QCD corrections to evolution equations for operator matrix elements

Sven-Olaf Moch

Universität Hamburg

A Loop Summit - new perturbative results and methods in precision physics, Cadenabbia , July 29, 2021

Based on work done in collaboration with:

- *Renormalization of non-singlet quark operator matrix elements for off-forward hard scattering*
  S. M. and S. Van Thurenhout, `arXiv:2107.02470`

- *Five-loop contributions to low-N non-singlet anomalous dimensions in QCD*
  F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt
  `arXiv:1812.11818`

- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`

- *Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1703.09532`

- *Two-loop conformal generators for leading-twist operators in QCD*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1601.05937`

- Many more papers of MVV and friends . . .
  `2001 - ...`

QCD evolution at 1% precision

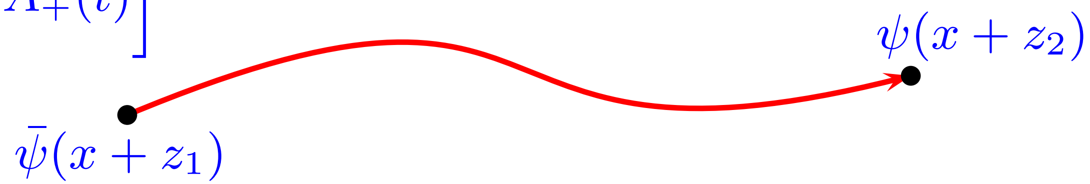
Operator matrix elements

- QCD applications to hard processes use nonlocal operators of partons at light-like separation

$$\mathcal{O}_\mu(x; z_1, z_2) = \bar{\psi}(x + z_1)\gamma_\mu[z_1, z_2]\psi(x + z_2)$$

  - quark and anti-quark fields joined by Wilson line along '$+$'-direction

$$[z_1, z_2] = \text{Pexp}\left[ig\int_{z_2}^{z_1} dt A_+(t)\right]$$



- Expansion of $\mathcal{O}_\mu(x; z_1, z_2)$ at short distances leads to local operators
  - (anti-)quark fields with covariant derivatives $D_\mu = \partial_+ - igA_+$

$$\bar{\psi}(x)(\overleftarrow{D}_+)^m\gamma_\mu(\overrightarrow{D}_+)^k\psi(x)$$

Applications

- (Generalized) parton distributions: PDFs and GPDs
- Hard exclusive reactions with identified hadrons $N(p)$ and $N(p')$ in initial and final state: $\gamma^* N(p) \to \gamma N(p')$ (DVCS)
- Meson-photon transition form factors $\gamma^* \to \gamma\pi$

Braun, Manashov, S.M., Schönleber '21, Gao, Huber, Ji, Wang '21

Light-ray operators

- Short-distance expansion yields light-ray operators $\mathcal{O}_\mu(x; z_1, z_2)$ with light-like direction n

$$[\mathcal{O}](x; z_1, z_2) \quad \equiv \quad \sum_{m,k} \frac{z_1^m z_2^k}{m!k!} \left[\bar{\psi}(x) (\overleftarrow{D} \cdot n)^m \not{n} (n \cdot \overrightarrow{D})^k \psi(x) \right]$$

- multiplicative renormalization $[\mathcal{O}] = Z\mathcal{O}$
- Light-ray operators satisfy renormalization group equation Balitsky, Braun '87

$$\Big(\mu^2 \partial_{\mu^2} + \beta(a_s) \partial_{a_s} + \mathcal{H}(a_s) \Big) [\mathcal{O}](x; z_1, z_2) = 0$$

- Integral operator $\mathcal{H}(a_s)$ acts on light-cone coordinates of fields $z_{12}^\alpha = z_1(1-\alpha) + z_2\alpha$

$$\mathcal{H}(a_s)[\mathcal{O}](z_1, z_2) \; = \; \int_0^1 d\alpha \int_0^1 d\beta \, h(\alpha, \beta) [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

- Evolution kernel $h(\alpha, \beta)$
- Mellin moments $\gamma_{N,N} = \int_0^1 d\alpha \int_0^1 d\beta \, (1 - \alpha - \beta)^{N-1} h(\alpha, \beta)$ are anomalous dimensions of leading-twist local operators with $N = m + k$ derivatives

Evolution equations

- Leading-order result for evolution kernel

$$\mathcal{H}^{(1)}\, f(z_1, z_2) \quad = 4C_F \Big\{ \int_0^1 d\alpha\, \frac{\bar{\alpha}}{\alpha} \Big[2 f(z_1, z_2) - f(z_{12}^{\alpha}, z_2) - f(z_1, z_{21}^{\alpha}) \Big]$$

$$- \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, f(z_{12}^{\alpha}, z_{21}^{\beta}) + \tfrac{1}{2} f(z_1, z_2) \Big\}$$

- Expression comprises all classical leading-order QCD evolution equations
  - PDFs Altarelli, Parisi ‘77; $\gamma_{N,N}^{(0)}$
  - meson light-cone distribution amplitudes

    Efremov, Radyushkin, Brodsky, Lepage
  - general evolution equation for GPDs Belitsky, Müller ‘99; $h^{(1)}(\alpha, \beta)$

