Effective Theory for Gravity in Binary Systems at 5th and 6th Post-Newtonian Order

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In collaboration with

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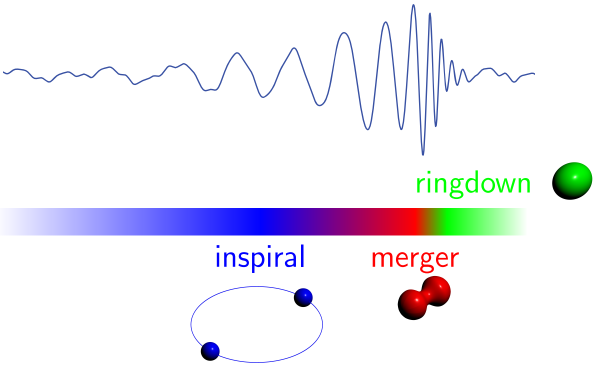
Gravitational waves from binary mergers



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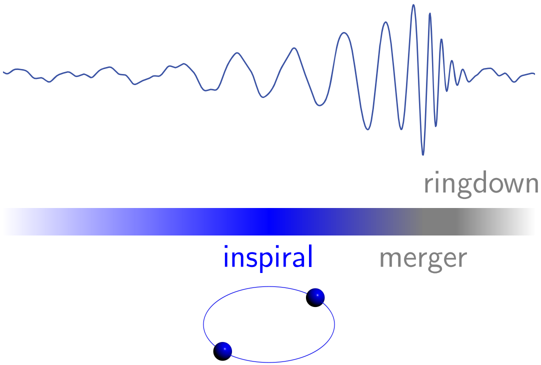
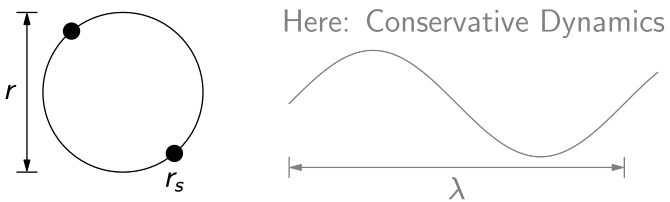


Compact binary systems

Post-Newtonian expansion



- Masses comparable: $m \equiv m_1 \sim m_2$
  Generalisation to different masses straightforward
- Nonrelativistic system: $v \ll 1$
- Virial theorem: $mv^2 \sim \frac{Gm^2}{r}$

> *Post-Newtonian (PN) expansion*:
> Combined expansion in $v \sim \sqrt{Gm/r} \ll 1$

General relativity

General relativity action:

$$S_{\mathrm{GR}}[g^{\mu\nu}] = S_{\mathrm{EH}} + S_{\mathrm{GF}} + S_{\mathrm{matter}}$$

With $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $g = \det(g^{\mu\nu})$:

- Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{1}{16G\pi}\int d^d x\sqrt{-g}R$$

- *Harmonic* gauge $\partial_\mu\sqrt{-g}g^{\mu\nu} = 0$:

$$S_{\mathrm{GF}} = -\frac{1}{32G\pi}\int d^d x\sqrt{-g}\Gamma_\mu\Gamma^\mu\,, \qquad \Gamma^\mu = g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}$$

- Assume point-like matter, no spin:

$$S_{\mathrm{matter}} = -\sum_{a=1}^{2} m_a\int d\tau_a$$

General relativity

Post-Newtonian expansion

- Parametrise metric: [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-c_d\phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}, \qquad c_d = 2\frac{d-2}{d-3}$$

Post-Newtonian expansion: $\phi, A_i, \sigma_{ij} \sim v \sim \sqrt{Gm/r} \ll 1$

General relativity

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Post-Newtonian expansion: $\phi, A_i, \sigma_{ij} \sim v \sim \sqrt{Gm/r} \ll 1$

