3–loop Off–Shell Operator Matrix Elements and Non–Singlet Anomalous Dimensions

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in collaboration with

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Outline

Introduction

Anonymous dimensions are fundamental quantities of importance for

- α_s measurement
- pdf evolution
- 3–loop results and higher are necessary for measurements at the 1% level.
- We apply the traditional method of off–shell massless operator matrix elements (OMEs) [Gross & Wilczek 1974 @ 1 loop]
- The off–shell OMEs are gauge dependent; there is no equation of motion (EOM) & new local operators (alien operators) contribute in the gluonic case
- results up to 2–loops: [Matiounine, Smith, van Neerven, 1998 a,b] ; various relations are not fully correct.

Introduction

- We consider both the unpolarized and the polarized case.
- There is no γ_5 problem in the polarized case because of a known Ward identity $\Longrightarrow$ anticommuting γ_5.
- even Mellin moments: unpolarized $\gamma_{\rm NS}^{(k),+}$
- odd Mellin moments: polarized $\gamma_{\rm NS}^{(k),-}$
- $\exists$ a 3rd anomalous dimension from 3–loop onward: $\gamma_{\rm NS}^{(2),s}$ each in the unpolarized and polarized case.
- The anomalous dimension are extracted from the $1/\varepsilon$ terms of the physical projections of the OMEs in the non–singlet cases.
- We also calculate the transversity anomalous dimensions $\gamma_{\rm NS,TR}^{(k),\pm}$.

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- diagrams generated with QGRAF [Nogueira 1993]
- algebra done using TFORM [Vermasern et al. 2000–]
- reduction to master integrals CRUSHER [Marquard and Seidel]

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Setup of the calculation

- diagrams generated with `QGRAF` [Nogueira 1993]
- algebra done using `TFORM` [Vermasern et al. 2000–]
- reduction to master integrals `CRUSHER` [Marquard and Seidel]
- setup of recursions for the method of arbitrary high moments [Blümlein & Schneider, 2017] through differential equations
- guessing the recurrences [Kauers et al.] using `Sage`
- maximal number of moments: 1537; largest difference eq.: o = 16, d =304.
- solving the recurrences using `Sigma` [Schneider 2000–]
- transforming to z space using `HarmonicSums` [Ablinger 2012–]
- the function spaces are either harmonic sums [Vermaseren 1998; Blümlein & Kurth 1998] or harmonic polylogarithms [Vermaseren & Remiddi 1999] .

Operators

OMEs are defined as expectation values of the local operators

$$O^{\rm NS}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms},$$

$$O^{\rm NS,5}_{q,r;\mu_1\ldots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\gamma_5\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms}$$

between quark (antiquark) states ψ ($\bar{\psi}$) of space–like momentum p, $p^2 < 0$, and are given by

$$\hat{A}^{\rm NS,(5)}_{qq} = \langle q(p)|O^{\rm NS,(5)}|q(p)\rangle.$$

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$$O^{\mathrm{NS,tr}}_{q,r;\mu\mu_1\dots\mu_N} = i^{N-1}\mathbf{S}\left[\overline{\psi}\sigma_{\mu\mu_1}D_{\mu_2}\dots D_{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{ trace terms },$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$.

General Structure and Projectors

The general structure of the OME is given by

$$\hat{A}_{qq}^{\rm NS} = \left[\not\Delta \hat{A}_{qq}^{\rm NS,phys} + \not p \frac{\Delta.p}{p^2} \hat{A}_{qq}^{\rm NS,EOM} \right] (\Delta.p)^{N-1}.$$

and the individual parts can be obtained using the projectors

$$\hat{A}_{qq}^{\rm NS,phys} = \frac{1}{4(\Delta.p)^N} \operatorname{tr} \left[\left(\not p - \frac{p^2}{\Delta.p} \not\Delta \right) \hat{A}_{qq}^{\rm NS} \right],$$

$$\hat{A}_{qq}^{\rm NS,EOM} = \frac{1}{4(\Delta.p)^N} \operatorname{tr} \left[\not\Delta \hat{A}_{qq}^{\rm NS} \right].$$