Task

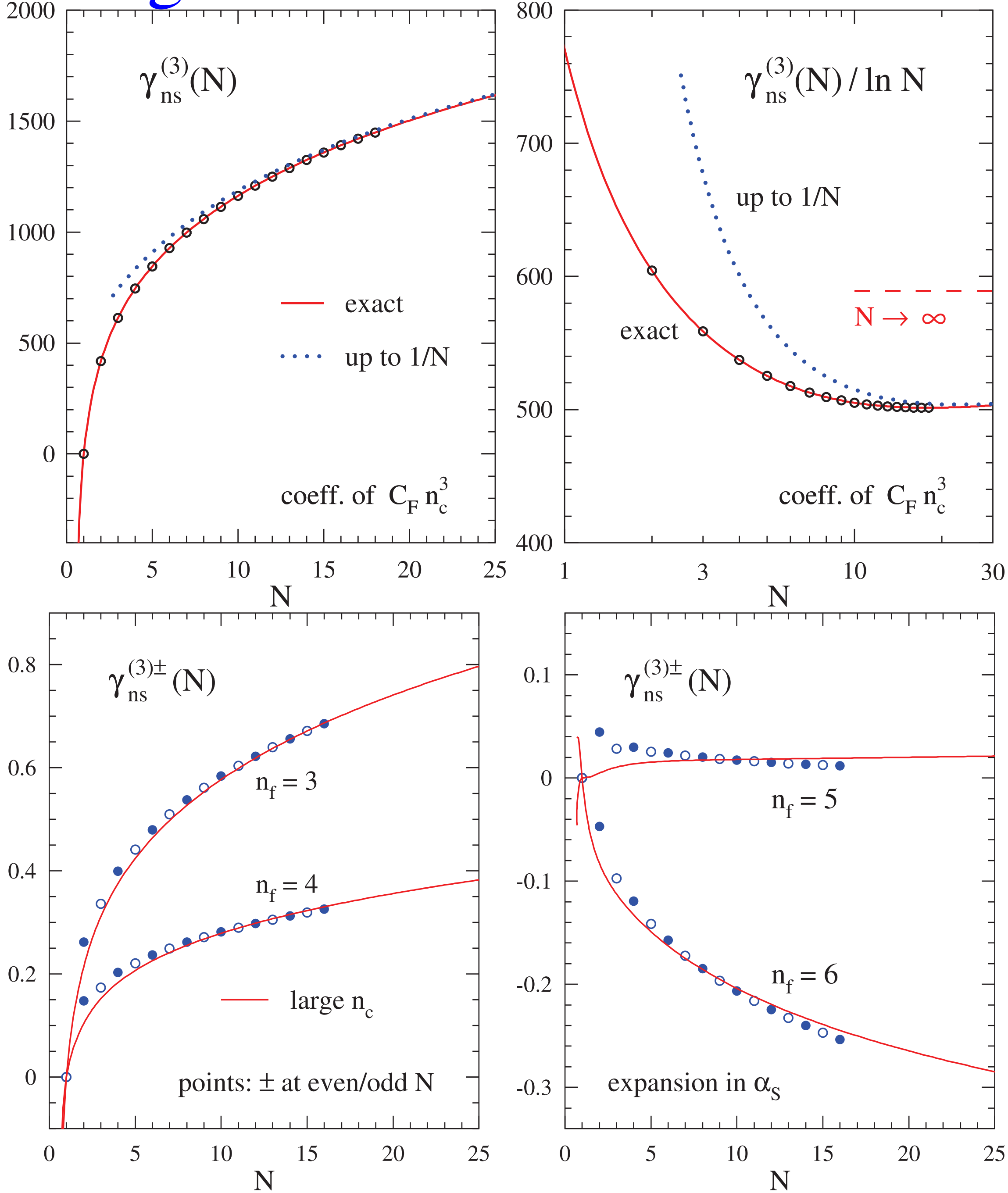
- Push accuracy of evolution equations to NNLO and beyond
- Computation of anomalous dimensions in forward and off-forward kinematics to three and four loops

Forward kinematics

Four-loop non-singlet anomalous dimensions

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$
  in large-n_c limit
  and large-N expansion

- Bottom: results for even-N ($\gamma_{\rm ns}^{(3)+}(N)$)
  and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$)
  in large-n_c limit
  for $n_f = 3, \dots 6$

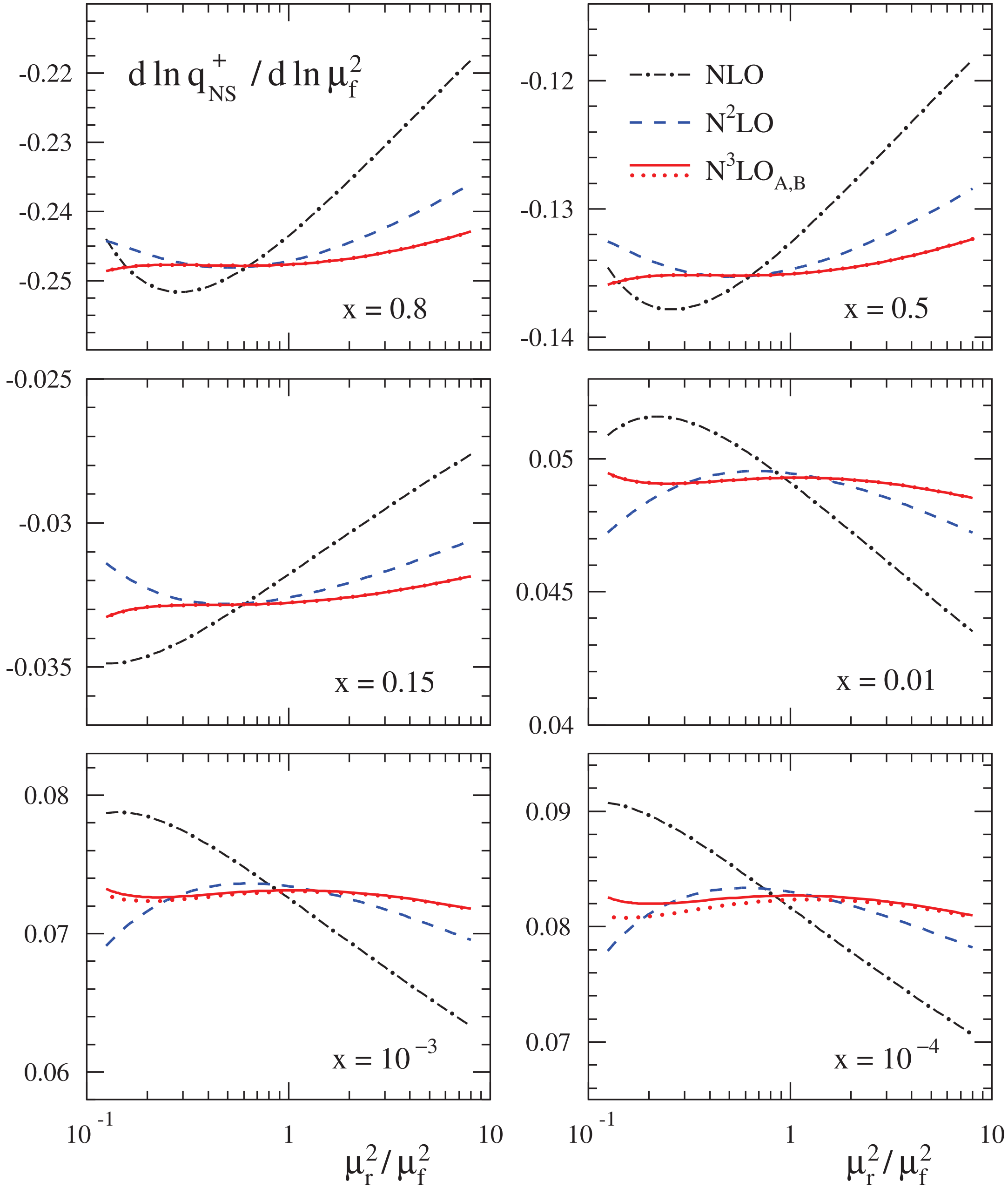


Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

$$xq_{\rm ns}^{+}(x, \mu_0^2) \;=\; x^{0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Five-loop non-singlet anomalous dimensions