- Decompose fields: [Beneke, Smirnov 1997]

$$\phi = \phi_{\text{rad}} + \phi_{\text{pot}}, \quad A = A_{\text{rad}} + A_{\text{pot}}, \quad \sigma = \sigma_{\text{rad}} + \sigma_{\text{pot}}$$

$$\partial_0(\phi_{\substack{\text{pot}\\\text{rad}}}, A_{\substack{\text{pot}\\\text{rad}}}, \sigma_{\substack{\text{pot}\\\text{rad}}}) \sim v(\phi_{\substack{\text{pot}\\\text{rad}}}, A_{\substack{\text{pot}\\\text{rad}}}, \sigma_{\substack{\text{pot}\\\text{rad}}}),$$

$$\partial_k(\phi_{\text{rad}}, A_{\text{rad}}, \sigma_{\text{rad}}) \sim v(\phi_{\text{rad}}, A_{\text{rad}}, \sigma_{\text{rad}}),$$

$$\partial_k(\phi_{\text{pot}}, A_{\text{pot}}, \sigma_{\text{pot}}) \sim (\phi_{\text{pot}}, A_{\text{pot}}, \sigma_{\text{pot}})$$

Effective field theory

Nonrelativistic general relativity [Goldberger, Rothstein 2004-2006]

$$S_{\mathrm{GR}}[\phi_{\mathrm{pot}}, A_{\mathrm{pot}}, \sigma_{\mathrm{pot}}, \phi_{\mathrm{rad}}, A_{\mathrm{rad}}, \sigma_{\mathrm{rad}}] \to S_{\mathrm{NRGR}}[\phi_{\mathrm{rad}}, A_{\mathrm{rad}}, \sigma_{\mathrm{rad}}]$$

- Keep radiation/ultrasoft modes
- Absorb potential modes into matching coefficients, e.g. near-zone potential

Ansatz:

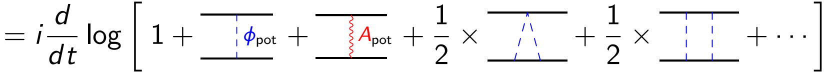
$$S_{\mathrm{NRGR}} = S_{\mathrm{matter}} + S_{\mathrm{mixed}} + S_{\mathrm{radiation}}$$

- $S_{\mathrm{matter}} = \int dt\, (M + T - V_{\mathrm{NZ}})$
- S_{mixed}: coupling of radiation modes to matter
- $S_{\mathrm{radiation}}$: pure radiation modes

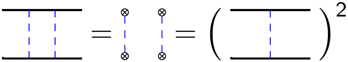
Nonrelativistic effective theory

Potential matching

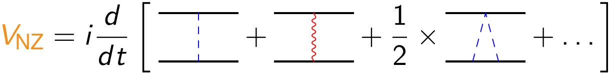
$$V_{\mathrm{NZ}} = i\frac{d}{dt}\log\left[\int \mathcal{D}\phi_{\mathrm{pot}}\mathcal{D}A_{\mathrm{pot}}\mathcal{D}\sigma_{\mathrm{pot}}\; e^{iS_{\mathrm{GR}} - i\int dt(M+T)}\Big|_{\phi_{\mathrm{rad}}=A_{\mathrm{rad}}=\sigma_{\mathrm{rad}}=0}\right]$$



Matter lines are no propagators:



No contributions from factorising diagrams: cf. [Fischler 1977]

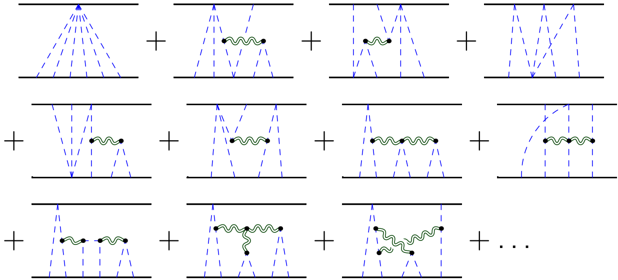


Potential matching

5PN calculation

1. Generate diagrams with up to 5 loops with `QGRAF` [Nogueira 1991]
2. Discard unwanted diagrams, e.g. graviton loops

$$-i\int dt\, V_{\mathrm{NZ}}^{\mathrm{5PN}} =$$



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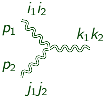

$$= \frac{i}{32 m_{\mathrm{Pl}}} \left(\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + \tilde{V}^{i_2 i_1, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \right)$$

$$\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_1 j_2, k_2 k_1}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_2 k_1}_{\sigma\sigma\sigma}$$

