Vector Space of Feynman Integrals and Multivariate Intersection Numbers

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Based on joined work w/ Seva Chestnov, Hjalte Frellesvig, Stefano Laporta, Manoj K. Mandal, Pierpaolo Mastrolia, Luca Mattiazzi, Sebastian Mizera & Henrik Munch

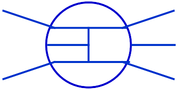
Feynman Integrals

Feynman Integrals are **ubiquitous**

$$\mathcal{I} = \int \prod_{j=1}^{\ell} d^d k_j \, \frac{1}{\mathcal{D}_1^{a_1} \ldots D_n^{a_n}}$$

- **Dimensional Regularization** employed: $d = 4 - 2\epsilon$ ['t Hooft & Veltman]
- Number of **Propagators**: $n = \ell(\ell+1)/2 + E\ell$ (Den. + ISPs)
- $\mathcal{D}_i$ **quadratic/linear** in **loop momenta**: $\mathcal{D}_i = l_i(k,p) - m_i^2$
- $a_i \in \mathbb{Z}$

Amplitude decomposition



$$= \sum_{i=1}^{\mathcal{O}(10^5)} c_i \, \mathcal{I}_i$$

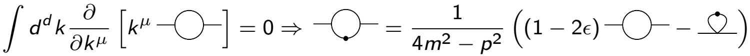
Integration by Parts Identities (IBPs)

Integrals are **not independent** $\Longrightarrow$ related by **Integration by Parts Identities** (IBPs)

[Chetyrkin & Tkachov]

$$\int d^d k_j \frac{\partial}{\partial k_j^{\mu}} \left(v^{\mu} \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}} \right) = 0, \quad \text{arbitrary vector } v^{\mu} \in \{p^{\mu}, k^{\mu}\}$$

E.g.:



Well established and **successful** strategy:
Build **over-constrained** system $\Rightarrow$ Solve $\Rightarrow$ **Basis** of **Master Integrals** $\mathcal{O}\left(10^2 - 10^3\right)$

[Laporta]

Many **public** & **private** implementations

AIR [Anastasiou & Lazopoulos] **Azurite** [Larsen & Zhang] **Crusher** [Marquard] **FineRed** [v. Mantueffel] **FiniteFlow** [Peraro] **FIRE** [Smirnov] **KIRA(+FireFly)** [Maierhofer & al.] **Reduze** [v. Mantueffel & Studerus] **LiteRed** [Lee] **Solve** [Remiddi] **SYS** [Laporta] ...

An Alternative: Intersection Theory

Despite their legacy IBPs suffer of **drawbacks**

E.g.:

- redundant equations & unwanted integrals
- (large) intermediate expressions

Always **interesting** studying **Mathematics** underpinning **Ideas** in **Physics**

In this talk **introduce New Framework** & **Computational Tools**

$\rightsquigarrow$ **Intersection Theory**

to capture **all relevant relations** among **Feynman Integrals**

Direct Decomposition into MIs   Diff. Equations & Dim. Rec. Relations for MIs

Outline

- Introduction (✓)
- Baikov representation
- Basic of Intersection Theory
- Intersection Theory & Master Integrals decomposition
- Univariate Intersection Number & Applications
- Multivariate Intersection Number & Applications
- Conclusions & Outlook

Baikov Representation

We employ **Baikov representation**

[Baikov]

$$\{\mathcal{D}_1, ..., \mathcal{D}_n\} \Rightarrow \{z_1, ..., z_n\} = \mathbf{z}$$

$$\mathcal{I} = \int \prod_{j=1}^{\ell} d^d\, k_j \,\frac{1}{\mathcal{D}_1^{a_1} \ldots \mathcal{D}_n^{a_n}} \Longrightarrow \int_{\mathcal{C}} d\mathbf{z}\, B^{\gamma}(\mathbf{z}) \,\frac{1}{z_1^{a_1} \ldots z_n^{a_n}}$$

