Dupont].

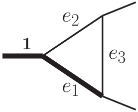
- **Diagrammatic coaction**

  1st claim: the output from the other two coactions are compatible with each other and can be repackaged as Feynman integrals.

  2nd claim: the coaction output can also be obtained by applying graphical operations before evaluating.

- Coaction is naturally compatible with discontinuities and differential operators, so we hope it can be applied to new computations
- Dimensional regularization is essential
- Formally, we distinguish motivic and de Rham MPLs, hypergeometric functions, etc. or use single-valued versions
- This talk: general principles and 1- and 2-loop examples

Example: 3 equivalent expressions, 3 compatible coactions



$$= \frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)}(m^2)^{-1-\epsilon}\,{}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$

$$= \frac{1}{p^2}\left[\frac{\log\left(\frac{m^2}{m^2-p^2}\right)}{\epsilon} + \mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + \log^2\left(1-\frac{p^2}{m^2}\right) + \log(m^2)\log\left(1-\frac{p^2}{m^2}\right)\right] + \mathcal{O}(\epsilon)$$

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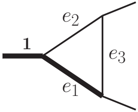

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$$\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log^2 z) &= 1 \otimes \log^2 z + 2\log z \otimes \log z + \log^2 z \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}$$

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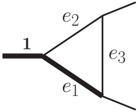

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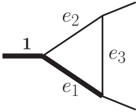
$$\Delta\left({}_2F_1(\alpha,\beta;\gamma;x)\right) = {}_2F_1(1+a\epsilon, b\epsilon; 1+c\epsilon; x) \otimes {}_2F_1(\alpha,\beta;\gamma;x)$$
$$- \frac{b\epsilon}{1+c\epsilon}\,{}_2F_1(1+a\epsilon, 1+b\epsilon; 2+c\epsilon; x)$$
$$\otimes \frac{\Gamma(1-\beta)\Gamma(\gamma)}{\Gamma(1-\beta+\alpha)\Gamma(\gamma-\alpha)} x^{1-\alpha}\,{}_2F_1\left(\alpha, 1+\alpha-\gamma; 1-\beta+\alpha; \frac{1}{x}\right)$$

Example: 3 equivalent expressions, 3 compatible coactions



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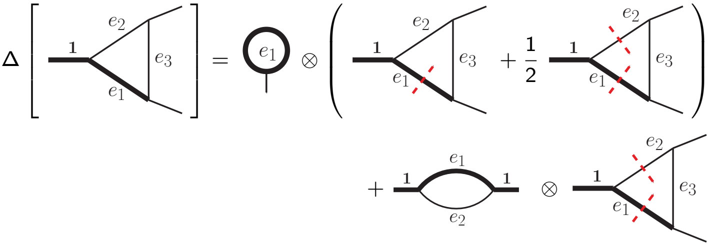
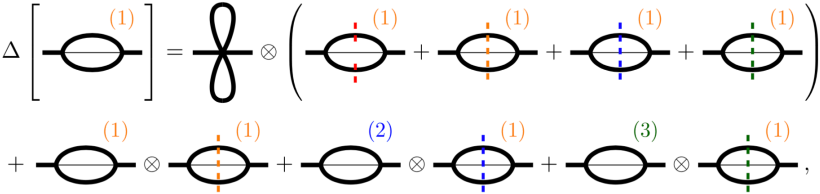


Sunset with two massive propagators.



General formula for coaction on integrals:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

General formula for coaction on integrals

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- All three coactions have this structure.
- Satisfies axioms of coaction.
- Claim of this formula: there exist sets $\{\omega_i\}$, $\{\gamma_j\}$, $\{c_{ij}\}$ to make it true.
- $\{\omega_i\}$ generate cohomology
- $\{\gamma_j\}$ generate homology
- $\{c_{ij}\}$ are rational in ϵ, algebraic in other parameters/kinematic variables; uniquely fixed by choices of $\{\omega_i\}$, and $\{\gamma_j\}$

General formula for coaction on integrals

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

Choice of bases:

- $\{\omega_i\}$
  Left entries $\int_\gamma \omega_i$ related to $\int_\gamma \omega$ by standard IBP reduction. Choose them to be pure.
- $\{\gamma_j\}$
  Right entries $\int_{\gamma_j} \omega$ related to $\int_\gamma \omega$ by change of contour. Can start with all possible cuts, but there are relations among them.