$$\begin{aligned}
V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \overset{v=0}{\equiv} & \, (\vec{p}_1^{\,2} + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^{\,2}) \Big(-\delta^{j_1 j_2} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \\
& \quad + 2 \big[\delta^{i_1 j_1} (4\delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2}) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} \big] \Big) \\
& + 2 \Big\{ 4 (p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 j_1} \delta^{j_2 k_1} \\
& \quad + 2 \big[(p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \big] \\
& \quad + \delta^{j_1 j_2} \big[p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2 (p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 k_1} - (p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{k_1 k_2} \big] \\
& \quad + p_2^{j_2} \Big(4 p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \\
& \qquad + 2 \big[\delta^{i_1 j_1} (p_1^{i_2} \delta^{k_1 k_2} - 2 p_1^{k_2} \delta^{i_2 k_1}) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} \big] \Big) \\
& \quad + p_1^{j_2} \Big(p_1^{j_1} (2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) - 4 p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \\
& \qquad + 2 \big[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} (2 p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2}) \big] \Big) \Big\}
\end{aligned}$$

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1. Generate diagrams with up to 5 loops with `QGRAF` [Nogueira 1991]
2. Discard unwanted diagrams, e.g. graviton loops
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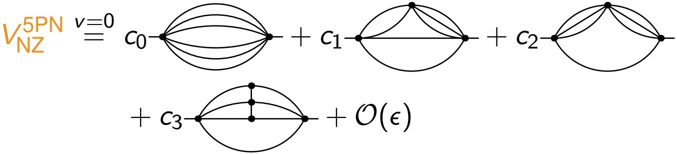
4. Reduce massless propagators to master integrals using Laporta's algorithm [Chetyrkin, Tkachov 1981, Laporta 2000] implemented in `crusher`



c_j: Laurent series in $\epsilon = \frac{3-d}{2}$,
polynomials in m_1, m_2, r^{-1}, G^{-1}

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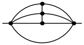
5. Insert known (factorising) master integrals

[Lee, Mingulov 2015; Damour, Jaranowski 2017]



$$= 6\pi^{7/2}\left[\frac{2}{\epsilon} - 4 - 4\ln(2) + \mathcal{O}(\epsilon^1)\right]$$

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   [Lee, Mingulov 2015; Damour, Jaranowski 2017]

$$V_{5\mathrm{PN}}^{\mathrm{NZ}} \overset{v=0}{=} \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

$V_{5\mathrm{PN}}^{\mathrm{NZ}}, v = 0$: [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 2019; Blümlein, Maier, Marquard 2019]

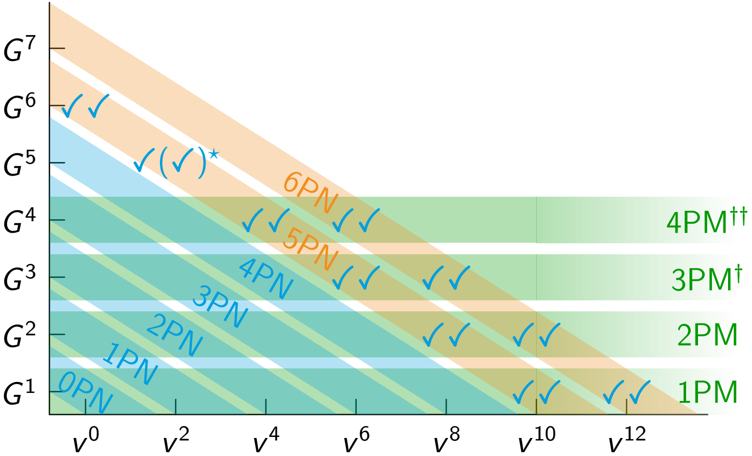
New:

- $V_{5\mathrm{PN}}^{\mathrm{NZ}}$ complete [Blümlein, Maier, Marquard, Schäfer, 2020]
  partial $V_{5\mathrm{PN}}^{\mathrm{NZ}}$ (parts of v^2G^5) [Foffa, Sturani, Torres Bobadilla 2020]
- $V_{6\mathrm{PN}}^{\mathrm{NZ}}$ up to 3 loops: v^6G^4, v^8G^3, $v^{10}G^2$, $v^{12}G$

  [Blümlein, Maier, Marquard, Schäfer, 2020-2021]

Near-zone potential

Cross checks



† [Bern, Cheung, Roiban, Shen, Solon 2019; Cheung Solon 2020; Kälin, Liu, Porto 2020]

†† [Bern, Martinez, Roiban, Ruf, Shen 2021; Dlapa, Kälin, Liu, Porto 2021]