- **B** is the **Gram Determinant** referred to as **Baikov Polynomial**

$$\mathbf{B}(\mathbf{z}) = \det \begin{pmatrix} k_1 \cdot k_1 & k_1 \cdot k_2 & \ldots & k_1 \cdot p_E \\ k_1 \cdot k_2 & k_2 \cdot k_2 & \ldots & k_2 \cdot p_E \\ \ldots & \ldots & \ldots & \ldots \\ p_E \cdot k_1 & p_E \cdot k_2 & \ldots & p_E \cdot p_E \end{pmatrix} \Bigg|_{\mathbf{s}=\mathbb{A}\,\mathbf{z}+\mathbf{c}}$$

- $\gamma = \frac{d-\ell-E-1}{2}$
- key property: $\mathbf{B}(\partial\mathcal{C})=0$
- Unitarity Cut: $\int dz \longrightarrow \oint dz$

[Bosma & al]

Basic of Intersection Theory

Similar type of **Integrals** studied within **Intersection Theory**

[Aomoto, Kita, Yoshida, Matsumoto, Goto, ...]

$$\mathcal{I} = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(z)$$

$\int_{\mathcal{C}} u(\mathbf{z}) \rightsquigarrow$ **Twisted Cycle** $\qquad \varphi(z) \rightsquigarrow$ **Twisted Co-Cycle**

u **multivalued** & $u(\partial\mathcal{C}) = 0$ $\qquad \varphi(z) = \hat{\varphi}(z)\, dz_1 \wedge \ldots \wedge dz_n$

Stokes Theorem gives

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\partial\mathcal{C}} u\,\xi = \int_{\mathcal{C}} u\,\big(d + d\log u \wedge\big)\xi = \int_{\mathcal{C}} u\nabla_\omega\,\xi$$

where $\nabla_\omega := d + \omega\wedge \equiv d + d\log u\wedge$

So

$$\int_{\mathcal{C}} u\,\varphi \equiv \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi) \qquad \varphi \sim \varphi + \nabla_\omega \xi$$

Repeat same construction for **Dual Integrals**:

$$\mathcal{I} \rightsquigarrow \mathcal{I}^\vee \qquad u \rightsquigarrow u^{-1} \qquad \nabla_\omega \rightsquigarrow \nabla_{-\omega}$$

Basic of Intersection Theory

We work with **equivalence class** of **forms**

$$\langle\varphi_L| : \varphi_L \sim \varphi_L + \nabla_\omega \xi$$

They belong to the **Twisted Co-Homology Groups**

$$\langle\varphi_L| \in H^n_\omega := \{n \text{ forms} \,|\, \nabla_\omega \varphi_n = 0\} \,/\, \{\nabla_\omega \varphi_{n-1}\} \rightsquigarrow \textbf{Vector Space}$$

We work with **equivalence class** of **dual forms**

$$|\varphi_R\rangle : \varphi_R \sim \varphi_R + \nabla_{-\omega} \xi$$

They belong to the **dual Twisted Co-Homology Groups**

$$|\varphi_R\rangle \in H^n_{-\omega} := \{n \text{ forms} \,|\, \nabla_{-\omega} \varphi_n = 0\} \,/\, \{\nabla_{-\omega} \varphi_{n-1}\} \rightsquigarrow \textbf{dual Vector Space}$$

Possible to define **equivalence class** for **(dual) contours** & **(dual) Twisted Homology Groups**

$$|\mathcal{C}_R] \in H^\omega_n \qquad\qquad [\mathcal{C}_L| \in H^{-\omega}_n$$

Basic of Intersection Theory

Define **pairing** among building blocks

$$\mathcal{I} = \langle \varphi_L | \mathcal{C}_R] \Longrightarrow \int_{\mathcal{C}_R} u \, \varphi_L \rightsquigarrow \textbf{Integrals}$$

$$\mathcal{I}^{\vee} = [\mathcal{C}_L | \varphi_R \rangle \Longrightarrow \int_{\mathcal{C}_L} u^{-1} \, \varphi_R \rightsquigarrow \textbf{dual Integrals}$$