General Structure and Projectors: Transversity

Here, we consider the Green's function

$$\hat{G}^{ij,\mathrm{NS},\mathrm{tr}}_{\mu,qQ} = \delta_{ij}(\Delta.p)^{N-1}\Bigg[\Delta_\rho\sigma^{\mu\rho}\Delta_T\hat{A}^{\mathrm{NS},\mathrm{phys}}_{qq}\left(\frac{-p^2}{\mu^2},\varepsilon,N\right)+c_1\Delta^\mu+c_2p^\mu$$
$$+c_3\gamma_\mu\not{p}+c_4\not{\Delta}\not{p}\Delta^\mu+c_5\not{\Delta}\not{p}p^\mu\Bigg],$$

with the projector

$$A^{\mathrm{TR}}_{qq} = -\frac{i}{4(\Delta.p)^{N+1}(D-2)}\left\{\mathrm{tr}\left[\not{\Delta}\not{p}p^\mu A^\mu_{qq}\right]-\Delta.p\,\mathrm{tr}\left[p^\mu A^\mu_{qq}\right]+i\Delta.p\,\mathrm{tr}\left[\sigma_{\mu\nu}p^\nu A^\mu_{qq}\right]\right\}$$

Resummation

all order Mellin-N solution

- resum N into auxiliary parameter t
- Nth moment corresponds to coefficient of the Nth term in the corresponding expansions

$$\sum_{N=0}^{\infty}(\Delta.k)^N\left(t^N \pm (-t)^N\right) = \left[\frac{1}{1-\Delta.k\ t} \pm \frac{1}{1+\Delta.k\ t}\right],$$

- Note: unpolarized and polarized anomalous dimensions only depend on even and odd moments
- take linear combinations of master integrals such that

$$\overline{M}_{\pm} = M_1 \pm t\, M_2 = \sum_{n=0}^{\infty} C_n^{\pm}(t^2)^n$$

Obtaining the all-N solution

- For the required 252 master integrals we derive diff. eqns. in t^2
- We insert the ansatz

$$\overline{M}^i_\pm = \sum_{n=0}^{\infty} C_n^{\pm,i} (t^2)^n$$

  into the diff. eqns. to obtain recursion relations for the coefficients $C_n^{\pm,i}$
- using the recursion relations a large number of expansion coefficients can be calculated
- inserting these expansions into the OME results in a large number of moments for the OME
- from this large number of moments we guess recursions for the final results
- finally, these recursions can be solved to get the full-N solution

Guessing: Example

- start with the sequence for C_i in $\sum C_i y^i$

$$-2, 0, -\frac{1}{6}, -\frac{1}{6}, -\frac{3}{20}, -\frac{2}{15}, -\frac{5}{42}, -\frac{3}{28}, -\frac{7}{72}, -\frac{4}{45}, -\frac{9}{110}, -\frac{5}{66}, -\frac{11}{156}, -\frac{6}{91}, -\frac{13}{210}, -\frac{7}{120}, -\frac{15}{272}, -\frac{8}{153}, -\frac{17}{342}, -\frac{9}{190}, -\frac{19}{420}, \dots$$

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  $n^2 C_n - (n-1)(n+2) C_{n+1} = 0$

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  $-\frac{6}{91}, -\frac{13}{210}, -\frac{7}{120}, -\frac{15}{272}, -\frac{8}{153}, -\frac{17}{342}, -\frac{9}{190}, -\frac{19}{420}, \ldots$
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  $C_n = -\frac{n-1}{n(n+1)}$

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- guess recurrence
  $n^2 C_n - (n-1)(n+2) C_{n+1} = 0$
- solve the recurrence
  $C_n = -\frac{n-1}{n(n+1)}$
- sum it
  $-2 - \sum_{n=1}^{\infty} \frac{n-1}{n(n+1)} y^n = -\frac{(y-2)\log(1-y)}{y} \overset{y \to 1-x}{=} \frac{(1+x)\log(x)}{1-x}$