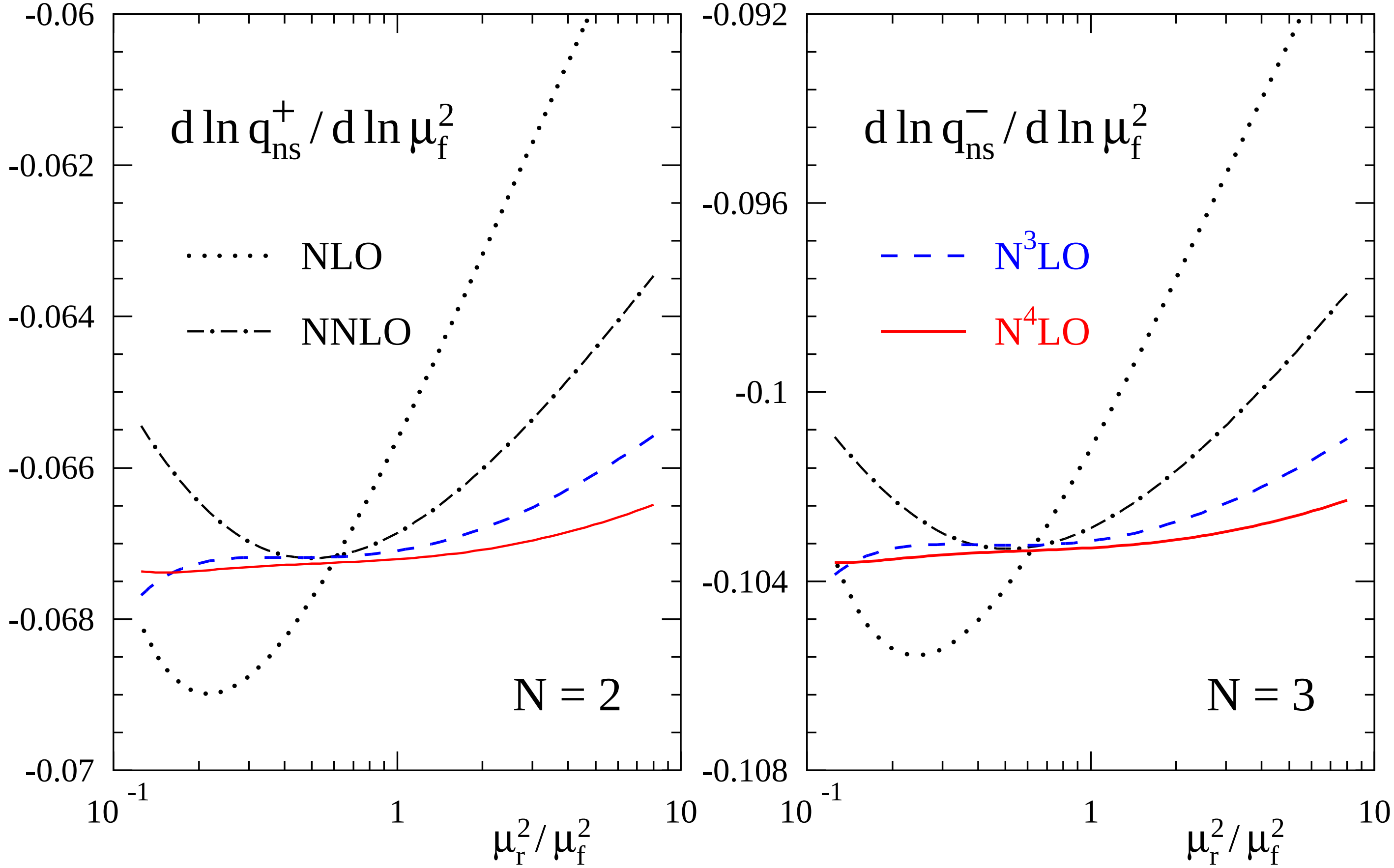
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions $\gamma_{\rm ns}^{\pm}$
  - implementation by Herzog, Ruijl '17 of local R^* operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \;=\;& \\
& C_F^5 \Big[\frac{9306376}{19683} - \frac{802784}{729}\zeta_3 - \frac{557440}{81}\zeta_5 + \frac{12544}{9}\zeta_3^2 + 8512\zeta_7 \Big] \\
& - C_A C_F^4 \Big[\frac{81862744}{19683} - \frac{1600592}{243}\zeta_3 + \frac{59840}{81}\zeta_4 - \frac{142240}{27}\zeta_5 + 3072\zeta_3^2 - \frac{35200}{9}\zeta_6 + 19936\zeta_7 \Big] \\
& + C_A^2 C_F^3 \Big[\frac{63340406}{6561} - \frac{1003192}{243}\zeta_3 - \frac{229472}{81}\zeta_4 + \frac{61696}{27}\zeta_5 + \frac{30976}{9}\zeta_3^2 - \frac{35200}{9}\zeta_6 + 15680\zeta_7 \Big] \\
& - C_A^3 C_F^2 \Big[\frac{220224724}{19683} + \frac{4115536}{729}\zeta_3 - \frac{170968}{27}\zeta_4 - \frac{3640624}{243}\zeta_5 + \frac{70400}{27}\zeta_3^2 + \frac{123200}{27}\zeta_6 + \frac{331856}{27}\zeta_7 \Big] \\
& + C_A^4 C_F \Big[\frac{266532611}{39366} + \frac{2588144}{729}\zeta_3 - \frac{221920}{81}\zeta_4 - \frac{3102208}{243}\zeta_5 + \frac{74912}{81}\zeta_3^2 + \frac{334400}{81}\zeta_6 + \frac{178976}{27}\zeta_7 \Big] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \Big[\frac{15344}{81} - \frac{12064}{27}\zeta_3 - \frac{704}{3}\zeta_4 + \frac{58400}{27}\zeta_5 - \frac{6016}{3}\zeta_3^2 - \frac{19040}{9}\zeta_7 \Big] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_F \Big[\frac{23968}{81} - \frac{733504}{81}\zeta_3 + \frac{176320}{81}\zeta_5 + \frac{6400}{3}\zeta_3^2 + \frac{77056}{9}\zeta_7 \Big] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_A \Big[\frac{82768}{81} - \frac{555520}{81}\zeta_3 + \frac{10912}{9}\zeta_4 - \frac{1292960}{81}\zeta_5 + \frac{84352}{27}\zeta_3^2 + \frac{140800}{27}\zeta_6 + 12768\zeta_7 \Big] \\
& + n_f C_F^4 \Big[\frac{1824964}{19683} - \frac{463520}{243}\zeta_3 + \frac{21248}{81}\zeta_4 - \frac{16480}{81}\zeta_5 + \frac{6656}{9}\zeta_3^2 - \frac{6400}{9}\zeta_6 + \frac{8960}{3}\zeta_7 \Big] \\
& - n_f C_A C_F^3 \Big[\frac{3375082}{6561} - \frac{420068}{243}\zeta_3 - \frac{48256}{81}\zeta_4 + \frac{458032}{81}\zeta_5 + \frac{3968}{3}\zeta_3^2 - \frac{8000}{3}\zeta_6 + \frac{4480}{3}\zeta_7 \Big] \\
& + n_f C_A^2 C_F^2 \Big[\frac{15291499}{13122} + \frac{1561600}{243}\zeta_3 - \frac{114536}{27}\zeta_4 - \frac{252544}{243}\zeta_5 + \frac{24896}{27}\zeta_3^2 + \frac{13600}{27}\zeta_6 + \frac{11200}{27}\zeta_7 \Big] \\
& - n_f C_A^3 C_F \Big[\frac{48846580}{19683} + \frac{4314308}{729}\zeta_3 - \frac{274768}{81}\zeta_4 - \frac{1389080}{243}\zeta_5 + \frac{27808}{81}\zeta_3^2 + \frac{184000}{81}\zeta_6 + \frac{39088}{27}\zeta_7 \Big] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \Big[\frac{22096}{27} + \frac{43712}{81}\zeta_3 - \frac{512}{9}\zeta_4 - \frac{217280}{81}\zeta_5 - \frac{25088}{27}\zeta_3^2 + \frac{25600}{27}\zeta_6 - 2464\zeta_7 \Big] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{170752}{81} - \frac{328832}{81}\zeta_3 + \frac{650240}{81}\zeta_5 - \frac{8192}{9}\zeta_3^2 - \frac{35840}{9}\zeta_7 \Big] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{207824}{81} + \frac{251392}{81}\zeta_3 - \frac{5632}{9}\zeta_4 - \frac{522880}{81}\zeta_5 + \frac{15872}{27}\zeta_3^2 + \frac{70400}{27}\zeta_6 - \frac{29120}{9}\zeta_7 \Big] \\
& + n_f^2 C_F^3 \Big[\frac{1082297}{6561} - \frac{145792}{243}\zeta_3 + \frac{1072}{81}\zeta_4 + \frac{55552}{81}\zeta_5 + \frac{1792}{9}\zeta_3^2 - \frac{3200}{9}\zeta_6 \Big] \\
& + n_f^2 C_A C_F^2 \Big[\frac{332254}{2187} - \frac{85016}{243}\zeta_3 + \frac{20752}{27}\zeta_4 - \frac{28544}{81}\zeta_5 - \frac{13952}{27}\zeta_3^2 + \frac{1600}{27}\zeta_6 \Big] \\
& + n_f^2 C_A^2 C_F \Big[\frac{631400}{6561} + \frac{214268}{243}\zeta_3 - 784\zeta_4 - \frac{53344}{243}\zeta_5 + \frac{25472}{81}\zeta_3^2 + \frac{22400}{81}\zeta_6 \Big] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{43744}{81} - \frac{35648}{81}\zeta_3 - \frac{1792}{9}\zeta_4 - \frac{52480}{81}\zeta_5 + \frac{2048}{27}\zeta_3^2 + \frac{12800}{27}\zeta_6 \Big] \\
& + n_f^3 C_F^2 \Big[\frac{265510}{19683} + \frac{11872}{729}\zeta_3 - \frac{128}{3}\zeta_4 + \frac{512}{27}\zeta_5 \Big] \\
& + n_f^3 C_A C_F \Big[\frac{168677}{19683} + \frac{11872}{729}\zeta_3 + \frac{2752}{81}\zeta_4 - \frac{4096}{81}\zeta_5 \Big] - n_f^4 C_F \Big[\frac{5504}{19683} + \frac{1024}{729}\zeta_3 - \frac{128}{81}\zeta_4 \Big]
\end{aligned}$$