And

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int \iota \left(\varphi_L \right) \wedge \varphi_R \rightsquigarrow \textbf{Co-Homology Intersetion Number}$$

$$\rightsquigarrow \textbf{The Scalar Product}$$

Also

$$[\mathcal{C}_L | \mathcal{C}_R] \rightsquigarrow \textbf{Homology Intersetion Number}$$

Postpone **computation algorithm** just assume they are **well defined**

Basic of Intersection Theory

Vector Space $\rightsquigarrow$ **Twisted Co-Homology Groups** $H^n_\omega \ni \langle \varphi_L |$

$\nu =$ number of **Master Integrals** [Laporta]

$=$ is **Finite** [Smirnov & Petuckhov]

$=$ number of **Critical Points** $\log u$ [Aomoto & Kita; Lee & Pomeranski]

i.e. $\#$ solutions $d \log u \equiv \omega = 0$

$=$ Number of **Independent Contours** [Zhang & al.; Primo & Tancredi]

$=$ Number of **Independent Forms**

$= \dim \left(H^n_{\pm\omega} \right)$ [Mastrolia & Mizera; FGLMMMM]

Intersection Theory & Master Integrals Decomposition

[Mastrolia & Mizera; FGLMMMM]

Ready to tackle **MIs decomposition** within **Intersection Theory**

$$\mathcal{I} = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i \qquad \text{where} \quad \mathcal{I} = \langle \varphi | \mathcal{C}] \quad \& \quad \mathcal{J} = \langle e_i | \mathcal{C}]$$

$$\rightsquigarrow \quad \langle \varphi | = \sum_{i=1}^{\nu} c_i \, \langle e_i | \qquad \text{where} \quad \langle \varphi | \rightsquigarrow \text{given} \quad \& \quad \langle e_i | \rightsquigarrow \text{chosen}$$

We have a **pairing** $\langle \varphi_L | \varphi_R \rangle$ $\rightsquigarrow$ introduce **auxiliary dual form** $|\psi\rangle$ & **basis** $|h_i\rangle$

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \dots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \dots & \langle e_1|h_\nu\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix} = \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$\rightsquigarrow$ $(\nu+1)\times(\nu+1)$ degenerate matrix

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[Mastrolia & Mizera; FGLMMMM]

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$\rightsquigarrow$ $(\nu{+}1)\times(\nu{+}1)$ degenerate matrix

Intersection Theory & Master Integrals Decomposition

[Mastrolia & Mizera; FGLMMMM]

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Intersection Theory & Master Integrals Decomposition

[Mastrolia & Mizera; FGLMMMM]

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$\rightsquigarrow (\nu+1) \times (\nu+1)$ degenerate matrix

Intersection Theory & Master Integrals Decomposition

$\langle\varphi| \rightsquigarrow$ given $\quad \langle e_i| \rightsquigarrow$ chosen **basis** $\quad |\psi\rangle \rightsquigarrow$ **auxiliary** dual form $\quad |h_i\rangle \rightsquigarrow$ **dual basis**

We built

$$\mathbf{M} = \begin{pmatrix} \langle\varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{C} & \mathbf{B} \end{pmatrix} \qquad (\nu+1)\times(\nu+1) \text{ degenrate matrix}$$

Then

$$0 = \det \mathbf{M} = \det \mathbf{C} \cdot \left(\langle\varphi|\psi\rangle - \mathbf{A}^\intercal \mathbf{C}^{-1} \mathbf{B} \right) \rightsquigarrow 0 \qquad \text{since } \det \mathbf{C}_{ij} = \det\langle e_i|h_j\rangle \neq 0$$