Outline

Renormalization

- Our algorithm produces the unrenormalized OMEs for the irreducible contributions, to which the reducible contributions have to be added.
- We then renormalize the strong coupling and the gauge parameter and obtain:

$$\begin{aligned}\tilde{A}_{qq}^{\mathrm{NS,phys}} &= 1 + a\left[\frac{a_{qq}^{\mathrm{NS},(1,-1)}}{\varepsilon} + a_{qq}^{\mathrm{NS},(1,0)} + a_{qq}^{\mathrm{NS},(1,1)}\varepsilon\right] \\ &\quad + a^2\left[\frac{a_{qq}^{\mathrm{NS},(2,-2)}}{\varepsilon^2} + \frac{a_{qq}^{\mathrm{NS},(2,-1)}}{\varepsilon} + a_{qq}^{\mathrm{NS},(2,0)}\right] \\ &\quad + a^3\left[\frac{a_{qq}^{\mathrm{NS},(3,-3)}}{\varepsilon^3} + \frac{a_{qq}^{\mathrm{NS},(3,-2)}}{\varepsilon^2} + \frac{a_{qq}^{\mathrm{NS},(3,-1)}}{\varepsilon}\right].\end{aligned}$$

- Here only the local operators have to be renormalized.

Renormalization

$$Z^{\rm NS} = 1 + a\frac{\gamma_{qq}^{(0),\rm NS}}{\varepsilon} + a^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS^2} + \beta_0\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{2\varepsilon}\gamma_{qq}^{(0),\rm NS}\right] + a^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{qq}^{(0),\rm NS^3} + \beta_0\gamma_{qq}^{(0),\rm NS^2} + \frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS}\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_0\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_1\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{3\varepsilon}\gamma_{qq}^{(2),\rm NS}\right].$$

- The renormalized OME is given by

$$A_{qq}^{\rm NS,phys} = \frac{\tilde{A}_{qq}^{\rm NS,phys}}{Z^{\rm NS}}$$

- The coefficients $a_{qq}^{\rm NS,(k,l)}$ are determined iteratively & one finds the anomalous dimensions in the $1/\varepsilon$ terms.

Renormalization

- The anomalous dimension $\gamma_{\rm NS}^{(2),s}$ is obtained in the following way

$$\frac{2}{1+(-1)^N}\hat{A}_{qq}^{\rm PS,(3),phys}\Big|_{d_{abc}d^{abc}} = \hat{a}_s^3\frac{1}{3\varepsilon}\gamma_{\rm NS}^{(2),s} + O(\varepsilon^0), \quad N\in\mathbb{N}, \text{ odd } , N\geq 1$$

- The calculation of $\Delta\gamma_{\rm NS}^{(2),s}$ is underway.
- It has been computed from the polarized structure function g_5 [Bluemlein & Kochelev, 1996] in [Moch, Vermaseren & Vogt, 2015] using the forward Compton amplitude.

