Local subtraction at NNLO

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A Loop Summit 2021

LHC is …

- a hadron machine → QCD-based processes
- a high-energy machine → complex processes
- entering a high-precision phase → theory must follow
- searching new physics → must control SM background

High precision computation in QCD needed

- PDFs, resumation, parton shower, hadronization and …
- … fixed order computations

Ambitious goal: Automatic NNLO QCD computations

- Loop computations and …
- … cancellation of soft and collinear singularities → this talk

Well established subtraction schemes at NLO

- Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer
- Catani-Seymour (CS) Dipole subtraction — Catani, Seymour
- Nagy-Soper subtraction — Nagy, Soper

Many methods available at NNLO

- Antenna subtraction — Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.
- CoLoRFul subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- Sector-improved residue subtraction — Czakon et al.
- Nested soft-collinear subtraction — Melnikov et al.
- Local analytic sector subtraction — Magnea, Maina, Torrielli, U. et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- $\mathcal{E}$-prescription — Frixione, Grazzini
- Unsubtraction — Rodrigo et al.
- Geometric — Herzog

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Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

VV and RV have poles in ϵ, RV and RR diverge in phase space

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VV and *RV* have poles in ϵ, *RV* and *RR* diverge in phase space

Let's insert counterterms !

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \Big(VV \qquad\qquad\qquad\qquad \Big)\,\delta_{X_n} \\ & + \int d\Phi_{n+1} \Big[\Big(RV \qquad\qquad \Big)\,\delta_{X_{n+1}} \qquad\qquad\qquad\qquad \Big] \\ & + \int d\Phi_{n+2} \Big[RR\,\delta_{X_{n+2}} \qquad\qquad\qquad\qquad\qquad\qquad \Big]\end{aligned}$$

Structure of subtraction at NNLO

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$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \Big(VV \qquad\qquad \Big)\,\delta_{X_n}$$
$$+ \int d\Phi_{n+1} \Big[\Big(RV \qquad \Big)\,\delta_{X_{n+1}} \qquad\qquad \Big]$$
$$+ \int d\Phi_{n+2} \Big[RR\,\delta_{X_{n+2}} \qquad \text{Counterterms for } RR \qquad \Big]$$

Counterterms for *RR*

Counterterms for *RR*

- Partition of phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^\alpha} \frac{1}{(\mathcal{E}_k + \delta_{kj}\, \mathcal{E}_i)\, w_{kl}} \qquad \alpha > 1$$

$$\sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1 \qquad RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

$\mathcal{E}_i$ → (adimensional) energy of particle i

$w_{ij} = \dfrac{1-\cos\theta_{ij}}{2}$, θ_{ij} → angle between particles i and j

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In each sector **"few"** singular limits survive

$$\mathcal{W}_{ijjk} \; : \quad \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkj} \; : \quad \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \quad \mathbf{SC}_{kij}$$

$$\mathcal{W}_{ijkl} \; : \quad \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl} \quad \mathbf{SC}_{kij}$$

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$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^{\alpha}} \frac{1}{(\mathcal{E}_k + \delta_{kj}\, \mathcal{E}_i)\, w_{kl}} \qquad \alpha > 1$$

$$\sigma = \sum_{\substack{i,\, j \neq i \\ k \neq i,\, l \neq i,k}} \sigma_{ijkl}$$

$\mathbf{S}_i : \quad \mathcal{E}_i \to 0$

$\mathbf{C}_{ij} : \quad w_{ij} \to 0$

$\mathbf{S}_{ij} : \quad \mathcal{E}_i, \mathcal{E}_j \to 0$ uniformly

$\mathbf{C}_{ijk} : \quad w_{ij}, w_{jk}, w_{ij} \to 0$ uniformly

$\mathbf{C}_{ijkl} : \quad w_{ij}, w_{kl} \to 0$ uniformly

$\mathbf{SC}_{ijk} : \quad \mathcal{E}_i, w_{jk} \to 0$ uniformly

In each sector

		1-unresolved limits		2-unresolved limits			
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

1-unresolved limits

2-unresolved limits

Counterterms for *RR*

- Partition of phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^\alpha} \frac{1}{(\mathcal{E}_k + \delta_{kj}\, \mathcal{E}_i)\, w_{kl}} \qquad \alpha > 1$$

$$\sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\mathbf{S}_i : \quad \mathcal{E}_i \to 0$$

$$\mathbf{C}_{ij} : \quad w_{ij} \to 0$$

$$\mathbf{S}_{ij} : \quad \mathcal{E}_i, \mathcal{E}_j \to 0 \;\; \text{uniformly}$$

$$\mathbf{C}_{ijk} : \quad w_{ij}, w_{jk}, w_{ij} \to 0 \;\; \text{uniformly}$$

$$\mathbf{C}_{ijkl} : \quad w_{ij}, w_{kl} \to 0 \;\; \text{uniformly}$$

$$\mathbf{SC}_{ijk} : \quad \mathcal{E}_i, w_{jk} \to 0 \;\; \text{uniformly}$$

In each sector ”

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

1-unresolved limits

2-unresolved limits

All limits commute

Counterterms for RR

- Partition of phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\boxed{\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\, w_{ij})^\alpha}\,\frac{1}{(\mathcal{E}_k + \delta_{kj}\,\mathcal{E}_i)\,w_{kl}} \qquad \alpha > 1}$$

$$\sigma = \sum_{\substack{i,\,j\neq i \\ k\neq i,\,l\neq i,k}} \sigma_{ijkl} \qquad \sum_{i,j\neq i}\sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1 \qquad RR = \sum_{i,j\neq i}\sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}$$

In each sector **"few"** singular limits survive

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
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$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

ALL limits commute

Get rid of sector functions for analytical integration !!