Plugging back $\mathbf{A}^\intercal$ and $\mathbf{B}$

$$\langle\varphi|\psi\rangle = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\psi\rangle$$

Intersection Theory & Master Integrals Decomposition

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$$\langle\varphi|\psi\rangle = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\psi\rangle = \langle\varphi|\,\mathbb{I}\,|\psi\rangle$$

$$\mathbb{I} = \sum_{i,j=1}^{\nu} |h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i| \rightsquigarrow \textbf{Resolution of Identity}$$

Intersection Theory & Master Integrals Decomposition

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$$\langle\varphi|\psi\rangle = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|\psi\rangle \rightsquigarrow \textbf{Drop arbitrary } |\psi\rangle$$

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$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i| \rightsquigarrow \textbf{Direct Decomposition into MIs}$$

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Intersection Theory & Master Integrals Decomposition

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Intersection Theory & Integral Relations

Within **Intersection Theory MIs decomposition** reads

$$\mathcal{I} = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i \qquad \langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \, \langle e_i | \rightsquigarrow \textbf{Master Decomposition Formula}$$

so **coefficients** are **directly computed** [Mastrolia & Mizera; FGLMMMM]

$$c_i = \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \qquad \text{where } \mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

Remark: c_i do **not depend choice** of $|h_i\rangle$

System of **Differential Equations** [Kotikov; Remiddi; Ghermann & Remiddi; Henn]

$$\partial_s \, \mathcal{J}_k = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i \qquad \text{where} \qquad \partial_s \, \mathcal{J}_k = \partial_s \int u \, e_k = \int u \, (\partial_s + \sigma) \, e_k \quad \text{with } \sigma := \partial_s \log u$$

so **coefficients** are **directly computed** [Mastrolia & Mizera; FGLMMMM; Mizera & Pokraka]

$$c_i = \sum_{j=1}^{\nu} \langle (\partial_s + \sigma) \, e_k | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \qquad \text{where } \mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

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System of **Differential Equations** (✓) & System of **Dimension-shift Equations** [Tarasov]

$$\mathcal{J}_k^{d+2} = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i^{d} \qquad \text{where} \qquad \mathcal{J}_k^{d+2} = \int u \, B \, e_k \qquad \text{with } u = B^{\gamma}$$

so **coefficients** are **directly computed** [Mastrolia & Mizera; FGLMMMM]

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System of **Differential Equations** (✓) & System of **Dimension-shift Equations** (✓)

$$\partial_s \, \mathcal{J}_k = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i \qquad \mathcal{J}_k^{d+2} = \sum_{i=1}^{\nu} c_i \, \mathcal{J}_i^d$$

$$c_i = \sum_{j=1}^{\nu} \langle (\partial_s + \sigma) \, e_k | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \qquad c_i = \sum_{j=1}^{\nu} \langle B \, e_k | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji}$$

The Univariate Intersection Number

[Cho & Matsumoto; Matsumoto; Mizera]

We have to define the **Intersection Number**

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int \varphi_L \wedge \varphi_R \rightsquigarrow \text{“}0/0\text{”} \quad \textbf{since} \quad dz \wedge dz/\textbf{pole}$$

Slightly modify it

$$\begin{aligned} \langle \varphi_L | \varphi_R \rangle &= \frac{1}{2\pi i} \int \iota(\varphi_L) \wedge \varphi_R \\ &= \frac{1}{2\pi i} \int \left(\varphi_L - \sum_{p \in \mathcal{P}_\omega} \nabla_\omega h_p(z, \bar{z}) \psi_p \right) \wedge \varphi_R \\ &= \sum_{p \in \mathcal{P}_\omega} \text{Res}\left(\psi_p \, \varphi_R \right) \end{aligned}$$

where

$$\mathcal{P}_\omega := \{\text{poles of } \omega\}$$

and

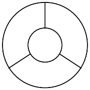
$$\psi_p \quad \textbf{local solution} \text{ of } \quad \nabla_\omega \psi_p \equiv (d + \omega \wedge) = \varphi_L \quad \text{around } p \in \mathcal{P}_\omega$$

i.e. fix coefficients in educated ansatz $\psi_p = \sum_{j=\min}^{\max} \alpha_p (z-p)^j + \mathcal{O}(z-p)^{\max+1}$