$$\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \;=\;& \\
& C_F^5 \Big[\frac{81472935625}{80621568} + \frac{99382175}{23328}\zeta_3 - \frac{3395975}{162}\zeta_5 - \frac{9650}{9}\zeta_3^2 + \frac{34685}{2}\zeta_7 \Big] \\
& - C_A C_F^4 \Big[\frac{286028134219}{80621568} - \frac{23916529}{7776}\zeta_3 - 4490\zeta_3^2 + \frac{134090}{81}\zeta_4 - \frac{2468075}{108}\zeta_5 - \frac{55000}{9}\zeta_6 + \frac{155155}{4}\zeta_7 \Big] \\
& + C_A^2 C_F^3 \Big[\frac{20173099267}{3359232} - \frac{15401281}{864}\zeta_3 + \frac{732787}{1296}\zeta_4 + \frac{1972075}{216}\zeta_5 - \frac{63830}{9}\zeta_3^2 - \frac{79750}{9}\zeta_6 + \frac{139895}{4}\zeta_7 \Big] \\
& - C_A^3 C_F^2 \Big[\frac{166662991819}{20155392} - \frac{36397493}{2916}\zeta_3 - \frac{103763}{54}\zeta_4 + \frac{30994565}{3888}\zeta_5 - \frac{133990}{27}\zeta_3^2 - \frac{72875}{54}\zeta_6 + \frac{2127335}{108}\zeta_7 \Big] \\
& + C_A^4 C_F \Big[\frac{75932079965}{10077696} - \frac{27693563}{23328}\zeta_3 - \frac{1791229}{1296}\zeta_4 - \frac{9417425}{1944}\zeta_5 - \frac{96700}{81}\zeta_3^2 + \frac{163625}{81}\zeta_6 + \frac{199640}{27}\zeta_7 \Big] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \Big[\frac{81725}{162} - \frac{33505}{18}\zeta_3 - \frac{1100}{3}\zeta_4 + \frac{52025}{18}\zeta_5 - \frac{7000}{3}\zeta_3^2 - \frac{48125}{36}\zeta_7 \Big] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_F \Big[\frac{231575}{36} + \frac{6351445}{324}\zeta_3 - \frac{2927225}{162}\zeta_5 + \frac{23210}{3}\zeta_3^2 - \frac{200410}{9}\zeta_7 \Big] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_A \Big[\frac{165871}{54} + \frac{1816625}{162}\zeta_3 - \frac{41800}{9}\zeta_4 - \frac{4456145}{162}\zeta_5 + \frac{196880}{27}\zeta_3^2 + \frac{200750}{27}\zeta_6 - \frac{7525}{4}\zeta_7 \Big] \\
& + n_f C_F^4 \Big[\frac{1776521549}{40310784} - \frac{1332919}{486}\zeta_3 + \frac{5000}{9}\zeta_3^2 + \frac{33290}{81}\zeta_4 - \frac{30325}{81}\zeta_5 - \frac{10000}{9}\zeta_6 + \frac{14000}{3}\zeta_7 \Big] \\
& - n_f C_A C_F^3 \Big[\frac{3737356319}{3359232} - \frac{2327111}{432}\zeta_3 + \frac{1280}{3}\zeta_3^2 + \frac{262069}{648}\zeta_4 + \frac{1693715}{162}\zeta_5 - \frac{14000}{3}\zeta_6 + \frac{7000}{3}\zeta_7 \Big] \\
& + n_f C_A^2 C_F^2 \Big[\frac{5637513931}{3359232} + \frac{2711207}{486}\zeta_3 - \frac{5020}{27}\zeta_3^2 - \frac{457499}{108}\zeta_4 + \frac{508820}{243}\zeta_5 - \frac{20375}{27}\zeta_6 + \frac{50155}{108}\zeta_7 \Big] \\
& - n_f C_A^3 C_F \Big[\frac{8766012215}{2519424} + \frac{45697231}{5832}\zeta_3 + \frac{1195}{81}\zeta_3^2 - \frac{2848403}{648}\zeta_4 - \frac{1808870}{243}\zeta_5 + \frac{222250}{81}\zeta_6 + \frac{250915}{108}\zeta_7 \Big] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{24385}{27} - \frac{334010}{81}\zeta_3 - \frac{8480}{9}\zeta_3^2 + \frac{1622600}{81}\zeta_5 - \frac{135380}{9}\zeta_7 \Big] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \Big[\frac{297889}{162} + \frac{154970}{81}\zeta_3 - \frac{62600}{27}\zeta_3^2 + \frac{3700}{9}\zeta_4 - \frac{122780}{81}\zeta_5 - \frac{36500}{27}\zeta_6 - 910\zeta_7 \Big] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{241835}{162} + \frac{333487}{81}\zeta_3 + \frac{30560}{27}\zeta_3^2 - 10780/9\,\zeta_4 - \frac{316900}{81}\zeta_5 + \frac{110000}{27}\zeta_6 - \frac{71960}{9}\zeta_7 \Big] \\
& + n_f^2 C_F^3 \Big[\frac{512848319}{1679616} - \frac{57109}{54}\zeta_3 + \frac{2800}{9}\zeta_3^2 + \frac{9118}{81}\zeta_4 + \frac{86440}{81}\zeta_5 - \frac{5000}{9}\zeta_6 \Big] \\
& + n_f^2 C_A C_F^2 \Big[\frac{1080083}{5832} - \frac{296729}{972}\zeta_3 - \frac{21800}{27}\zeta_3^2 + \frac{56327}{54}\zeta_4 - \frac{42860}{81}\zeta_5 + \frac{2500}{27}\zeta_6 \Big] \\
& + n_f^2 C_A^2 C_F \Big[\frac{61747877}{419904} + \frac{2496811}{1944}\zeta_3 + \frac{39800}{81}\zeta_3^2 - \frac{3503}{3}\zeta_4 - \frac{88990}{243}\zeta_5 + \frac{35000}{81}\zeta_6 \Big] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \Big[\frac{19435}{27} - \frac{53366}{81}\zeta_3 + \frac{3200}{27}\zeta_3^2 - \frac{3160}{9}\zeta_4 - \frac{70000}{81}\zeta_5 + \frac{20000}{27}\zeta_6 \Big] \\
& + n_f^3 C_F^2 \Big[\frac{28758139}{1259712} + \frac{21673}{729}\zeta_3 - \frac{610}{9}\zeta_4 + \frac{800}{27}\zeta_5 \Big] \\
& + n_f^3 C_A C_F \Big[\frac{13729181}{1259712} + \frac{14947}{729}\zeta_3 + \frac{4390}{81}\zeta_4 - \frac{6400}{81}\zeta_5 \Big] - n_f^4 C_F \Big[\frac{259993}{629856} + \frac{1660}{729}\zeta_3 - \frac{200}{81}\zeta_4 \Big]
\end{aligned}$$