- Partition of phase s

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma}$$

$$\sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

In each sector **"few"**

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	C
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	C
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	C

Get rid of sector functions for analytical integration !!

Sum rules

$$\mathbf{S}_{ik}\left(\sum_{j\neq i}\sum_{l\neq i,k}\mathcal{W}_{ijkl}+\sum_{j\neq k}\sum_{l\neq i,k}\mathcal{W}_{kjil}\right)=1$$

$$\mathbf{C}_{ijk}\left(\mathcal{W}_{ijjk}+\mathcal{W}_{ikkj}\right)+\text{perm. of } i,j,k=1$$

$$\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\left(\mathcal{W}_{ikkj}+\mathcal{W}_{kiij}+\mathcal{W}_{ijkj}+\mathcal{W}_{kjij}\right)=1$$

$$\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}+\mathcal{W}_{jikl}+\mathcal{W}_{jilk}+\mathcal{W}_{klij}+\mathcal{W}_{klji}+\mathcal{W}_{lkij}+\mathcal{W}_{lkji}\right)=1$$

$$\mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{klij}\right)=1$$

$$\mathbf{C}_{ijkl}\,\mathbf{SC}_{ikl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}+\mathcal{W}_{klij}+\mathcal{W}_{lkij}\right)=1$$

$$\mathbf{SC}_{ijk}\left(\sum_{l\neq i}\mathcal{W}_{iljk}+\sum_{l\neq i}\mathcal{W}_{ilkj}+\sum_{l\neq i,j}\mathcal{W}_{jkil}+\sum_{l\neq i,k}\mathcal{W}_{kjil}\right)=1$$

$$\mathbf{S}_{ik}\,\mathbf{SC}_{ijk}\left(\sum_{l\neq i}\mathcal{W}_{ilkj}+\sum_{l\neq i,k}\mathcal{W}_{kjil}\right)=1$$

$$\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\left(\mathcal{W}_{ijjk}+\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}+\mathcal{W}_{ikjk}+\mathcal{W}_{jkik}+\mathcal{W}_{kjij}\right)=1$$

$$\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\left(\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}+\mathcal{W}_{kjij}\right)=1$$

commu

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$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$

$$\mathcal{W}_{ijjk} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

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All limits commute

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All limits commute

- Identify counterterms through IR limits

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})RR\,\mathcal{W}_{ijjk} = \text{finite}$$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})(1-\mathbf{SC}_{kij})\,RR\,\mathcal{W}_{ijkj} = \text{finite}$$

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$$1-\mathbf{L}^{(\mathbf{2})}_{ijjk}$$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})(1-\mathbf{SC}_{kij})\,RR\,\mathcal{W}_{ijkj} = \text{finite}$$

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$$1-\mathbf{L}^{(\mathbf{1})}_{ij} \qquad\qquad 1-\mathbf{L}^{(\mathbf{2})}_{ijkl}$$

Counterterms for *RR*

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}$$

$$\begin{array}{lcll} \mathcal{W}_{ijjk} & : & \mathbf{S}_i \quad \mathbf{C}_{ij} & \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \\ \mathcal{W}_{ijkj} & : & \mathbf{S}_i \quad \mathbf{C}_{ij} & \mathbf{S}_{ik} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \quad \mathbf{SC}_{kij} \\ \mathcal{W}_{ijkl} & : & \mathbf{S}_i \quad \mathbf{C}_{ij} & \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl} \quad \mathbf{SC}_{kij} \end{array}$$

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$$1-\mathbf{L}^{(2)}_{ijjk}$$

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$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})(1-\mathbf{SC}_{ikl})(1-\mathbf{SC}_{kij})\,RR\,\mathcal{W}_{ijkl} = \text{finite}$$

$$1-\mathbf{L}^{(1)}_{ij} \qquad\qquad 1-\mathbf{L}^{(2)}_{ijkl}$$

$$\left(1-\mathbf{L}^{(1)}_{ij}\right)\left(1-\mathbf{L}^{(2)}_{ijkl}\right)RR\,\mathcal{W}_{ijkl} = \left[RR-\mathbf{L}^{(1)}_{ij}RR-\mathbf{L}^{(2)}_{ijkl}RR+\mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl}\,RR\right]\mathcal{W}_{ijkl} = \text{finite}$$

Counterterms for *RR*

- Partition of phase space through sector functions

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}$$

- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(\mathbf{1})}_{ij}\right)\left(1-\mathbf{L}^{(\mathbf{2})}_{ijkl}\right)RR\,\mathcal{W}_{ijkl} \;=\; \left[RR-\mathbf{L}^{(\mathbf{1})}_{ij}RR-\mathbf{L}^{(\mathbf{2})}_{ijkl}RR+\mathbf{L}^{(\mathbf{1})}_{ij}\mathbf{L}^{(\mathbf{2})}_{ijkl}\,RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