  An important relation for 1-loop integrals:

  $$\sum_i \mathcal{C}_i I_n + \sum_{i<j} \mathcal{C}_{ij} I_n = -\epsilon\, I_n \mod i\pi$$

  applies to subgraphs of multiloop groups.

  Principle: no uncut loops needed in right entries. Use only "genuine L-loop cuts."

Dual bases

Compact version of master formula:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

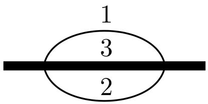
Look for bases such that

$$\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$$

Principles of the diagrammatic coaction

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- Left entries related by IBP, only L-loop diagrams
- Right entries are genuine L-loop cuts
- Coefficients c_{ij} are explicit for $L = 1$, ad hoc for $L > 1$ as of now.
- Can dualize bases either before or after initial choice
- Consistent with degenerate limits
- UV/IR pole cancellation
- γ can be a cut contour. Gives coaction on cut integrals.



$$S(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5;D;p^2,m^2) = \left(\frac{e^{\gamma_E\epsilon}}{i\pi^{D/2}}\right)^2 \int d^Dk\, d^Dl \frac{[(k+l)^2]^{-\nu_4}[(l+p)^2]^{-\nu_5}}{[k^2]^{\nu_1}[l^2]^{\nu_2}[(k+l+p)^2-m^2]^{\nu_3}}$$

2 master integrals, normalized to $1+\mathcal{O}(\epsilon)$.

$$\begin{aligned} S^{(1)} &= \epsilon^2\left(p^2-m^2\right) S(1,1,1,0,0;2-2\epsilon;p^2,m^2) \\ &= (m^2)^{-2\epsilon}\left(1-\frac{p^2}{m^2}\right) e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)\,{}_2F_1\left(1+2\epsilon,1+\epsilon;1-\epsilon;\frac{p^2}{m^2}\right) \\ S^{(2)} &= -\epsilon^2 S(1,1,1,-1,0;2-2\epsilon;p^2,m^2) \\ &= (m^2)^{-2\epsilon} e^{2\gamma_E\epsilon}\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(1+\epsilon)\,{}_2F_1\left(2\epsilon,\epsilon;1-\epsilon;\frac{p^2}{m^2}\right) \end{aligned}$$

Hence only two 2 independent integration contours, e.g. the maximal cuts.

First 2-loop example: 1-mass sunset

Maximal cut integral:

$$\mathcal{C}_{123} S^{(1)} \sim \int dk_0\, k_0^{-1-2\epsilon} \left(p^2 - m^2 + 2\sqrt{p^2} k_0\right)^{-1-2\epsilon} \left(p^2 + 2\sqrt{p^2} k_0\right)^{2\epsilon}$$

After taking three residues, there is one integration left, of the hypergeometric form ${}_2F_1$.
Can choose two independent contours:

$$\Gamma_{123}^{(1)} :\ k_0 \in \left[-\frac{\sqrt{p^2}}{2}, 0\right] \qquad \Gamma_{123}^{(2)} :\ k_0 \in \left[\frac{m^2 - p^2}{2\sqrt{p^2}}, 0\right]$$

Results:

$$\int_{\Gamma_{123}^{(1)}} \omega^{(1)} = 2\epsilon\, e^{2\gamma_E \epsilon} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} (p^2 - m^2)^{-2\epsilon} {}_2F_1\left(-2\epsilon, 1+2\epsilon; 1-\epsilon; \frac{p^2}{p^2 - m^2}\right)$$