$$\begin{aligned}
\gamma_{\rm ns}^{(4)\,{\rm v}}(N=3) \;=\;& \gamma_{\rm ns}^{(4)-}(N=3) \\
& + n_f \frac{d_{abc}d^{abc}}{N_F} \Big\{ C_F^2 \Big[\frac{79906955}{46656} + \frac{246955}{54}\zeta_3 - \frac{504550}{81}\zeta_5 \Big] \\
& \quad - C_A C_F \Big[\frac{9797321}{3888} - \frac{475655}{54}\zeta_3 + \frac{17600}{9}\zeta_4 + \frac{516950}{81}\zeta_5 - \frac{500}{9}\zeta_3^2 + \frac{2800}{9}\zeta_7 \Big] \\
& \quad + C_A^2 \Big[\frac{166535}{486} - \frac{1783913}{324}\zeta_3 + \frac{5555}{9}\zeta_4 + \frac{507515}{81}\zeta_5 - \frac{2035}{27}\zeta_3^2 - \frac{5500}{27}\zeta_6 - \frac{2765}{18}\zeta_7 \Big] \\
& \quad + n_f C_A \Big[\frac{285985}{3888} + \frac{41954}{81}\zeta_3 + \frac{160}{27}\zeta_3^2 - \frac{1010}{9}\zeta_4 - \frac{56480}{81}\zeta_5 + \frac{1000}{27}\zeta_6 \Big] \\
& \quad + n_f C_F \Big[\frac{1098323}{3888} - \frac{49720}{81}\zeta_3 + \frac{3200}{9}\zeta_4 \Big] - n_f^2 \Big[\frac{21823}{1944} \Big] \Big\}
\end{aligned}$$