Counterterms for *RR*

- Partition of phase space through sector functions $$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$$
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$$\left(1-\mathbf{L}_{ij}^{(\mathbf{1})}\right)\left(1-\mathbf{L}_{ijkl}^{(\mathbf{2})}\right) RR\, \mathcal{W}_{ijkl} = \left[RR - \mathbf{L}_{ij}^{(\mathbf{1})} RR - \mathbf{L}_{ijkl}^{(\mathbf{2})} RR + \mathbf{L}_{ij}^{(\mathbf{1})} \mathbf{L}_{ijkl}^{(\mathbf{2})}\, RR\right] \mathcal{W}_{ijkl} = \text{finite}$$

- Remapping of momenta

  * to have momentum-conserving and on-sheel momenta in matrix elements of counterterms

  * to have factorisation of (n+2) phase space

$$\mathbf{L}_{ij}^{(\mathbf{1})},\ \mathbf{L}_{ijkl}^{(\mathbf{2})},\ \mathbf{L}_{ij}^{(\mathbf{1})}\mathbf{L}_{ijkl}^{(\mathbf{2})} \longrightarrow \overline{\mathbf{L}}_{ij}^{(\mathbf{1})},\ \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})},\ \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}$$

Barred limits = Limits + Remapping

Counterterms for *RR*

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$$\left(1-\mathbf{L}^{(1)}_{ij}\right)\left(1-\mathbf{L}^{(2)}_{ijkl}\right) RR\,\mathcal{W}_{ijkl} = \left[RR - \mathbf{L}^{(1)}_{ij} RR - \mathbf{L}^{(2)}_{ijkl} RR + \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} = \text{finite}$$

- Remapping of momenta DELICATE !!!

  * to have momentum-conserving and on-sheel momenta in matrix elements of counterterms

  * to have factorisation of (n+2) phase space

$$\mathbf{L}^{(1)}_{ij},\ \mathbf{L}^{(2)}_{ijkl},\ \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl} \longrightarrow \overline{\mathbf{L}}^{(1)}_{ij},\ \overline{\mathbf{L}}^{(2)}_{ijkl},\ \overline{\mathbf{L}}^{(1)}_{ij}\overline{\mathbf{L}}^{(2)}_{ijkl}$$

Barred limits = Limits + Remapping

* must fulfil
$$\left[RR - \overline{\mathbf{L}}^{(1)}_{ij} RR - \overline{\mathbf{L}}^{(2)}_{ijkl} RR + \overline{\mathbf{L}}^{(1)}_{ij}\overline{\mathbf{L}}^{(2)}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} = \text{finite}$$

* carry the **same symmetries** as unbarred limits

  → to use sum rules of sector functions

Counterterms for *RR*

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- Identify counterterms through IR limits

$$\left(1-\mathbf{L}^{(1)}_{ij}\right)\left(1-\mathbf{L}^{(2)}_{ijkl}\right) RR\, \mathcal{W}_{ijkl} \;=\; \left[RR - \mathbf{L}^{(1)}_{ij} RR - \mathbf{L}^{(2)}_{ijkl} RR + \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijkl}\, RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Remapping of momenta DELICATE !!!

NESTED CATANI-SEYMOUR REMAPPINGS

$$\{k\} \to \{\bar{k}\}^{(abc)} \to \{\bar{k}\}^{(abc,def)}$$

$$\bar{k}^{(abc)}_b = k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}}\, k_c \qquad \bar{k}^{(abc)}_c = \frac{s_{abc}}{s_{ac}+s_{bc}}\, k_c \qquad k_a \to w, y, z$$

$$d\Phi_{n+1}\big(\{k\}\big) = d\Phi_n\big(\{\bar{k}\}^{(abc)}\big)\, d\Phi_{\rm rad}\big(\bar{s}^{(abc)}_{bc}; w, y, z\big)$$

$$\int d\Phi_{\rm rad}(s; w, y, z) = N_\epsilon\, s^{1-\epsilon} \int_0^1 dw \int_0^1 dy \int_0^1 dz\, \left[w(1-w)\right]^{-\epsilon-\frac{1}{2}} \left[y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y)$$

$$s_{ab} = y\, \bar{s}^{(abc)}_{bc} \qquad s_{ac} = z(1-y)\bar{s}^{(abc)}_{bc} \qquad s_{bc} = (1-z)(1-y)\bar{s}^{(abc)}_{bc} \qquad s_{cd} = (1-y)s^{(abc)}_{cd}$$

$$s_{ad} = y(1-z)s^{(abc)}_{cd} + z s^{(abc)}_{bd} - 2\,(1-2w)\left[yz(1-z)s^{(abc)}_{bd}\, s^{(abc)}_{cd}\right]^{1/2}$$

$$s_{bd} = yz s^{(abc)}_{cd} + (1-z)s^{(abc)}_{bd} + 2\,(1-2w)\left[yz(1-z)s^{(abc)}_{bd}\, s^{(abc)}_{cd}\right]^{1/2}$$

Counterterms for *RR*

- Partition of phase space through sector functions $\quad RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl}$
- Identify counterterms through IR limits

$$\left(1-\mathbf{L}_{ij}^{(1)}\right)\left(1-\mathbf{L}_{ijkl}^{(2)}\right)RR\,\mathcal{W}_{ijkl} \;=\; \left[RR-\mathbf{L}_{ij}^{(1)}RR-\mathbf{L}_{ijkl}^{(2)}RR+\mathbf{L}_{ij}^{(1)}\mathbf{L}_{ijkl}^{(2)}\,RR\right]\mathcal{W}_{ijkl} \;=\; \text{finite}$$

- Remapping of momenta DELICATE !!!