Decomposition by Univariate Intersection

[Mastrolia & Mizera; FGLMMMM]

Univariate Intersection $\rightsquigarrow$ decomposition on the **Maximal Cut** i.e. subtopologies $\to 0$



$$u = \left(\frac{z}{2} - \frac{3z^2}{16} \right)^{(d-5)/2}$$

$$\omega = d \log u = 0 \qquad 1 \text{ sol.} \Rightarrow 1 \text{ MI} \qquad \nu = 1 \qquad \mathcal{P}_\omega = \{0, 1, 8/3\}$$

MI chosen as

$$\mathcal{J} = I_{1,1,1,1,1,1,1,1,1;0} = \langle 1 | \mathcal{C}] \quad \Longrightarrow \quad \langle e | := \langle 1 |$$

Decompose

$$\mathcal{I} = I_{1,1,1,1,1,1,1,1,1;-2} = \langle z^2 | \mathcal{C}] \quad \Longrightarrow \quad \langle \varphi | := \langle z^2 |$$

Dual Basis as $|h\rangle \equiv |e\rangle = |1\rangle$ & compute

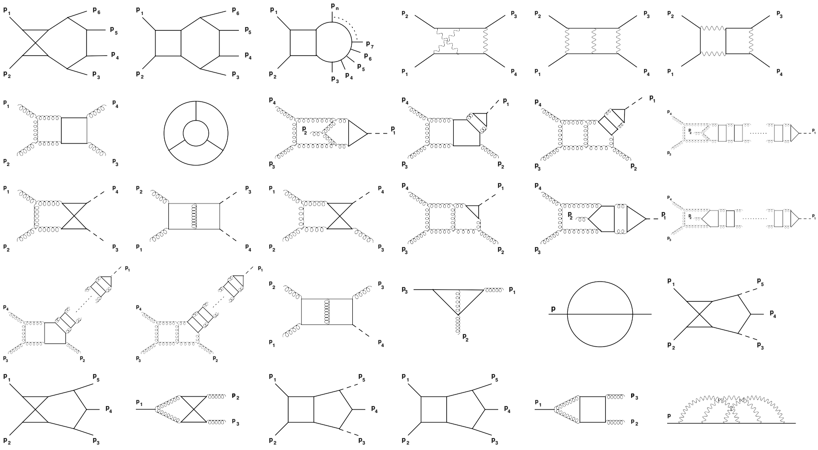
$$\mathbf{C} := \langle e | h \rangle = \langle 1 | 1 \rangle = \frac{16(d-5)}{9(d-6)(d-4)} \qquad \langle \varphi | h \rangle = \langle z^2 | 1 \rangle = \frac{256(d-5)(d-1)}{81(d-6)(d-4)(d-2)}$$

Master Decomposition Formula gives

$$\mathcal{I} = c\,\mathcal{J} \qquad c = \frac{\langle z^2 | 1 \rangle}{\langle 1 | 1 \rangle} = \frac{16(d-1)}{9(d-2)}$$

Further Examples

Decomposition on the **Maximal Cut** [FGLMMMM]



Loop-by-Loop Baikov employed [Frellesvig & Papadopoulos]

The Multivariate Intersection Number

[Mizera; FGLMMMM]

Focus on 2 **vars** $\rightsquigarrow$ **Given** $\varphi_L(z_1, z_2)$ & $\varphi_R(z_1, z_2)$ $\rightsquigarrow$ **compute** $\langle \varphi_L | \varphi_R \rangle$