⋆ [Foffa, Sturani, Torres Bobadilla 2021]

Classical action

V_{NZ} is not physical:

- Gauge dependent
- Infrared divergence at $\geq$4PN

$\Rightarrow$ combine with contribution from radiation/ultrasoft modes

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Construct classical post-Newtonian action *without* any field:

$$S_{\mathrm{PN}} = \int dt\, L[\vec{x}_a, \vec{v}_a] = \int dt\, (M + T - V)$$

Absorb radiation modes radiation/ultrasoft modes into far-zone potential V_{FZ} ("tail")

$$V = V_{\mathrm{NZ}} + V_{\mathrm{FZ}}$$

$V_{\mathrm{FZ}}^{\mathrm{5PN}}$ from combination of known results?

[Ross 2012] [Marchand, Henry, Larrouturou, Marsat, Faye, Blanchet 2020] [Foffa, Sturani 2019–2021]

[Henry, Faye, Blanchet 2021]

Classical Hamiltonian

$$\mu = \frac{m_1 m_2}{M}\,, \quad p = \frac{\vec{p}_1}{\mu} = -\frac{\vec{p}_2}{\mu}\,, \quad r = \frac{|x_1 - x_2|}{GM}\,, \quad n = \frac{\vec{r}}{r}$$

$$\begin{aligned}
\frac{H_{5PN}^{\text{pole free}} - M}{\mu} = & -\frac{21p^{12}}{1024} + \frac{5}{16r^6} - \frac{125p^2}{16r^5} - \frac{499p^4}{64r^4} - \frac{161p^6}{32r^3} - \frac{445p^8}{256r^2} - \frac{77p^{10}}{256r} \\
& + \frac{\mu}{M}\Bigg[\frac{231p^{12}}{1024} - \frac{279775133}{529200r^6} - \frac{1450584679p^2}{2116800r^5} + \frac{2010713771p^4}{1411200r^4} + \frac{11206267p^6}{141120r^3} + \frac{937p^8}{32r^2} \\
& + \frac{805p^{10}}{256r} + \ln\left(\frac{r}{r_0}\right)\Bigg(\frac{64}{105r^6} - \frac{18944p^2}{105r^5} + \frac{1796p^4}{105r^4} + \frac{19136(p.n)^2}{105r^5} - \frac{10664p^2(p.n)^2}{105r^4} \\
& + \frac{2748(p.n)^4}{35r^4}\Bigg) + \pi^2\Bigg(\frac{70399}{1152r^6} + \frac{65291p^2}{1152r^5} - \frac{1328147p^4}{12288r^4} - \frac{7719p^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5} \\
& + \frac{5042575p^2(p.n)^2}{6144r^4} + \frac{58887p^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} - \frac{89625p^2(p.n)^4}{4096r^3} \\
& + \frac{42105(p.n)^6}{4096r^3}\Bigg) - \frac{34541593(p.n)^2}{2116800r^5} - \frac{2395722563p^2(p.n)^2}{282240r^4} - \frac{62196341p^4(p.n)^2}{78400r^3} \\
& - \frac{589p^6(p.n)^2}{16r^2} - \frac{35p^8(p.n)^2}{256r} + \frac{631107353(p.n)^4}{78400r^4} + \frac{31226291p^2(p.n)^4}{23520r^3} \\
& + \frac{8951p^4(p.n)^4}{384r^2} - \frac{563921(p.n)^6}{960r^3} - \frac{5117p^2(p.n)^6}{320r^2} + \frac{159(p.n)^8}{28r^2}\Bigg] + \frac{\mu^2}{M^2}\Bigg[-\frac{231p^{12}}{256} \\
& + \frac{72454}{225r^6} + \frac{1353196483p^2}{529200r^5} - \frac{787300061p^4}{264600r^4} + \frac{3605263p^6}{29400r^3} - \frac{11535p^8}{128r^2} - \frac{2865p^{10}}{256r} \\
& - \ln\left(\frac{r}{r_0}\right)\Bigg(\frac{256}{105r^6} + \frac{3392p^2}{105r^5} - \frac{432p^4}{35r^4} - \frac{2992(p.n)^2}{105r^5} - \frac{6824p^2(p.n)^2}{105r^4} + \frac{496(p.n)^4}{7r^4}\Bigg)
\end{aligned}$$