NESTED CATANI-SEYMOUR REMAPPINGS

$$\{k\} \to \{\bar{k}\}^{(abc)} \to \{\bar{k}\}^{(abc,def)}$$

* simple phase-space factorisation
* simple expressions for invariants
* flexible → freedom in choosing (abc,def)

To give an idea …

$$\overline{\mathbf{S}}_{ij}\, RR \;\equiv\; \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)} \mathcal{I}_{ef}^{(j)}\, \bar{B}_{cdef}^{(icd,jef)} + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)} \mathcal{I}_{ed}^{(j)}\, \bar{B}_{cded}^{(icd,jed)}$$

$$+ 2\, \mathcal{I}_{cd}^{(i)} \mathcal{I}_{cd}^{(j)}\, \bar{B}_{cdcd}^{(icd,jcd)} + \left(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2} \mathcal{I}_{cc}^{(ij)} - \frac{1}{2} \mathcal{I}_{dd}^{(ij)} \right) \bar{B}_{cd}^{(ijcd)} \Bigg]$$

$$\overline{\mathbf{S}}_{i}\, \overline{\mathbf{S}}_{ij}\, RR \;\equiv\; \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} \mathcal{I}_{cd}^{(i)}\, \bar{\mathcal{I}}_{ef}^{(j)(icd)}\, B_{cdef}^{(icd,jef)}$$

$$+ 2 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \bar{\mathcal{I}}_{ed}^{(j)(icd)} B_{cded}^{(icd,jed)} + 2 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \bar{\mathcal{I}}_{ed}^{(j)(idc)} B_{cded}^{(idc,jed)}$$

$$+ 2\, \mathcal{I}_{cd}^{(i)}\, \bar{\mathcal{I}}_{cd}^{(j)(icd)}\, B_{cdcd}^{(icd,jcd)}$$

$$- 2\, C_A \left(\mathcal{I}_{jc}^{(i)}\, \bar{\mathcal{I}}_{cd}^{(j)(icj)} + \mathcal{I}_{jd}^{(i)}\, \bar{\mathcal{I}}_{cd}^{(j)(ijd)} - \mathcal{I}_{cd}^{(i)}\, \bar{\mathcal{I}}_{cd}^{(j)(icd)} \right) B_{cd}^{(ijcd)} \Bigg]$$

They coincide in the soft limit $\mathbf{S}_i$

$$\mathbf{S}_i\, \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ij}\, RR \;=\; \mathbf{S}_i\, \overline{\mathbf{S}}_{ij}\, RR$$

Counterterms for *RR*

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left(VV \qquad\qquad\qquad\qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV \qquad\qquad \right) \delta_{X_{n+1}} \qquad\qquad\qquad\qquad \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]$$

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(1)}$ → 3 barred limits

$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(2)}$ → 11 barred limits

$$K^{(\mathbf{12})} = -\sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(12)}$ → 21 barred limits

Counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(VV \Big)\, \delta_{X_n} + \int d\Phi_{n+1} \Big[\Big(RV \Big)\, \delta_{X_{n+1}} \Big] + \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \Big(K^{(\mathbf{2})} + K^{(\mathbf{12})} \Big)\, \delta_{X_n} \Big]$$

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(1)}$ → 3 barred limits

$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(2)}$ → 11 barred limits

$$K^{(\mathbf{12})} = - \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}\, RR\, \mathcal{W}_{ijkl}$$

$K^{(12)}$ → 21 barred limits

We have verified analytically that

* Each barred limit carries the right symmetries ✓
* $RR + K^{(1)} + K^{(2)} + K^{(12)}$ → no IR singularities ✓

verified sector by sector

Integrated counterterms for *RR*

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} + \left(I^{(12)} \qquad \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} + K^{(12)} \right) \delta_{X_n} \right]$$

$$I^{(1)} = \int d\Phi_{\text{rad}}\, K^{(1)}$$

- Integration of $I^{(1)}$ trivial ✓

$$I^{(12)} = \int d\Phi_{\text{rad}}\, K^{(12)}$$

- Integration of $I^{(12)}$ easy ✓

$$I^{(2)} = \int d\Phi_{\text{rad},2}\, K^{(2)}$$

- Integration of $I^{(2)}$ feasible

$$I^{(2)} = I^{(2)}_{\text{ss}} + I^{(2)}_{\text{hcc}} + I^{(2)}_{\text{hchc}} + I^{(2)}_{\text{hsc}}$$

✓ ✓

ongoing

Integrated counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n} + \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} + \left(I^{(12)} \right) \delta_{X_n} \right]$$