Univariate known (✓) $\rightsquigarrow$ **recursive strategy**

$$\langle \varphi_L | = \langle e_i^{(1)} | \wedge \langle \varphi_{L,i}^{(2)} | \qquad \textit{where} \qquad \langle \varphi_{L,i}^{(2)} | = \langle \varphi_L | h_j^{(1)} \rangle \left(\mathbf{C}_1^{-1} \right)_{ji} \quad \checkmark$$

$$| \varphi_R \rangle = | h_i^{(1)} \rangle \wedge | \varphi_{R,i}^{(2)} \rangle \qquad \textit{where} \qquad \langle \varphi_{R,i}^{(2)} | = \left(\mathbf{C}_1^{-1} \right)_{ij} \langle e_j^{(1)} | \varphi_R \rangle \quad \checkmark$$

Then

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L | \mathbb{I}_1 | \varphi_R \rangle = \langle \varphi_L | h_j^{(1)} \rangle \left(\mathbf{C}_1^{-1} \right)_{ji} \langle e_i^{(1)} | \varphi_R \rangle$$
$$= \sum_{p \in \mathcal{P}} \text{Res} \left[\psi_{p,i} \, \varphi_{R,j}^{(2)} \, \mathbf{C}_{1,ij} \right]$$

with ψ_p **local solution**

$$\partial_2 \psi_{p,i} + \psi_{p,j} \, \Omega_{ji}^{(2)} = \varphi_{L,i}^{(2)}$$

where

$$\Omega_{ji}^{(2)} := \langle (\partial_2 + \omega_2) \, e_j^{(1)} | h_k^{(1)} \rangle \left(\mathbf{C}_1^{-1} \right)_{ki} \quad \checkmark$$

The Multivariate Intersection Number

[Mizera; FGLMMMM]

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$$| \varphi_R \rangle = | h_i^{(1)} \rangle \wedge | \varphi_{R,i}^{(2)} \rangle \qquad \textit{where} \qquad \langle \varphi_{R,i}^{(2)} | = \left(\mathbf{C}_1^{-1} \right)_{ij} \langle e_j^{(1)} | \varphi_R \rangle \quad ✓$$

Then

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L | \mathbb{I}_1 | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \text{Res} \left[\psi_{p,i} \, \varphi_{R,j}^{(2)} \, \mathbf{C}_{1,ij} \right]$$

where

$$\Omega_{ji}^{(2)} := \langle (\partial_2 + \omega_2) \, e_j^{(1)} | h_k^{(1)} \rangle \left(\mathbf{C}_1^{-1} \right)_{ki} \quad ✓$$

Intuition

$$\int dz_2 \, \varphi_L^{(1)}(z_2) \int dz_1 \, u(z_1, z_2) \, e^{(1)} = \int dz_2 \, \varphi_L^{(1)}(z_2) \, u(z_2)$$

new $u(z_2) \Longrightarrow$ **new** $(d + \Omega \wedge) \neq \nabla_\omega$

The Multivariate Intersection Number

[Mizera; FGLMMMM]

Focus on **n vars** $\rightsquigarrow$ **Given** $\varphi_L(z_1,...,z_n)$ & $\varphi_R(z_1,...,z_n)$ $\rightsquigarrow$ **compute** $\langle\varphi_L|\varphi_R\rangle$

n-1 known (✓) $\rightsquigarrow$ **recursive strategy**

$$\langle\varphi_L| = \langle e_i^{(n-1)}| \wedge \langle\varphi_{L,i}^{(n)}| \quad \textit{where} \quad \langle\varphi_{L,i}^{(n)}| = \langle\varphi_L|h_j^{(n-1)}\rangle \left(\mathbf{C}_{n-1}^{-1}\right)_{ji} \qquad ✓$$

$$|\varphi_R\rangle = |h_i^{(n-1)}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle \quad \textit{where} \quad |\varphi_{R,i}^{(n)}\rangle = \left(\mathbf{C}_{n-1}^{-1}\right)_{ij} \langle e_j^{(n-1)}|\varphi_R\rangle \qquad ✓$$