Outline

Results $\gamma_{\rm NS}^{(2),+}$

$$\gamma_{\rm NS}^{(2),+} = \frac{1}{2}\left[1+(-1)^N\right]$$
$$\times\Big\{C_F^2\Big\{C_A\Big[\frac{72P_3}{N^2(1+N)^2}\zeta_3 + \frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4}$$
$$+\Big(-\frac{16P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320S_4 - 1024S_{3,1}$$
$$+\frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1}\Big)S_1 + \Big(256S_3$$
$$+1792S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1}\Big)S_2 + \frac{352}{3}S_2^2$$
$$+\frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Big(-\frac{16P_{22}}{9N^2(1+N)^3} + (-\frac{64P_9}{9N^2(1+N)^2} - 256S_2\Big)S_1$$
$$+\frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3\Big)S_{-2}$$
$$+\Big(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1\Big)S_{-2}^2 + \Big(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16P_{16}}{9N^2(1+N)^2}$$
$$-1152S_1^2 + 2624S_2 + 960S_{-2}\Big)S_{-3} + \Big(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1\Big)S_{-4}$$
$$+2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768S_{4,1}$$
$$+\frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728S_{-4,1}$$
$$-5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2}$$
$$-5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1}\Big] + T_FN_F\Big[-\frac{16P_5}{9N^2(1+N)^2}S_2 + \frac{4P_{34}}{9N^4(1+N)^4}$$
$$+\Big(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3\Big)S_1 - \frac{128}{3}S_2^2$$
$$+\frac{64(12+29N+29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \Big(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9}S_1 - \frac{256}{3}S_2\Big)$$
$$\times S_{-2} + \Big(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3}S_1\Big)S_{-3} - \frac{256}{3}S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)}S_{-2,1}$$
$$+\frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\}$$
$$+C_F\Big\{T_F^2N_F^2\Big[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\Big] + C_A^2\Big[-\frac{24P_3}{N^2(1+N)^2}\zeta_3$$
$$-\frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \Big(\frac{4P_{35}}{3N^4(1+N)^4}$$
$$-\frac{16(-8+11N+11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)}S_{-2,1}$$
$$-1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1}\Big)S_1 + \Big(-128S_3 - 512S_{-2,1}\Big)S_1^2 + \Big(-\frac{8344}{27}$$
$$+384S_3 + 1536S_{-2,1}\Big)S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)}S_4 + 64S_5 + \Big(\frac{32P_{10}}{9N^2(1+N)^2}S_1$$
$$+\frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3\Big)S_{-2}$$
$$+\Big(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1\Big)S_{-2}^2 + \Big(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)}S_1$$
$$+256S_1^2 - 768S_2 - 320S_{-2}\Big)S_{-3} + \Big(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1\Big)S_{-4}$$
$$-704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1}$$
$$-\frac{32(-48+11N+11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1}$$
$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1}$$
$$+S_{-3,1,1}] - 3072S_{-2,1,1,1}\Big] + C_AT_FN_F\Big[-\frac{8P_{31}}{27N^3(1+N)^3} + \Big(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3$$
$$+\frac{256}{3}S_{-2,1} - 128\zeta_3\Big)S_1 + \frac{5344}{27}S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \Big(-\frac{1280}{9}S_1$$
$$+\frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3}S_2\Big)S_{-2} + \Big(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3}S_1\Big)S_{-3}$$
$$+\frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$
$$+\frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\} + C_F^3\Big\{-\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5}$$
$$+\Big(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2}$$
$$-\frac{384(-4+N+N^2)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1}\Big)S_1 + \Big(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3}$$
$$-1536S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1}\Big)S_2 - \frac{32(2+3N+3N^2)}{N(1+N)}S_2^2$$
$$-\frac{32(2+15N+15N^2)}{N(1+N)}S_4 + (\frac{32P_{24}}{N^3(1+N)^3} + \Big(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2\Big)S_1$$
$$-\frac{64(4+3N+3N^2)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3\Big)S_{-2} + \Big(\frac{128}{N(1+N)}$$
$$-256S_1\Big)S_{-2}^2 + \Big(\frac{32(8+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2$$
$$-640S_{-2}\Big)S_{-3} + \Big(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_1\Big)S_{-4} - 1792S_{-5} - 384S_{2,3}$$
$$-2304S_{2,-3} + \frac{128(-2+3N+3N^2)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2}S_{-2,1}$$
$$-\frac{64(-26+3N+3N^2)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1}$$
$$+4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768(-4+N+N^2)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2}$$
$$+4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1}\Big\}\Big\}$$