To give an idea …

$$\int d\Phi_{\rm rad,2}\, \overline{\mathbf{S}}_{ij}\, RR = \delta_{f_i g} \delta_{f_j g} \sum_{\substack{c \neq i,j \\ d \neq i,j,c}} \left[\frac{1}{2} \sum_{\substack{e \neq i,j,c,d \\ f \neq i,j,c,d,e}} J^{(4)}_{\rm s\otimes s}(\bar{s}_{cd}, \bar{s}_{ef}) \bar{B}_{cdef} + 2 \sum_{e \neq i,j,c,d} J^{(3)}_{\rm s\otimes s}(\bar{s}_{cd}, \bar{s}_{ed}) \bar{B}_{cded} + J^{(2)}_{\rm s\otimes s}(\bar{s}_{cd}) \bar{B}_{cdcd} - C_A\, J^{\rm (gg)}_{\rm ss}(\bar{s}_{cd}) \bar{B}_{cd} \right] + \delta_{\{f_i f_j\}\{q\bar{q}\}} \sum_{\substack{c \neq i,j \\ d \neq i,j,c}} T_R\, J^{(\rm q\bar{q})}_{\rm ss}(\bar{s}_{cd}) \bar{B}_{cd}$$

$$\int d\Phi_{\rm rad}\, \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ij}\, RR = \delta_{f_i g} \frac{\mathcal{N}_1}{2} \sum_{\substack{l \neq j \\ m \neq j,l}} \bar{\mathcal{I}}^{(j)}_{lm} \left[\sum_{\substack{c \neq j,l \\ d \neq j,l,c}} J_s(\bar{s}_{cd})\, \bar{B}^{(jlm)}_{cdlm} + 2 \sum_{c \neq j,m} J_s(\bar{s}_{cm})\, \bar{B}^{(jlm)}_{cmlm} - 2C_A \Big(J_s(\bar{s}_{jl}) + J_s(\bar{s}_{jm}) - J_s(\bar{s}_{lm}) \Big) \bar{B}^{(jlm)}_{lm} \right]$$

Integrated counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

To give an idea …

$$\int d\Phi_{\rm rad,2}\, \overline{\mathbf{S}}_{ij}\, RR = \delta_{f_i g}\delta_{f_j g} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\frac{1}{2} \sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} J^{(4)}_{\rm s\otimes s}(\bar{s}_{cd}, \bar{s}_{ef}) \bar{B}_{cdef} + 2 \sum_{e\neq i,j,c,d} J^{(3)}_{\rm s\otimes s}(\bar{s}_{cd}, \bar{s}_{ed}) \bar{B}_{cded}$$

$$+ J^{(2)}_{\rm s\otimes s}(\bar{s}_{cd}) \bar{B}_{cdcd} - C_A J^{(\rm gg)}_{\rm ss}(\bar{s}_{cd}) \bar{B}_{cd} \Bigg]$$

$$+ \delta_{\{f_i f_j\}\{q\bar{q}\}} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} T_R\, J^{(\rm q\bar{q})}_{\rm ss}(\bar{s}_{cd}) \bar{B}_{cd}$$

$$\int d\Phi_{\rm rad}\, \overline{\mathbf{S}}_i\, \overline{\mathbf{S}}_{ij}\, RR = \delta_{f_i g} \frac{\mathcal{N}_1}{2} \sum_{\substack{l\neq j \\ m\neq j,l}} \bar{\mathcal{I}}^{(j)}_{lm} \Bigg[\sum_{\substack{c\neq j,l \\ d\neq j,l,c}} J_s(\bar{s}_{cd}) \bar{B}^{(jlm)}_{cdlm} + 2 \sum_{c\neq j,m} J_s(\bar{s}_{cm})\, \bar{B}^{(jlm)}_{cmlm}$$

$$- 2C_A \Big(J_s(\bar{s}_{jl}) + J_s(\bar{s}_{jm}) - J_s(\bar{s}_{lm}) \Big) \bar{B}^{(jlm)}_{lm} \Bigg]$$

Integrated counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n} + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + I^{(\mathbf{12})} \right) \delta_{X_n} \Big]$$

To give an idea …

$$J^{(4)}_{\rm s\otimes s}(s,s') = \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(16 - \frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2} + \left(60 - \frac{14}{3}\pi^2 - \frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} + 216 - \frac{56}{3}\pi^2 - \frac{200}{3}\zeta_3 + \frac{29}{120}\pi^4\right]$$

$$J^{(3)}_{\rm s\otimes s}(s,s') = \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{ss'}{\mu^4}\right)^{-\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(17 - \frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(70 - \frac{16}{3}\pi^2 - \frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} + 284 - \frac{68}{3}\pi^2 - \frac{272}{3}\zeta_3 + \frac{13}{90}\pi^4\right]$$

$$J^{(2)}_{\rm s\otimes s}(s) = \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(18 - \frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2} + \left(76 - 6\pi^2 - \frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} + 312 - 27\pi^2 - \frac{308}{3}\zeta_3 + \frac{49}{120}\pi^4\right]$$