Classical Hamiltonian

$$
\begin{aligned}
&+\pi^2\Bigg(\frac{5453}{768r^6}-\frac{121315p^2}{768r^5}+\frac{2076041p^4}{12288r^4}+\frac{29987p^6}{4096r^3}+\frac{200359(p.n)^2}{768r^5}-\frac{172311p^4(p.n)^2}{4096r^3}\\
&-\frac{5962205p^2(p.n)^2}{6144r^4}+\frac{2617363(p.n)^4}{4096r^4}+\frac{127125p^2(p.n)^4}{4096r^3}+\frac{14175(p.n)^6}{4096r^3}\Bigg)\\
&-\frac{857318207(p.n)^2}{264600r^5}+\frac{34200172759p^2(p.n)^2}{2116800r^4}-\frac{5034763p^4(p.n)^2}{9800r^3}+\frac{4969p^6(p.n)^2}{64r^2}\\
&+\frac{275p^8(p.n)^2}{256r}-\frac{4989943687(p.n)^4}{352800r^4}+\frac{2674877p^2(p.n)^4}{7840r^3}+\frac{925p^4(p.n)^4}{24r^2}+\frac{15p^6(p.n)^4}{128r}\\
&-\frac{25649(p.n)^6}{3360r^3}-\frac{8331p^2(p.n)^6}{160r^2}+\frac{751(p.n)^8}{28r^2}\Bigg]+\frac{\mu^3}{M^3}\Bigg[\frac{1617p^{12}}{1024}-\frac{238966727p^2}{151200r^5}\\
&+\frac{127702733p^4}{84672r^4}+\frac{108551131p^6}{4233600r^3}+\frac{16283p^8}{256r^2}+\frac{3995p^{10}}{256r}+\pi^2\Bigg(-\frac{2339p^2}{192r^5}+\frac{98447p^4}{3072r^4}\\
&-\frac{20259p^6}{1024r^3}-\frac{16111(p.n)^2}{192r^5}+\frac{131231p^2(p.n)^2}{1536r^4}+\frac{106947p^4(p.n)^2}{1024r^3}-\frac{361499(p.n)^4}{1024r^4}\\
&-\frac{30075p^2(p.n)^4}{1024r^3}-\frac{65625(p.n)^6}{1024r^3}\Bigg)+\frac{758233181(p.n)^2}{151200r^5}-\frac{10374288811p^2(p.n)^2}{705600r^4}\\
&-\frac{2207947669p^4(p.n)^2}{1411200r^3}+\frac{177p^6(p.n)^2}{256r^2}-\frac{221p^8(p.n)^2}{64r}+\frac{12810612439(p.n)^4}{705600r^4}\\
&+\frac{355111837p^2(p.n)^4}{94080r^3}-\frac{125225p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{128r}-\frac{13905527(p.n)^6}{4480r^3}\\
&+\frac{136977p^2(p.n)^6}{1280r^2}-\frac{15p^4(p.n)^6}{128r}-\frac{289839(p.n)^8}{4480r^2}-\frac{35p^2(p.n)^8}{256r}\Bigg]
\end{aligned}
$$

Classical Hamiltonian

$$+\frac{\mu^4}{M^4}\Big[-\frac{1155p^{12}}{1024}-\frac{593p^6}{32r^3}+\frac{6649p^8}{256r^2}-\frac{1615p^{10}}{256r}+\frac{549p^4(p.n)^2}{32r^3}-\frac{62143p^6(p.n)^2}{256r^2}+\frac{867p^8(p.n)^2}{256r}$$

$$-\frac{5749p^2(p.n)^4}{96r^3}+\frac{652381p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{64r}-\frac{17623(p.n)^6}{240r^3}-\frac{1178329p^2(p.n)^6}{1280r^2}$$

$$-\frac{45p^4(p.n)^6}{128r}+\frac{1443091(p.n)^8}{4480r^2}+\frac{105p^2(p.n)^8}{128r}\Big]+\frac{\mu^5}{M^5}\Big[\frac{231p^{12}}{1024}-\frac{63p^{10}}{256r}-\frac{35p^8(p.n)^2}{256r}$$

$$-\frac{15p^6(p.n)^4}{128r}-\frac{15p^4(p.n)^6}{128r}-\frac{35p^2(p.n)^8}{256r}-\frac{63(p.n)^{10}}{256r}\Big]+\frac{17(p.n)^2}{4r^5}+\frac{29p^2(p.n)^2}{8r^4}$$

$$+\frac{21p^4(p.n)^2}{16r^3}+\frac{5p^6(p.n)^2}{32r^2}-\frac{(p.n)^4}{8r^4}+V^{\text{5PN}}_{\text{FZ,fin, non-log}}$$