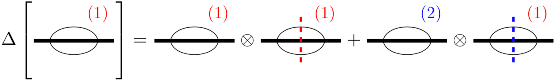
$$\int_{\Gamma_{123}^{(2)}} \omega^{(1)} = 4\epsilon\, e^{2\gamma_E \epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-4\epsilon)} (p^2)^{2\epsilon} (p^2 - m^2)^{-4\epsilon} {}_2F_1\left(-2\epsilon, -\epsilon; -4\epsilon; 1 - \frac{m^2}{p^2}\right)$$

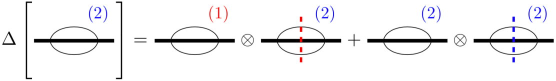
First 2-loop example: 1-mass sunset

Can arrive at compact/dual form $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ by requiring $\int_{\gamma_j} \omega_i = \delta_{ij} + \mathcal{O}(\epsilon)$.
Solution:

$$\gamma_{123}^{(1)} = \frac{1}{4\epsilon}\Gamma_{123}^{(2)}, \qquad \gamma_{123}^{(2)} = \frac{1}{2\epsilon}\left(\Gamma_{123}^{(1)} - \frac{1}{2}\Gamma_{123}^{(2)}\right).$$

$$\Delta S^{(1)} = S^{(1)} \otimes \mathcal{C}_{123}^{(1)} S^{(1)} + S^{(2)} \otimes \mathcal{C}_{123}^{(2)} S^{(1)}$$
$$\Delta S^{(2)} = S^{(1)} \otimes \mathcal{C}_{123}^{(1)} S^{(2)} + S^{(2)} \otimes \mathcal{C}_{123}^{(2)} S^{(2)}$$



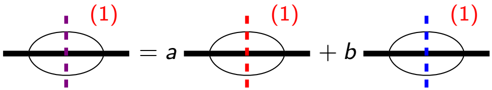


Check agreement with $\Delta[{}_2F_1]$.

1-mass sunset: comments on cuts

- γ_1 and γ_2 generate full homology, including uncut contour

$$\int_{\Gamma_0} \omega^{(i)} = a \int_{\gamma_1} \omega^{(i)} + b \int_{\gamma_2} \omega^{(i)} \quad \text{mod } i\pi$$

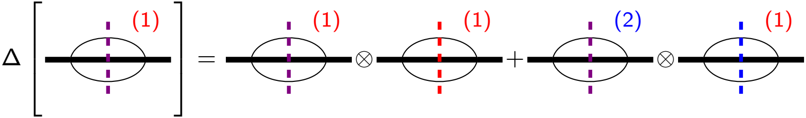


- Discontinuities can be recovered

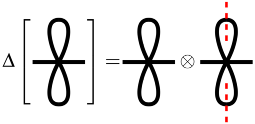
$$\mathrm{Disc}_{m^2} S^{(i)} \sim 2\epsilon \left(\mathcal{C}_{123}^{(1)} S^{(i)} - \mathcal{C}_{123}^{(2)} S^{(i)} \right)$$

$$\mathrm{Disc}_{p^2} S^{(i)} \sim -4\epsilon\, \mathcal{C}_{123}^{(1)} S^{(i)}$$

- Coaction takes the same form on cut integrals

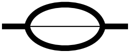


Diagrammatic coaction at two loops



$$\Delta(fg) = (\Delta f)\,(\Delta g)$$

Two-mass sunset



$$S^{(1)} = -\epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon}k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon}l}{i\pi^{1-\epsilon}} \frac{\sqrt{\lambda\left(p^2, m_1^2, m_2^2\right)}}{(k^2-m_1^2)(l^2-m_2^2)(k+l+p)^2}$$

$$S^{(2)} = \epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon}k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon}l}{i\pi^{1-\epsilon}} \frac{m_2^2-(k+p)^2}{(k^2-m_1^2)(l^2-m_2^2)(k+l+p)^2}$$