$$J^{(q\bar{q})}_{\rm ss}(s) = \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{6}\frac{1}{\epsilon^3} + \frac{17}{18}\frac{1}{\epsilon^2} + \left(\frac{116}{27} - \frac{7}{36}\pi^2\right)\frac{1}{\epsilon} + \frac{1474}{81} - \frac{131}{108}\pi^2 - \frac{19}{9}\zeta_3\right]$$

$$J^{(gg)}_{\rm ss}(s) = \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} \left[\frac{1}{2}\frac{1}{\epsilon^4} + \frac{35}{12}\frac{1}{\epsilon^3} + \left(\frac{487}{36} - \frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{1562}{27} - \frac{269}{72}\pi^2 - \frac{77}{6}\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81} - \frac{3829}{216}\pi^2 - \frac{1025}{18}\zeta_3 - \frac{23}{240}\pi^4\right]$$

$$J_s(s) = \frac{\alpha_{\rm S}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-\epsilon}\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 6 - \frac{7}{12}\pi^2\right]$$

Integrated counterterms for *RR*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n} + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} \right) \delta_{X_n} \right] + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

We have verified analytically that

- $RV + I^{(1)}$ → no ε poles ✓
- $I^{(1)} + I^{(12)}$ → no IR singularities ✓

verified sector by sector

Counterterm for *RV*

$$\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \qquad\qquad\qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} \qquad\qquad\qquad \right) \delta_{X_n} \right]$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

The counterterm $K^{(RV)}$ must satisfy

- $RV - K^{(RV)}$ → no IR singularities
- $I^{(12)} - K^{(RV)}$ → no ε poles

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

- Partition of phase space through sector functions

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, w_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij} \qquad \sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad RV = \sum_{i,j\neq i} RV\, \mathcal{W}_{ij}$$

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

- Partition of phase space through sector functions

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, w_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij} \qquad \sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad RV = \sum_{i,j\neq i} RV\, \mathcal{W}_{ij}$$

- Identify counterterms through IR limits

$$(1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij}) RV\, \mathcal{W}_{ij} = \left(RV - \mathbf{L}_{ij}^{(\mathbf{1})}\, RV \right) \mathcal{W}_{ij} = \text{finite}$$

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(VV + I^{(\mathbf{2})} \qquad\qquad\qquad \Big)\, \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(\mathbf{1})} \Big)\, \delta_{X_{n+1}} + \Big(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \Big)\, \delta_{X_n} \Big]$$
$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \Big(K^{(\mathbf{2})} + K^{(\mathbf{12})} \Big)\, \delta_{X_n} \Big]$$

- Partition of phase space through sector functions

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, w_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij} \qquad \sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad RV = \sum_{i,j\neq i} RV\ \mathcal{W}_{ij}$$

- Identify counterterms through IR limits

$$(1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij}) RV\ \mathcal{W}_{ij} = \Big(RV - \mathbf{L}_{ij}^{(\mathbf{1})}\, RV \Big) \mathcal{W}_{ij} = \text{finite}$$

- Remapping of momenta

$$\Big(RV - \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RV \Big) \mathcal{W}_{ij} = RV\, \mathcal{W}_{ij} - \Big[\overline{\mathbf{S}}_i RV + \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i) RV \Big] \mathcal{W}_{ij} = \text{finite}$$

The interesting case of $\mathbf{C}_{ij}$ *RV*

Somogyi-Trócsányi
hep-ph: 0609043

$$\mathbf{C}_{ij}RV = \frac{\mathcal{N}_1}{s_{ij}}\left[P_{ij}^{\mu\nu}\,V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_S}{2\pi}\Gamma_\epsilon\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\hat{P}_{ij}^{\mu\nu}\,B_{\mu\nu}(\{k\}_{[ij]})\right] - \frac{\alpha_S}{2\pi}\frac{\beta_0}{2\epsilon}\mathbf{C}_{ij}\,R$$

$$+\frac{\alpha_S}{2\pi}\frac{\Gamma_\epsilon}{\epsilon^2}\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\left[C_{f_{[ij]}}-C_{f_i}-C_{f_j}+F_i\left(C_{f_{[ij]}}+C_{f_i}-C_{f_j}\right)+F_j\left(C_{f_{[ij]}}-C_{f_i}+C_{f_j}\right)\right]\mathbf{C}_{ij}R$$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})RV\,\mathcal{W}_{ij} = \left(RV - \mathbf{L}_{ij}^{(\mathbf{1})}\,RV\right)\mathcal{W}_{ij} = \text{finite}$$

- Remapping of momenta

$$\left(RV - \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\,RV\right)\mathcal{W}_{ij} = RV\,\mathcal{W}_{ij} - \left[\overline{\mathbf{S}}_iRV + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)RV\right]\mathcal{W}_{ij} = \text{finite}$$

The interesting case of $\mathbf{C}_{ij}$ *RV*

Somogyi-Trócsányi
hep-ph: 0609043

$$\mathbf{C}_{ij}RV = \frac{\mathcal{N}_1}{s_{ij}}\left[P^{\mu\nu}_{ij}\,V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_S}{2\pi}\,\Gamma_\epsilon\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\hat{P}^{\mu\nu}_{ij}\,B_{\mu\nu}(\{k\}_{[ij]})\right] - \frac{\alpha_S}{2\pi}\frac{\beta_0}{2\epsilon}\,\mathbf{C}_{ij}\,R$$
$$+ \frac{\alpha_S}{2\pi}\frac{\Gamma_\epsilon}{\epsilon^2}\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\left[C_{f_{[ij]}} - C_{f_i} - C_{f_j} + F_i\left(C_{f_{[ij]}} + C_{f_i} - C_{f_j}\right) + F_j\left(C_{f_{[ij]}} - C_{f_i} + C_{f_j}\right)\right]\mathbf{C}_{ij}R$$