Observables

Energy

Parts of far-zone contribution are non-instantaneous ("non-local"):

$$E = E_{\text{loc}} + E_{\text{nl}}$$

Evaluate time integral in E_{nl}, e.g. for circular orbit:

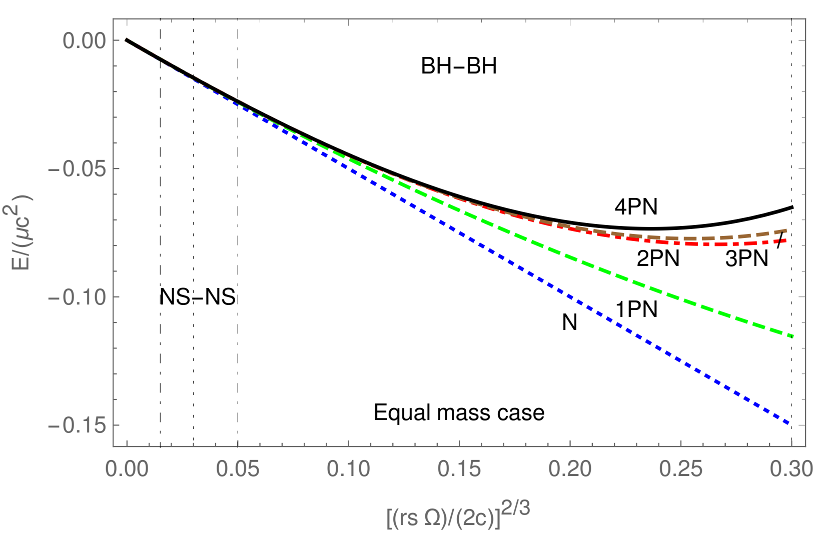
$$\nu = \frac{\mu}{M}\,, \qquad j = \frac{J}{GM}$$

$$\begin{aligned}\frac{E^{\text{circ}}_{\text{loc}}(j)}{\mu} = &-\frac{1}{2j^2} + \left(-\frac{\nu}{8} - \frac{9}{8}\right)\frac{1}{j^4} + \left(-\frac{\nu^2}{16} + \frac{7\nu}{16} - \frac{81}{16}\right)\frac{1}{j^6} + \left[-\frac{5\nu^3}{128} + \frac{5\nu^2}{64} + \left(\frac{8833}{384}\right.\right.\\ &\left.\left.-\frac{41\pi^2}{64}\right)\nu - \frac{3861}{128}\right]\frac{1}{j^8} + \left[-\frac{7\nu^4}{256} + \frac{3\nu^3}{128} + \left(\frac{41\pi^2}{128} - \frac{8875}{768}\right)\nu^2 + \left(\frac{989911}{3840}\right.\right.\\ &\left.\left.-\frac{6581\pi^2}{1024}\right)\nu - \frac{53703}{256}\right]\frac{1}{j^{10}} + \left[-\frac{21\nu^5}{1024} + \frac{5\nu^4}{1024} + \left(\frac{41\pi^2}{512} - \frac{3769}{3072}\right)\nu^3\right.\\ &\left.\left(r^E_{\nu^2} + \frac{132979\pi^2}{2048}\right)\nu^2 + \left(\frac{3747183493}{1612800} - \frac{31547\pi^2}{1536}\right)\nu - \frac{1648269}{1024}\right]\frac{1}{j^{12}} + \mathcal{O}\left(\frac{1}{j^{14}}\right)\end{aligned}$$

$$\begin{aligned}\frac{E^{\text{circ}}_{\text{nl}}}{\mu} = \nu\Bigg\{&\left[-\frac{64}{5}(\ln(j) - \gamma_E) + \frac{128}{5}\ln(2)\right]\frac{1}{j^{10}} + \left[\frac{32}{5} + \frac{28484}{105}\ln(2) + \frac{243}{14}\ln(3) - \frac{15172}{105}(\ln(j) - \gamma_E)\right.\\ &\left.+\nu\left(\frac{32}{5} + \frac{112}{5}(\ln(j) - \gamma_E) + \frac{912}{35}\ln(2) - \frac{486}{7}\ln(3)\right)\right]\frac{1}{j^{12}} + \mathcal{O}\left(\frac{1}{j^{14}}\right)\Bigg\}\end{aligned}$$