Scale stability of evolution

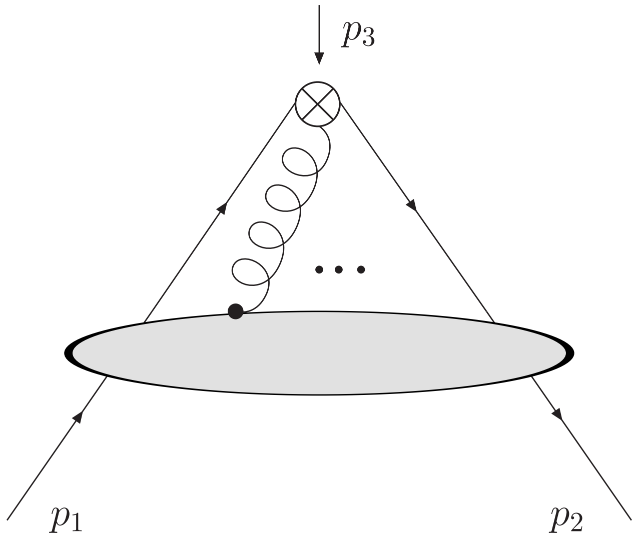


- Renormalization-scale dependence of $d \ln q_{\rm ns}^{\pm} / d \ln \mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N^{3}LO and N^{4}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Off-forward kinematics

Operator matrix elements

- Off-forward kinematics considers matrix elements with general momentum assignments $\langle \psi(p_1) | \mathcal{O}^{NS}_{\mu_1 \dots \mu_N}(p_3) | \overline{\psi}(p_2) \rangle$
  - standard local non-singlet quark opera
    $\mathcal{O}^{NS}_{\mu_1 \dots \mu_N} = \mathcal{S}\, \overline{\psi} \lambda^\alpha \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi$



- Short-distance expansion of light-ray operators uses basis of local operators in terms of Gegenbauer polynomials

$$\mathcal{O}^{\mathcal{G}}_{N,k} = (\partial_{z_1} + \partial_{z_2})^k C^{3/2}_N \left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}} \right) \mathcal{O}(z_1, z_2) \Big|_{z_1 = z_2 = 0}$$

- Evolution equation for renormalized operators $[\mathcal{O}^{\mathcal{G}}_{N,k}]$

$$\left(\mu^2 \partial_{\mu^2} + \beta(a_s) \partial_{a_s} \right) [\mathcal{O}^{\mathcal{G}}_{N,k}] = \sum_{j=0}^{N} \gamma^{\mathcal{G}}_{N,j} [\mathcal{O}^{\mathcal{G}}_{j,k}] \qquad \text{with } \gamma^{\mathcal{G}}_{N,j} = 0 \text{ if } j > N$$

Conformal symmetry

- Full conformal algebra in 4 dimensions includes fifteen generators
  Mack, Salam '69; Treiman, Jackiw, Gross '72

$\mathbf{P}_\mu$ (4 translations)	$\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
$\mathbf{D}$ (dilatation)	$\mathbf{K}_\mu$ (4 special conformal transformations)

Collinear subgroup $SL(2,\mathbb{R})$

- Leading order evolution operator $\mathcal{H}^{(1)}$ commutes with (canonical) generators of collinear conformal transformations
- Evolution kernel $h^{(1)}(\alpha,\beta) = \bar{h}(\tau)$ effectively function of one variable
  $\tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$ (conformal ratio) Braun, Derkachov, Korchemsky, Manashov '99

$$h^{(1)}(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right],$$

- Conformal symmetry is broken in any realistic four-dimensional QFT
  - $\beta(a_s) \neq 0$

QCD in conformal window

- Instead of considering consequences of broken conformal symmetry in QCD make use of exact conformal symmetry of modified theory
  - $\beta(a_s) = 2a_s(-\epsilon - \beta_0 a_s - \beta_1 a_s^2 - \dots)$ with $a_s = \dfrac{\alpha_s}{4\pi}$
  - large-n_f QCD in $4 - 2\epsilon$ dimensions at critical coupling a_* with $\beta(a_*) = 0$ Banks, Zaks '82
- Maintain exact conformal symmetry, but the generators of $SL(2,\mathbb{R})$ are modified by quantum corrections

Results

- $\hat{\gamma}^{\mathcal{G}}$ in Gegenbauer basis Müller '93, Belitsky, Müller '99

$$\hat{\gamma}^{\mathcal{G}}(a_s) \;=\; \mathbf{G}\left\{[\hat{\gamma}^{\mathcal{G}}(a_s), \hat{b}]\left(\frac{1}{2}\hat{\gamma}^{\mathcal{G}}(a_s) + \beta(a_s)\right) + [\hat{\gamma}^{\mathcal{G}}(a_s), \hat{w}(a_s)]\right\}$$

  - matrix commutators denoted as $[*,*]$ and $\mathbf{G}\{\hat{M}\}_{N,k} = -\frac{M_{N,k}}{a(N,k)}$

$$a(N,k) \;=\; (N-k)(N+k+3)\,,$$

$$\hat{b}_{N,k} \;=\; -2k\delta_{N,k} - 2(2k+3)\vartheta_{N,k}$$

  - Conformal anomaly $\hat{w}(a_s)$ up to two-loops Braun, Manashov '13, Braun, Manashov, S.M., Strohmaier '16, Braun, Manashov, S.M., Strohmaier '17

Total derivative basis (I)

- Total derivative basis

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{S}\,\partial^{\mu_1}\dots\partial^{\mu_p}\,((D^{\nu_1}\dots D^{\nu_q}\overline{\psi})\,\lambda^{\alpha}\gamma_{\mu}\,(D^{\sigma_1}\dots D^{\sigma_r}\psi))$$

  - expansion in terms of powers of derivatives (left, right and total)
- Total derivative basis used in computations of operator correlation functions for DIS Gracey '09, Kniehl, Veretin '20
  - renormalization schemes $\overline{\text{MS}}$ and RI (for comparison to lattice QCD)
- Action of partial derivatives

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1}$$

  - left and right derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{p,0,r} = \sum_{j=0}^{r} Z_{r,r-j}\,[\mathcal{O}^{\mathcal{D}}_{p+j,0,r-j}]$$

- Anomalous dimensions $\gamma^{\mathcal{D}}_{N,k}$ govern scale dependence

$$\gamma^{\mathcal{D}}_{N,k} = -\left(\frac{d}{d\ln\mu^2}\,Z_{N,j}\right)Z^{-1}_{j,k} \qquad \text{with } \gamma^{\mathcal{D}}_{N,j} = 0 \text{ if } j > N$$