Alternative formulation → more appropriate to match $I^{(12)}$ ε poles

$$\mathbf{C}_{ij}RV = \frac{\mathcal{N}_1}{s_{ij}}\left[P^{\mu\nu}_{ij}\,V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_S}{2\pi}\,\Gamma_\epsilon\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\hat{P}^{\mu\nu}_{ij}\,B_{\mu\nu}(\{k\}_{[ij]})\right] - \frac{\alpha_S}{2\pi}\frac{\beta_0}{2\epsilon}\,\mathbf{C}_{ij}\,R$$
$$+ \frac{\alpha_S}{2\pi}\frac{\Gamma_\epsilon}{\epsilon^2}\left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon}\left[2\,\mathbf{C}_{ij}\,R_{ij} - 2\sum_{c\neq i,j}\left(F_i\,\mathbf{C}_{ij}\,R_{ic} + F_j\,\mathbf{C}_{ij}\,R_{jc}\right)\right]$$

Confirmed by factorisation

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})RV\,\mathcal{W}_{ij} = \left(RV - \mathbf{L}^{(1)}_{ij}\,RV\right)\mathcal{W}_{ij} = \text{finite}$$

- Remapping of momenta

$$\left(RV - \overline{\mathbf{L}}^{(1)}_{ij}\,RV\right)\mathcal{W}_{ij} = RV\,\mathcal{W}_{ij} - \left[\overline{\mathbf{S}}_i RV + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)RV\right]\mathcal{W}_{ij} = \text{finite}$$

The interesting case of $\mathbf{C}_{ij}\, RV$

Somogyi-Trócsányi
hep-ph: 0609043

$$\mathbf{C}_{ij} RV = \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}^{\mu\nu} V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_S}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}_{ij}^{\mu\nu} B_{\mu\nu}(\{k\}_{[ij]}) \right] - \frac{\alpha_S}{2\pi} \frac{\beta_0}{2\epsilon} \mathbf{C}_{ij} R$$
$$+ \frac{\alpha_S}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[C_{f_{[ij]}} - C_{f_i} - C_{f_j} + F_i \left(C_{f_{[ij]}} + C_{f_i} - C_{f_j} \right) + F_j \left(C_{f_{[ij]}} - C_{f_i} + C_{f_j} \right) \right] \mathbf{C}_{ij} R$$

Alternative formulation → more appropriate to match $I^{(12)}$ ε poles

$$\mathbf{C}_{ij} RV = \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}^{\mu\nu} V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_S}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}_{ij}^{\mu\nu} B_{\mu\nu}(\{k\}_{[ij]}) \right] - \frac{\alpha_S}{2\pi} \frac{\beta_0}{2\epsilon} \mathbf{C}_{ij} R$$
$$+ \frac{\alpha_S}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[2\, \mathbf{C}_{ij} R_{ij} - 2 \sum_{c \neq i,j} \left(F_i\, \mathbf{C}_{ij} R_{ic} + F_j\, \mathbf{C}_{ij} R_{jc} \right) \right]$$

Confirmed by factorisation

Corresponding barred limit

$$\overline{\mathbf{C}}_{ij} RV \equiv \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}^{\mu\nu} V_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right) + \frac{\alpha_S}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}_{ij}^{\mu\nu} B_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right) \right] - \frac{\alpha_S}{2\pi} \frac{\beta_0}{2\epsilon} \overline{\mathbf{C}}_{ij} R$$
$$+ \frac{\alpha_S}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[2\, \overline{\mathbf{C}}_{ij} R_{ij} - 2 \sum_{c \neq i,j} \left(\bar{F}_i^{(c)} \overline{\mathbf{C}}_{ij} R_{ic} + \bar{F}_j^{(c)} \overline{\mathbf{C}}_{ij} R_{jc} \right) \right]$$

$$\left(RV - \overline{\mathbf{L}}_{ij}^{(\mathbf{1})} RV \right) \mathcal{W}_{ij} = RV\, \mathcal{W}_{ij} - \left[\overline{\mathbf{S}}_i RV + \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i) RV \right] \mathcal{W}_{ij} = \text{finite}$$

The interesting case of $\mathbf{C}_{ij}$ *RV*

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$$\mathbf{C}_{ij} RV = \frac{\mathcal{N}_1}{s_{ij}} \left[P^{\mu\nu}_{ij} V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_s}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}^{\mu\nu}_{ij} B_{\mu\nu}(\{k\}_{[ij]}) \right] - \frac{\alpha_s}{2\pi} \frac{\beta_0}{2\epsilon} \mathbf{C}_{ij} R$$
$$+ \frac{\alpha_s}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[C_{f_{[ij]}} - C_{f_i} - C_{f_j} + F_i \left(C_{f_{[ij]}} + C_{f_i} - C_{f_j} \right) + F_j \left(C_{f_{[ij]}} - C_{f_i} + C_{f_j} \right) \right] \mathbf{C}_{ij} R$$