Observables

Circular energy



Observables

Periastron advance

$$K = K_{\text{loc}} + K_{\text{nl}}$$

K_{loc}, K_{nl} calculated in general, but remaining time integral in K_{nl}

For circular orbits:

$$K^{\text{circ}}_{\text{loc}}(j) = 1 + 3\frac{1}{j^2} + \left(\frac{45}{2} - 6\nu\right)\frac{1}{j^4} + \left[\frac{405}{2} + \left(-202 + \frac{123}{32}\pi^2\right)\nu + 3\nu^2\right]\frac{1}{j^6}$$
$$+ \left[\frac{15795}{8} + \left(\frac{185767}{3072}\pi^2 - \frac{105991}{36}\right)\nu + \left(-\frac{41}{4}\pi^2 + \frac{2479}{6}\right)\nu^2\right]\frac{1}{j^8} + \left[\frac{161109}{8}\right.$$
$$\left. + \left(-\frac{18144676}{525} + \frac{488373}{2048}\pi^2\right)\nu - \left(r^K_{\nu^2} + \frac{1379075}{1024}\pi^2\right)\nu^2 + \left(-\frac{1627}{6} + \frac{205}{32}\pi^2\right)\nu^3\right]\frac{1}{j^{10}} + O\left(\frac{1}{j^{12}}\right)$$
$$K^{\text{circ}}_{\text{nl}}(j) = -\frac{64}{10}\nu\left\{\frac{1}{j^8}\left[-11 - \frac{157}{6}(\ln(j) - \gamma_E) + \frac{37}{6}\ln(2) + \frac{729}{16}\ln(3)\right]\right.$$
$$+ \frac{1}{j^{10}}\left[-\frac{59723}{336} - \frac{9421}{28}[\ln(j) - \gamma_E] + \frac{7605}{28}\ln(2) + \frac{112995}{224}\ln(3)\right.$$
$$\left.\left. + \left(\frac{2227}{42} + \frac{617}{6}[\ln(j) - \gamma_E] - \frac{7105}{6}\ln(2) + \frac{54675}{112}\ln(3)\right)\nu\right] + O\left(\frac{1}{j^{12}}\right)\right.$$

Cross checks

- ✓ $\frac{\mu^0}{M^0}$ agrees with Schwarzschild limit
- ✓ $\frac{\mu^1}{M^1}$, $\frac{\mu^2}{M^2}$: poles cancel between near zone and far zone
- ✓ $\frac{\mu^3}{M^3}$, $\frac{\mu^4}{M^4}$, $\frac{\mu^5}{M^5}$ agree with [Bini, Damour, Geralico 2020]
- $\frac{\mu^1}{M^1}$ agrees with [Bini, Damour, Geralico 2020] after finite renormalisation of gravitomagnetic quadrupole moment $\dddot{J}^2 \to \left(1 - \frac{1}{3}\epsilon\right) \dddot{J}^2$
- Results for coefficients of $\pi^2 \frac{\mu^2}{M^2}$ confirmed by [Bini, Damour, Geralico 2021]

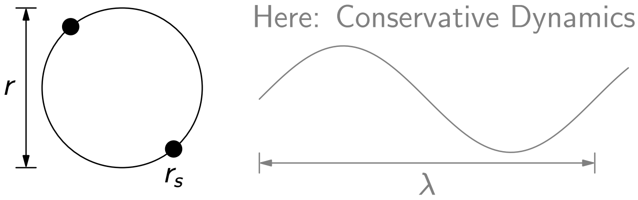
Conclusion

- Powerful methods from particle physics
  for post-Newtonian expansion of compact binaries:
  - Effective field theories
  - Techniques for multiloop calculations
- Latest results:
  - Complete 5PN (near-zone) potential
  - Partial 6PN potential
- Open questions in combination with far zone/tail

Backup

Post-Newtonian expansion

Scales



- $\omega_{GW} = \frac{2v}{r} \Rightarrow$ $\lambda \sim \frac{r}{v}$
- $r_s = 2Gm \Rightarrow$ $r_s \sim r v^2$