Results $\gamma_{\rm NS}^{(2),+}$

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\begin{aligned}
\gamma_{\rm NS}^{(2),+} &= \frac{1}{2}\left[1+(-1)^N\right] \\
&\times\Bigg\{C_F^2\Bigg\{C_A\Bigg[\frac{72P_3}{N^2(1+N)^2}\zeta_3+\frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1}-\frac{16P_{17}}{9N^2(1+N)^2}S_3+\frac{P_{33}}{18N^4(1+N)^4} \\
&+\Bigg(-\frac{16P_{29}}{9N^4(1+N)^4}-\frac{4288}{9}S_2+\frac{64\big(-12+31N+31N^2\big)}{3N(1+N)}S_3+320S_4-1024S_{3,1} \\
&+\frac{64\big(-84+31N+31N^2\big)}{3N(1+N)}S_{-2,1}+3712S_{-2,2}+3840S_{-3,1}-7168S_{-2,1,1}\Bigg)S_1+\Bigg(256S_3 \\
&+1792S_{-2,1}\Bigg)S_1^2+\Bigg(\frac{4P_{19}}{9N^2(1+N)^2}-832S_3-5248S_{-2,1}\Bigg)S_2+\frac{352}{3}S_2^2 \\
&+\frac{16\big(-30+151N+151N^2\big)}{3N(1+N)}S_4+\Bigg(-\frac{16P_{22}}{9N^2(1+N)^3}+\Big(-\frac{64P_9}{9N^2(1+N)^2}-256S_2\Bigg)S_1 \\
&+\frac{32\big(12+31N+31N^2\big)S_2}{3N(1+N)}+64S_3+5376S_{2,1}-384S_{-2,1}+576\zeta_3\Bigg)S_{-2} \\
&+\Bigg(-\frac{32\big(8+3N+3N^2\big)}{N(1+N)}+512S_1\Bigg)S_{-2}^2+\Bigg(\frac{32\big(108+31N+31N^2\big)}{3N(1+N)}S_1-\frac{16P_{16}}{9N^2(1+N)^2} \\
&-1152S_1^2+2624S_2+960S_{-2}\Bigg)S_{-3}+\Bigg(\frac{16\big(138+35N+35N^2\big)}{3N(1+N)}-1472S_1\Bigg)S_{-4}
\end{aligned}
$$

Comparison to the Literature

- We confirm the results of [Moch, Vermaseren and Vogt, Nucl.Phys.B 688 (2004) 101.] and previous moment calculations by Blümlein, Larin, Nogueira, Retey, von Ritbergen & Vermaseren.
- For transversity we confirm partial results on the moments by Bagaev, Bednyakov, Gracey, Pikelner, Velizhanin, up to a missing factor N_F^2 in the first moment and series of sign errors in some terms for several moments.
- An attached expression on the transversity anomalous dimensions, derived under special assumptions, in [Velizhanin Nucl. Phys. B864 (2012) 113] is correct.
- All contributions $\propto T_F$ have been calculated in [Ablinger et al. Nucl. Phys. B886 (2014) 733.]
- The terms $\propto T_F^2 N_F^2$ have been predicted in [Gracey, Phys. Lett. B 322 (1994) 141.]
- The leading term in the small x limit has been predicted by [Kirschner & Lipatov, Nucl. Phys. B 213 (1983) 122] and we agree after correcting errors there in [Blümlein & Vogt, Phys. Lett. B 370 (1996) 149] & [Bartels, Ermolaev & Ryskin, Z. Phys. C 70 (1996) 273.]

$$
\begin{aligned}
P_{\mathrm{NS}}^{(2),+} &\simeq 3.16049 H_0^4 + 45.0370 H_0^3 + 407.565 H_0^2 + 1684.87 H_0 + 3469.02, \\
P_{\mathrm{NS}}^{(2),-} &\simeq 2.86420 H_0^4 + 52.1481 H_0^3 + 570.854 H_0^2 + 1973.93 H_0 + 3769.92.
\end{aligned}
$$

for $N_F = 3$ shows, that the leading small x terms are not dominant.

Outline

Conclusions

- We calculated the three–loop non–singlet anomalous dimensions using off-shell OMEs within a gauge–dependent framework.
- Previous calculations were performed in massless or massive gauge–independent frameworks (forward Compton amplitude or massive OMEs).
- Used the method of arbitrary high Mellin moments, guessing & difference ring theory in the calculation, which could be automated in this way.
- The computation time amounted to about 20 days.
- For transversity it is the first calculation ab initio.
- We agree with available results in the literature and corrected some smaller errors in the case of transversity.