Total derivative basis (II)

- Operator bases related using light-ray operators as generating functions

$$\mathcal{O}(z_1, z_2) = \sum_{m,k} \frac{z_1^m \, z_2^k}{m! \, k!} \, \mathcal{O}^{\mathcal{D}}_{0,m,k}$$

  - expansion of Gegenbauer polynomials yields

$$\mathcal{O}^{\mathcal{G}}_{N,k} = \frac{1}{2N!} \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \, \mathcal{O}^{\mathcal{D}}_{k-l,0,l}$$

- Evolution equations relate anomalous dimension matrices $\gamma^{\mathcal{G}}_{N,j}$ and $\gamma^{\mathcal{D}}_{N,j}$

$$\sum_{j=0}^{N} (-1)^j \frac{(j+2)!}{j!} \, \gamma^{\mathcal{G}}_{N,j} = \frac{1}{N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{(N+j+2)!}{(j+1)!} \sum_{l=0}^{j} \gamma^{\mathcal{D}}_{j,l}$$

Task

- Exploit relation of $\gamma^{\mathcal{G}}_{N,j}$ and $\gamma^{\mathcal{D}}_{N,j}$ for known results
- Analyze constraints on $\gamma^{\mathcal{D}}_{N,j}$ in total derivative basis

Constraints on the anomalous dimensions (I)

- Recursion for bare operators $\mathcal{O}^{\mathcal{D}}_{p,q,r} = \sum_{i=0}^{p} \binom{p}{i} \mathcal{O}^{\mathcal{D}}_{0,p+q-i,r+i}$ leads to relation between sums of elements of the mixing matrix $\hat{\gamma}^{\mathcal{D}}_N$

$$\forall k : \qquad \sum_{j=k}^{N} \left\{ (-1)^k \binom{j}{k} \gamma^{\mathcal{D}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,k} \right\} = 0$$

  - $k = N-1$ relates next-to-diagonal elements to forward anomalous dimensions $\gamma_{N,N}$

$$\gamma^{\mathcal{D}}_{N,N-1} = \frac{N}{2} \left(\gamma_{N-1,N-1} - \gamma_{N,N} \right)$$

  - mixing matrix

$$\hat{\gamma}^{\mathcal{D}} = \begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \dots & \gamma^{\mathcal{D}}_{N,N-k} & \dots & \gamma^{\mathcal{D}}_{N,0} & \\ 0 & \gamma_{N-1,N-1} & \dots & \gamma^{\mathcal{D}}_{N-1,N-k} & \dots & \gamma^{\mathcal{D}}_{N-1,0} & \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \gamma^{\mathcal{D}}_{N-k,N-k} & \dots & \gamma^{\mathcal{D}}_{N-k,0} & \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \end{pmatrix}$$

Constraints on the anomalous dimensions (II)

- $k = 0$ relates sum of elements in N-th row to the conjugate $\mathcal{C}\,\gamma^{\mathcal{D}}_{N,0}$

$$\sum_{j=0}^{N} \left\{ \gamma^{\mathcal{D}}_{N,j} - (-1)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right\} = 0$$

  - mixing matrix

$$\hat{\gamma}^{\mathcal{D}} = \begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \cdots & \gamma^{\mathcal{D}}_{N,N-k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma^{\mathcal{D}}_{N-1,N-k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \gamma^{\mathcal{D}}_{N-k,N-k} & \cdots & \gamma^{\mathcal{D}}_{N-k,0} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

- arbitrary k

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}$$

  - bootstrapping $\gamma^{\mathcal{D}}_{N,k}$
  - solution of sums with ansatz for $\gamma^{\mathcal{D}}_{N,k}$ by means of **Sigma** Schneider '07

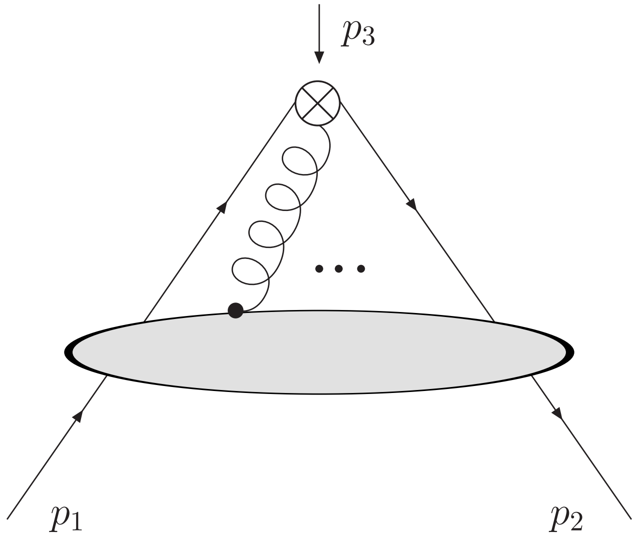
Calculation (I)

- OMEs in off-forward kinematics
  - momentum-flow through the operator vertex
  - choice $p_2 = 0$ maps OMEs to two-point functions

$$\Delta^{\mu_1} \dots \Delta^{\mu_N} \langle \psi(p_1) | O^{NS}_{\mu_1 \dots \mu_N}(-p_1) | \overline{\psi}(0) \rangle$$

- OMEs

$$\mathcal{O}_N \equiv \Delta^{\mu_1} \dots \Delta^{\mu_N} \mathcal{O}^{NS}_{\mu_1 \dots \mu_N}$$
$$\mathcal{O}_1 = \Delta^{\mu} \overline{\psi} \lambda^{\alpha} \gamma_{\mu} \psi$$



Work flow

- Anomalous dimensions $\gamma^{\mathcal{D}}_{N,k}$ from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17

Calculation (II)

- Renormalized OMEs in off-forward kinematics
  $$\mathcal{O}_{N+1} = Z_\psi \left(Z_{N,N} [O_{N+1}] + Z_{N,N-1} [\partial \mathcal{O}_N] + \cdots + Z_{N,0} [\partial^N \mathcal{O}_1] \right)$$

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial \mathcal{O}_N \\ \vdots \\ \partial^k \mathcal{O}_{N+1-k} \\ \vdots \\ \partial^N \mathcal{O}_1 \end{pmatrix} = Z_\psi \begin{pmatrix} Z_{N,N} & \dots & Z_{N,N-k} & \dots & Z_{N,0} & \\ 0 & \dots & Z_{N-1,N-k} & \dots & Z_{N-1,0} & \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & Z_{N-k,N-k} & \dots & Z_{N-k,0} & \\ \vdots & \dots & \vdots & \dots & \vdots & \\ 0 & \dots & 0 & \dots & 1 & \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial \mathcal{O}_N] \\ \vdots \\ [\partial^k \mathcal{O}_{N+1-k}] \\ \vdots \\ [\partial^N \mathcal{O}_1] \end{pmatrix}$$