Then

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p\in\mathcal{P}} \text{Res}\left[\psi_{i,p}\,\varphi_{R,j}^{(n)}\,\mathbf{C}_{n-1,ij}\right]$$

with ψ_p **local solution**

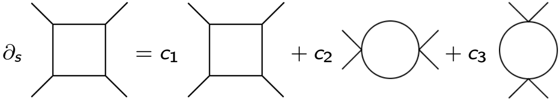
$$\partial_n\,\psi_{i,p} + \psi_{j,p}\Omega_{ji}^{(n)} = \varphi_{L,i}^{(n)}$$

where

$$\Omega_{ji}^{(n)} := \langle(\partial_n + \omega_n)\, e_j^{(n-1)}|h_k^{(n-1)}\rangle \left(\mathbf{C}_{n-1}^{-1}\right)_{ki} \qquad ✓$$

Decomposition by Multivariate Intersection

$n = 4$ vars

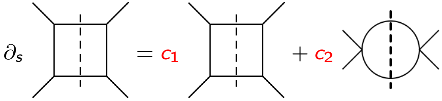


Decomposition by Multivariate Intersection

$n = 2$ vars **Unitarity Cuts** $\rightsquigarrow$ **less** variables & **preserve** relations

$$u = K\, B^{(d-5)/2} \cdot z_2^{\rho_2}\, z_4^{\rho_4} \rightsquigarrow \text{regulators} \qquad \text{with} \quad B = st^2 + s(z_2 - z_4)^2 - 2t\big(s(z_2+z_4) + 2 z_2 z_4\big)$$

$$K = (-st(s+t))^{(4-d)/2}$$



$$\partial_s \mathcal{J}_\square \rightsquigarrow \langle \varphi | = \left\langle \frac{\partial_s \log u}{z_2 z_4} \right| \qquad \mathcal{J}_\square \rightsquigarrow \langle e_1 | = \left\langle \frac{1}{z_2 z_4} \right| \qquad \mathcal{J}_\bigcirc \rightsquigarrow \langle e_2 | = \langle 1 |$$

Choose **arbitrary dual basis** e.g.: $|h_i\rangle = |e_i\rangle$

$$c_1 = \sum_{j=1}^{2} \langle \varphi | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{j1} = \frac{(d-6)\, t - 2s}{2s\,(s+t)} \quad \checkmark$$

$$c_2 = \sum_{j=1}^{2} \langle \varphi | h_j \rangle \, \left(\mathbf{C}^{-1}\right)_{j2} = \frac{2(d-3)}{s^2\,(s+t)} \quad \checkmark$$

Canonical Basis by Intersection Number

Intersection Theory $\rightsquigarrow$ **Differential Equations** in **Canonical Basis** [Chen & al.]

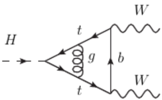
- γ_j' and β_j' **integers** $\rightsquigarrow$ $u = \frac{\mathcal{K}_1^\epsilon}{\mathcal{K}_0} \prod_{j=0}^{\nu} (z_j - c_j)^{-\gamma_j' - \beta_j' \epsilon}$

$$e_i = \frac{\mathcal{K}_0}{z - c_i} \prod_{j=0}^{\nu} (z_j - c_j)^{-\gamma_j'}$$

- γ_1 **half integers** $\rightsquigarrow$ $u = \frac{\mathcal{K}_1^\epsilon}{\mathcal{K}_0} [(z-c_0)(z-c_1)]^{-\gamma_1 - \beta_1 \epsilon} \prod_{j=2}^{\nu} (z_j - c_j)^{-\gamma_j' - \beta_j' \epsilon}$

$$e_1 = \frac{\mathcal{K}_0}{[(z-c_0)(z-c_1)]^{1/2-\gamma_1}} \prod_{j=2}^{\nu} (z-c_j))^{\gamma_j'} \qquad e_i = \frac{\mathcal{K}_0}{z-c_i} \frac{\sqrt{(c_0-c_i)(c_1-c_j)}}{[(z-c_0)(z-c_1)]^{1/2-\gamma_1}} \prod_{j=2}^{\nu} (z-c_j))^{\gamma_j'}$$

Applied to HW^+W^- **vertex** [Ma & al.]