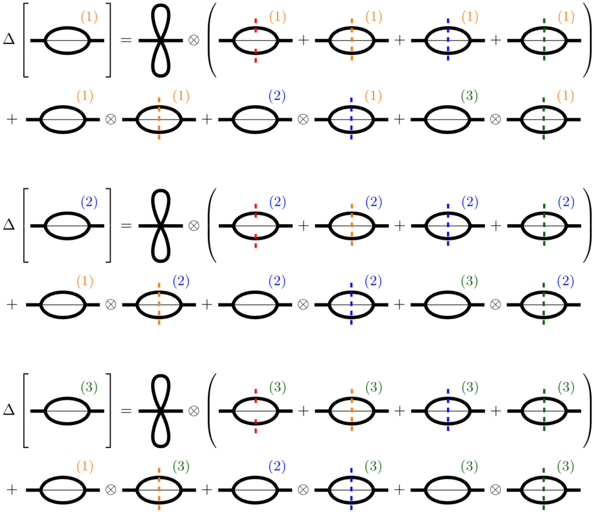
$$S^{(3)} = \epsilon^2 e^{2\gamma_E \epsilon} \int \frac{d^{2-2\epsilon}k}{i\pi^{1-\epsilon}} \int \frac{d^{2-2\epsilon}l}{i\pi^{1-\epsilon}} \frac{m_1^2-(l+p)^2}{(k^2-m_1^2)(l^2-m_2^2)(k+l+p)^2}$$

4th master integral is the double tadpole.

Expressions involve Appell F_4.

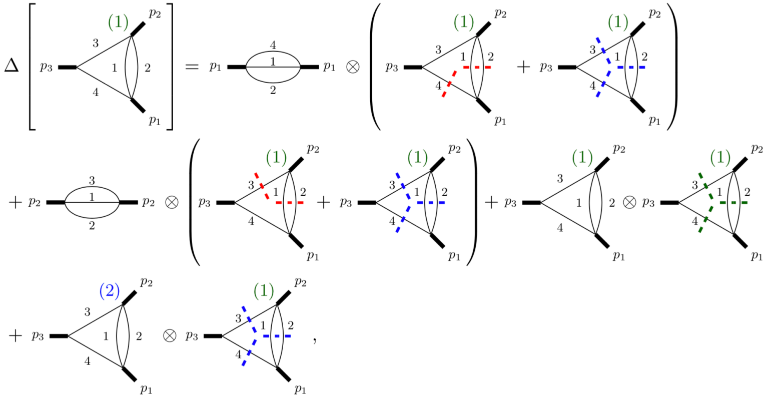
Basis of cuts: 3 maximal cuts $\Gamma_{123}^{(i)}$, and one 2-line cut Γ_{12} that is the max. cut of the double tadpole.

Two-mass sunset



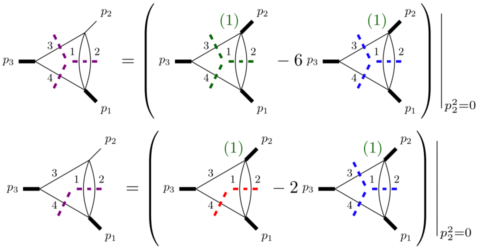
Double-edged triangle

4 master integrals:
2 in top topology, Appell F_4
2 0-mass sunsets in p_1^2, p_2^2

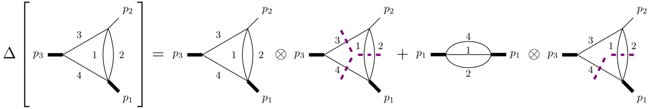


Examples of degenerate limits

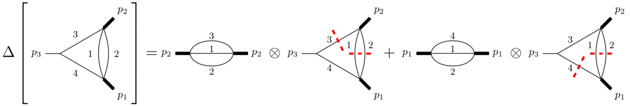
Take $p_2^2 \to 0$. Basis of master integrals collapses to 2: 1 in top topology, 1 sunset.
Need to construct new dual integration contours.