- Disentangle elements of anomalous dimensions matrix $\gamma^{\mathcal{D}}_{N,k}$ from
- Use additional constraints on $\gamma^{\mathcal{D}}_{N,k}$ in total derivative basis

Results (I)

Gegenbauer basis

- One-loop results for $\gamma^{\mathcal{G}}_{N,k}$ Makeenko '81

$$\gamma^{\mathcal{G},(0)}_{N,k} = 0$$

- Matrix for $N = 5$

$$\hat{\gamma}^{\mathcal{G},(0)}_{N=5} = C_F \begin{pmatrix} \frac{91}{15} & 0 & 0 & 0 & 0 \\ 0 & \frac{157}{30} & 0 & 0 & 0 \\ 0 & 0 & \frac{25}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Results (II)

Total derivative basis

- One-loop results for $\gamma^{\mathcal{D}}_{N,k}$

$$\gamma^{\mathcal{D},(0)}_{N,k} = C_F \Big(\frac{2}{N+2} - \frac{2}{N-k} \Big)$$

- Matrix for $N = 5$

$$\hat{\gamma}^{\mathcal{D},(0)}_{N=5} = C_F \begin{pmatrix} \frac{91}{15} & -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{157}{30} & -\frac{8}{5} & -\frac{3}{5} & -\frac{4}{15} \\ 0 & 0 & \frac{25}{6} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{8}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Results (III)

- General structure of matrix for $N = 5$

$$\begin{pmatrix} \gamma_{4,4} & \gamma_{4,3} & \gamma_{4,2} & \gamma_{4,1} & \gamma_{4,0} \\ 0 & \gamma_{3,3} & \gamma_{3,2} & \gamma_{3,1} & \gamma_{3,0} \\ 0 & 0 & \gamma_{2,2} & \gamma_{2,1} & \gamma_{2,0} \\ 0 & 0 & 0 & \gamma_{1,1} & \gamma_{1,0} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Three-loop results for $\gamma^{\mathcal{D}}_{N,k}$

$$\begin{aligned}
\gamma_{2,2} &= 5.55556\, a_s + (70.8848 - 5.12346\, n_f)\, a_s^2 \\
&\quad +(1244.91 - 199.637\, n_f - 1.762\, n_f^2)\, a_s^3 + O(a_s^4) \\
\gamma^{\mathcal{D}}_{2,1} &= -2\, a_s + (-22.5556 + 1.96296\, n_f)\, a_s^2 \\
&\quad +(-385.466 + 66.1992\, n_f + 0.532922\, n_f^2)\, a_s^3 + O(a_s^4) \\
\gamma^{\mathcal{D}}_{2,0} &= -0.666667\, a_s + (-9.50617 + 0.481481\, n_f)\, a_s^2 \\
&\quad +(-170.654 + 24.8232\, n_f + 0.3107\, n_f^2)\, a_s^3 + O(a_s^4)
\end{aligned}$$

- comparision with Kniehl, Veretin '20

$$\gamma^{\mathcal{D},(2)}_{2,0}\Big|_{\text{Kniehl, Veretin '20}} = -170.641(12) + 24.822(2)\, n_f + 0.3107(1)\, n_f^2$$

Results (IV)

- Four- and five-loop results for $\gamma^{\mathcal{G}}_{N,k}$ and $\gamma^{\mathcal{D}}_{N,k}$ in large n_f limit
  - forward anomalous dimension known $\gamma_{N,N}$ Gracey '94

$$
\begin{aligned}
\gamma^{\mathcal{D},(3)}_{N,k} = \frac{8}{27} n_f^3 C_F \Bigg\{ & \frac{1}{3}\Big(S_1(N) - S_1(k)\Big)^3 \Big(\frac{1}{N+2} - \frac{1}{N-k}\Big) \\
& + \Big(S_1(N) - S_1(k)\Big)^2 \Big(\frac{5}{3}\frac{1}{N-k} + \frac{2}{N+1} - \frac{11}{3}\frac{1}{N+2} + \frac{1}{(N+2)^2}\Big) \\
& + \Big(S_1(N) - S_1(k)\Big)\Big(S_2(N) - S_2(k)\Big)\Big(\frac{1}{N+2} - \frac{1}{N-k}\Big) \\
& + 2\Big(S_1(N) - S_1(k)\Big)\Big(\frac{1}{3}\frac{1}{N-k} - \frac{13}{3}\frac{1}{N+1} + \frac{2}{(N+1)^2} + \frac{4}{N+2} - \frac{11}{3}\frac{1}{(N+2)^2} \\
& + \frac{1}{(N+2)^3}\Big) + \Big(S_2(N) - S_2(k)\Big)\Big(\frac{5}{3}\frac{1}{N-k} + \frac{2}{N+1} - \frac{11}{3}\frac{1}{N+2} + \frac{1}{(N+2)^2}\Big) \\
& + \frac{2}{3}\Big(S_3(N) - S_3(k)\Big)\Big(\frac{1}{N+2} - \frac{1}{N-k}\Big) + \frac{2}{3}\frac{1}{N-k} + \frac{2}{N+1} - \frac{26}{3}\frac{1}{(N+1)^2} \\
& + \frac{4}{(N+1)^3} - \frac{8}{3}\frac{1}{N+2} + \frac{8}{(N+2)^2} - \frac{22}{3}\frac{1}{(N+2)^3} + \frac{2}{(N+2)^4} \Bigg\} + n_f^3 C_F \zeta_3 \ldots
\end{aligned}
$$

Summary

- QCD radiative corrections to evolution equations for operator matrix elements
  - necessary for precision of non-perturbative quantities: PDFs, GPDs, DAs
  - hard scattering in forward and off-forward kinematics
- Short distance expansion of light-ray operators generate set of local operators
  - Gegenbauer basis realizes a "hidden" conformal symmetry
  - total derivative basis with simple power counting
- OMEs of non-singlet local operators of twist two
  - relation between anomalous dimension $\gamma_{N,k}^{\mathcal{G}}$ and $\gamma_{N,k}^{\mathcal{D}}$
  - constraints on $\gamma_{N,k}^{\mathcal{D}}$ allow for bootstrap
- Novel results hard scattering in off-forward kinematics at four loops and beyond