Alternative formulation → more appropriate to match $I^{(12)}$ ε poles

$$\mathbf{C}_{ij} RV = \frac{\mathcal{N}_1}{s_{ij}} \left[P^{\mu\nu}_{ij} V_{\mu\nu}(\{k\}_{[ij]}) + \frac{\alpha_s}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}^{\mu\nu}_{ij} B_{\mu\nu}(\{k\}_{[ij]}) \right] - \frac{\alpha_s}{2\pi} \frac{\beta_0}{2\epsilon} \mathbf{C}_{ij} R$$
$$+ \frac{\alpha_s}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[2\, \mathbf{C}_{ij} R_{ij} - 2 \sum_{c \neq i,j} \left(F_i\, \mathbf{C}_{ij} R_{ic} + F_j\, \mathbf{C}_{ij} R_{jc} \right) \right]$$

Confirmed by factorisation

$$F_i = \epsilon \ln \frac{s_{ir}}{s_{ir} + s_{jr}} + \mathcal{O}(\epsilon^2) \qquad \bar{F}_i^{(c)} = \epsilon \ln \frac{s_{ic}}{\bar{s}_{jc}^{(ijr)}} + \mathcal{O}(\epsilon^2)$$

$$\overline{\mathbf{C}}_{ij} RV \equiv \frac{\mathcal{N}_1}{s_{ij}} \left[P^{\mu\nu}_{ij} V_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right) + \frac{\alpha_s}{2\pi} \Gamma_\epsilon \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \hat{P}^{\mu\nu}_{ij} B_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right) \right] - \frac{\alpha_s}{2\pi} \frac{\beta_0}{2\epsilon} \overline{\mathbf{C}}_{ij} R$$
$$+ \frac{\alpha_s}{2\pi} \frac{\Gamma_\epsilon}{\epsilon^2} \left(\frac{s_{ij}}{\mu^2}\right)^{-\epsilon} \left[2\, \overline{\mathbf{C}}_{ij} R_{ij} - 2 \sum_{c \neq i,j} \left(\bar{F}_i^{(c)} \overline{\mathbf{C}}_{ij} R_{ic} + \bar{F}_j^{(c)} \overline{\mathbf{C}}_{ij} R_{jc} \right) \right]$$

$$\left(RV - \overline{\mathbf{L}}^{(\mathbf{1})}_{ij} RV \right) \mathcal{W}_{ij} = RV\, \mathcal{W}_{ij} - \left[\overline{\mathbf{S}}_i RV + \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i) RV \right] \mathcal{W}_{ij} = \text{finite}$$

Counterterm for *RV*

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n} + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right] + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

$$K^{(\mathbf{RV})} = \sum_{i, j \neq i} \left[\overline{\mathbf{L}}_{ij}^{(\mathbf{1})} \, RV + \Delta_{ij}^{\text{mapping}} \right] \mathcal{W}_{ij}$$

Counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})} \right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})} \right) \delta_{X_n} \right]$$

$$K^{(\mathbf{RV})} = \sum_{i,j\neq i} \left[\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RV + \Delta^{\rm mapping}_{ij} \right] \mathcal{W}_{ij}$$

We have verified analytically that

- RV - $K^{(RV)}$ → no IR singularities ✓

done sector by sector

$$\mathbf{S}_i\, \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RV = \mathbf{S}_i\, RV \qquad \mathbf{S}_i\, \Delta^{\rm mapping}_{ij} = 0$$
$$\mathbf{C}_{ij}\, \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RV = \mathbf{C}_{ij}\, RV \qquad \mathbf{C}_{ij}\, \Delta^{\rm mapping}_{ij} = 0$$

- $I^{(12)}$ - $K^{(RV)}$ → no ε poles ✓

Integrated counterterm for *RV*

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right) \delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]$$

$$I^{(\mathbf{RV})} = \int d\Phi_{\text{rad}'}\, K^{(\mathbf{RV})}$$

Integrated counterterm for *RV*

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} + \left(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\right) \delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]$$

$$I^{(\mathbf{RV})} = \int d\Phi_{\rm rad'}\, K^{(\mathbf{RV})}$$

- Integration of $I^{(\rm RV)}$ feasible $I^{(\mathbf{RV})} = I_{\rm s}^{(\mathbf{RV})} ✓ + I_{\rm hc}^{(\mathbf{RV})} ✓ + I_{\rm mapping}^{(\mathbf{RV})}$

  ongoing

- Final check

$VV + I^{(2)} - I^{(RV)}$ → no ε poles

ongoing

State-of-the-art at NLO

- Initial and final state radiation in the massless case
- Dumping factors to improve convergence
- Final state radiation with massive particles

Ongoing:

- Numerical implementation
- Complete treatment of the massive cases

State-of-the-art at NNLO

massless final state radiation

- Verified cancellation of phase space singularities for *RR* and *RV*
- Verified cancellation of virtual singularities for *RV*

Ongoing:

- Verify cancellation of virtual singularities for *VV*

Outlook

- Initial state radiation at NNLO
- The massive case at NNLO

Thanks for your attention