Obtain coaction by taking the limit, or directly.

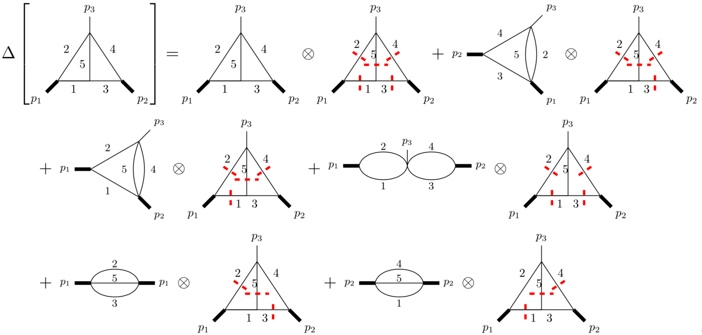


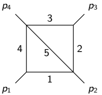
In the limit $p_3^2 \to 0$, the integral is reducible to the sunsets. 4-line cuts vanish.



$$
\begin{aligned}
&e^{2\gamma_E \epsilon}\epsilon \left\{ \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(2-2\epsilon)} \frac{p_1^2-p_2^2}{p_2^2}(-p_1^2)^{-2\epsilon}\, {}_2F_1\left(1-\epsilon, 1-2\epsilon; 2-2\epsilon; 1-\frac{p_1^2}{p_2^2}\right)\right. \\
&- \frac{\Gamma(1+2\epsilon)\Gamma^3(1-\epsilon)}{2(1-2\epsilon)\Gamma(1-3\epsilon)} \left[\frac{p_1^2-p_2^2}{p_1^2}(-p_2^2)^{-2\epsilon}\, {}_3F_2\left(1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1-\frac{p_2^2}{p_1^2}\right)\right. \\
&\left.\left. + \frac{p_1^2-p_2^2}{p_2^2}(-p_1^2)^{-2\epsilon}\, {}_3F_2\left(1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1-\frac{p_1^2}{p_2^2}\right)\right]\right\}
\end{aligned}
$$

6 master integrals.



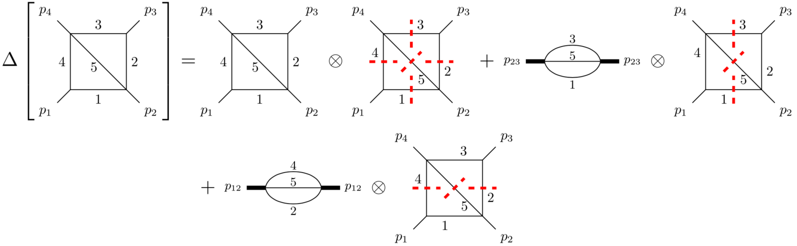


$$= -\,e^{2\gamma_E \epsilon}\,\frac{\epsilon(s+t)}{2(1-2\epsilon)}\,\frac{\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}$$

$$\left[\frac{t^{-2\epsilon}}{s}\,{}_2F_1\left(1-2\epsilon, 1-2\epsilon; 2-2\epsilon; 1+\frac{t}{s}\right)\right.$$

$$\left.+\frac{s^{-2\epsilon}}{t}\,{}_2F_1\left(1-2\epsilon, 1-2\epsilon; 2-2\epsilon; 1+\frac{s}{t}\right)\right]$$

3 master integrals, 3 natural cuts.



Summary

- Diagrammatic coaction is conjectured to exist, compatible with
  - local coaction on MPLs, eMPLs, ...
  - global coaction on hypergeometric functions

  Explicitly known at 1-loop. Beyond 1-loop, we find representations for various examples but lack a precise prediction.

- Coaction of L-loop graph has
  - L-loop master integrals in left entries
  - genuine L-loop cuts in right entries

  and exhibits a pairing between these objects.