General relativity

General relativity action:

$$S_{\mathrm{GR}}[g^{\mu\nu}] = S_{\mathrm{EH}} + S_{\mathrm{GF}} + S_{\mathrm{matter}}$$

With $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, $g = \det(g^{\mu\nu})$:

- Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{1}{16G\pi} \int d^d x \sqrt{-g} R$$

- *Harmonic* gauge $\partial_\mu \sqrt{-g} g^{\mu\nu} = 0$:

$$S_{\mathrm{GF}} = -\frac{1}{32G\pi} \int d^d x \sqrt{-g} \Gamma_\mu \Gamma^\mu \,, \qquad \Gamma^\mu = g^{\alpha\beta} \Gamma^\mu{}_{\alpha\beta}$$

- Assume point-like matter, no spin:

$$S_{\mathrm{matter}} = -\sum_{a=1}^{2} m_a \int d\tau_a$$

Far-zone potential

Matching equation for far-zone potential:

$$V_{\mathrm{FZ}} = i\frac{d}{dt}\log\left[\int \mathcal{D}\phi_{\mathrm{rad}}\mathcal{D}A_{\mathrm{rad}}\mathcal{D}\sigma_{\mathrm{rad}}\; e^{i(S_{\mathrm{mixed}}+S_{\mathrm{radiation}})}\right]$$

Matter-radiation interaction in NRGR:

$$S_{\mathrm{mixed}} = \frac{1}{2}\int d^dx\, T^{\mu\nu}\delta g_{\mu\nu} + \mathcal{O}(\delta g_{\mu\nu}^2), \qquad \delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

$\delta g_{\mu\nu}$ multipole expanded

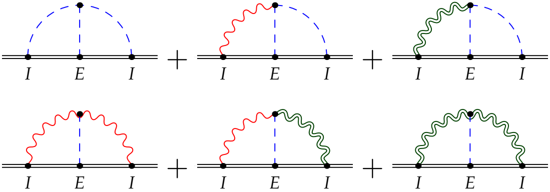
$\Rightarrow$ ϕ, A_i, σ_{ij} coupling to multipole moments $E, P_i, L_i, I_{ij}, \dots$ after integration by parts

e.g. [Ross 2012]

Far-zone potential

Matching at 4PN, conservative part:

$$-i \int dt\, V_{\mathrm{FZ}}^{\mathrm{4PN}} =$$



Far-zone potential

Matching at 4PN, conservative part:

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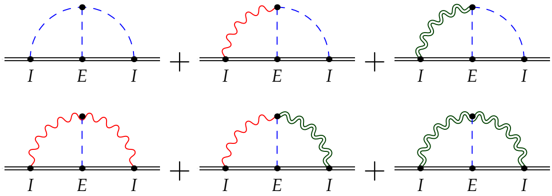

At 5PN: [Foffa, Sturani 2019–2021]

- Additional multipole moments J_{ij}, O_{ijk}
- More 2-loop diagrams
- 1PN corrections to E, I_{ij} in d dimensions

  [Marchand, Henry, Larrouturou, Marsat, Faye, Blanchet 2020]

Classical Hamiltonian

- Combine potentials $V = V_{\text{NZ}} + V_{\text{FZ}}$:

$$L_{\text{PN}}[\vec{x}_a, \vec{v}_a, \vec{a}_a, \vec{x}_a^{(3)}, \ldots] = M + T - V$$

- Eliminate higher time derivatives:
  - Total derivatives $L_{\text{PN}} \to L_{\text{PN}} + \frac{d}{dt}F$
  - Multiple zeroes $L_{\text{PN}} \to L_{\text{PN}} + FZ_1Z_2\cdots$
    with $\delta(FZ_1Z_2\cdots) = 0$ at current PN order
  - Coordinate shifts $\vec{x}_a \to \vec{x}_a + \Delta\vec{x}_a$ with $\Delta\vec{x}_a = \mathcal{O}(1\text{PN})$
- Transform to classical Hamiltonian $H = \vec{p}_a\vec{v}_a - L$
  in centre-of-mass frame
- Canonical transformations with generator g for comparisons
  and simpler expressions

$$H \to e^{D_g}H\,, \qquad D_g f = \{f, g\}$$