The Multivariate Intersection Number :: Properties

Intersection Numbers enjoy interesting properties $\rightsquigarrow$ simplify computation
Remarkable Simplification for $d\log$ forms $\rightsquigarrow$ no recursion [Mizera]

$$\langle \varphi_L | \varphi_R \rangle = (-1)^n \sum_{(z_1^*,\dots,z_n^*)} \det{}^{-1} \begin{pmatrix} \partial_1 \omega_1 & \cdots & \partial_n \omega_1 \\ \vdots & \ddots & \vdots \\ \partial_1 \omega_n & \cdots & \partial_n \omega_n \end{pmatrix} \varphi_L \, \varphi_R \Big|_{z=z^*} \quad \text{with } z^* := \text{zeros of } \omega$$

Invariance within the same **equivalence class** [FGLMMMM]

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi_R \rangle = \langle \varphi_L | \varphi'_R \rangle = \langle \varphi'_L | \varphi'_R \rangle \quad \text{with} \qquad \varphi'_{L/R} = \varphi_{L/R} + \nabla_{\pm\omega} \, \xi_{L/R}$$

Invariance within the same **equivalence class** in the **recursion** [Weinzierl]

$$\varphi^{(n)}_{L,i} \sim \varphi'^{(n)}_{L,i} + \left(\partial_n \, \xi_{L,i} + \xi_{L,j} \, \Omega^{(n)}_{ji} \right)$$
$$\varphi^{(n)}_{R,i} \sim \varphi'^{(n)}_{R,i} + \left(\partial_n \, \xi_{R,i} - \Omega^{\vee\,(n)}_{ij} \, \xi_{R,j} \right)$$

Provided $\Omega^{(n)}$ simple pole $\rightsquigarrow$ $\varphi^{(n)}_{L/R,i}$ **simple pole** $\rightsquigarrow$ **Global Residue Th.**

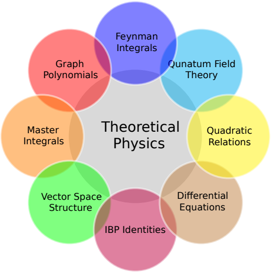
Conclusions & Outlook

- Evidence that **algebra** of **Feynman Integrals** controlled by **Intersection Theory** (✓)
- **Intersection Number** acts as **scalar product** $\rightsquigarrow$ **direct projection** into **MIs** (✓)

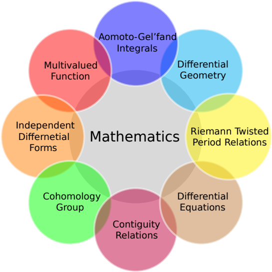
$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|$$

- Obtain **Differential Equations** & **Dimensional Recurrence Relations** (✓)
- Further explore the framework & simplify computational algorithms
  Promising $\rightsquigarrow$ **Relative Co-Homology** ($\rho = 0$ & $\mathbf{C} = \mathbb{I}$) [Caron-Huot & Pokraka]
- **Improve algorithm** for **Multivariate Intersection Number**
- Combine with **Finite Fields Arithmetics** [Peraro]
- **Quadratic Relations** for **Feynman Integrals** as **Riemann Twisted Period Relations** [Cho & Matsumoto]

$$\langle\varphi_L|\varphi_R\rangle = \langle\varphi_L|\mathcal{C}_R]\,[\mathcal{C}_R|\mathcal{C}_L]^{-1}\,[\mathcal{C}_L|\varphi_R\rangle$$



Intersection Theory
for
Twisted de Rham
Co-homology



Thank You!