Loop-Tree Duality and higher-orders.

German F. R. SBORLINI

Deutsches Elektronen-Synchrotron
DESY

“A Loop Summit – new perturbative results
and methods in precision physics”
Cadenabbia (Italy) - 30.07.2021

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LTD team

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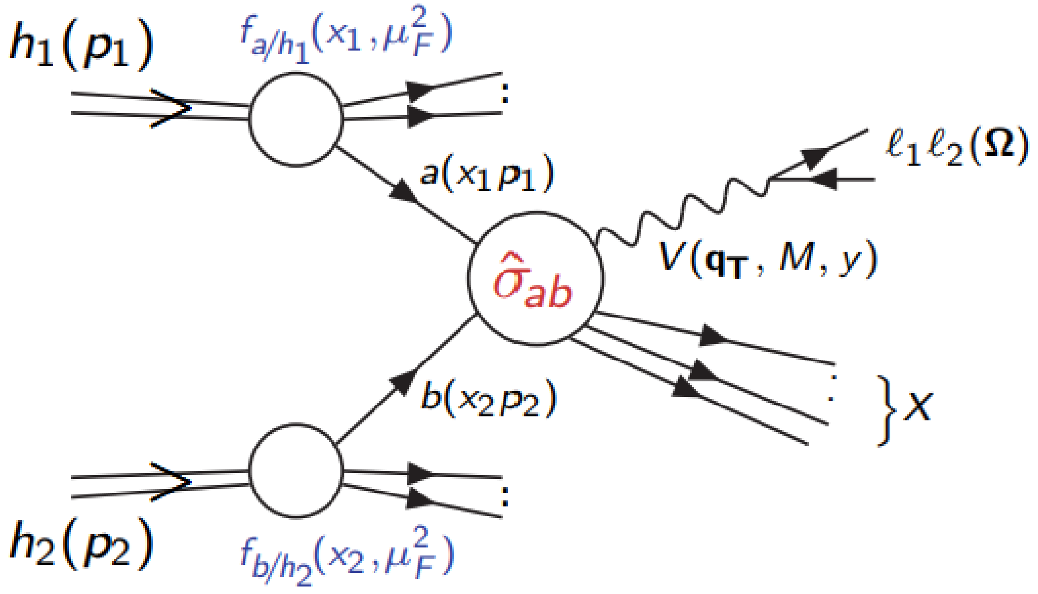
Motivation

- **What we need to calculate? Cross-sections and production/decay rates at colliders**
- How to calculate? Use the parton model and SM (or other QFT…)

$$\frac{d\sigma}{d^2\vec{q}_T\, dM^2 d\Omega dy} = \sum_{a,b}\int dx_1 dx_2\, f_a^{h_1}(x_1)\, f_b^{h_2}(x_2)\, \frac{d\hat{\sigma}_{ab\to V+X}}{d^2\vec{q}_T\, dM^2 d\Omega dy}$$

PDFs
(non-perturbative)

Partonic cross-section
(perturbative)

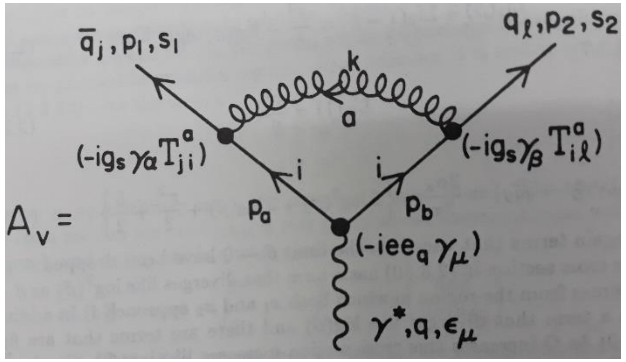


- Intermediate steps contain mathematical issues
- Need for regularization ⟹ **DREG**
- It changes the number of **space-time dimensions** in order to **achieve integrability**

$$\mathcal{O}_d[F] = \int d^d\mathrm{x}\, F(\mathrm{x}) \qquad d = 4 - 2\varepsilon$$

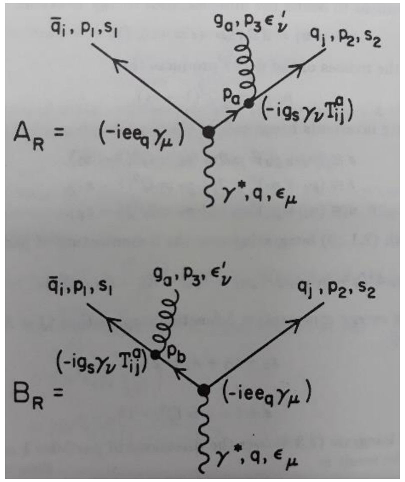
Motivation

- **Parton Distribution Functions:**
  - Extracted from data (fits, neural networks, etc)
  - Scale dependence determined by DGLAP equations (perturbative kernels)
  - Several PDFs sets available in the market (different datasets, models, approximations, etc)
- **Partonic Cross Sections:**
  - Directly obtained from QFT (applying perturbative methods)
  - Several ingredients required (for higher-orders)



Loop contributions (quantum fluctuations of vacuum)

+



Real corrections (additional particles)

+

Appears ***after*** *integration*

$$\frac{C_r}{\epsilon} \times d\sigma^{(0)}$$

Counter-terms (fix the problems of the other two)

=

FINITE NUMBER (compare to experiments)

CANCELLATION *AFTER* INTEGRATION

Motivation

- **Loop amplitudes are a bottleneck in current high-precision computations**
- Presence of **singularities and thresholds** prevents direct numerical implementations
- **Well-known theorems (KLN) guarantee the cancellation of singularities for physical observables**
- **Real-radiation** contributions are defined in **Euclidean space** (i.e. phase-space integrals)

LOOP AMPLITUDES

- *Virtual internal momenta*
- *Defined in Minkowski space-time*

Loop-Tree Duality

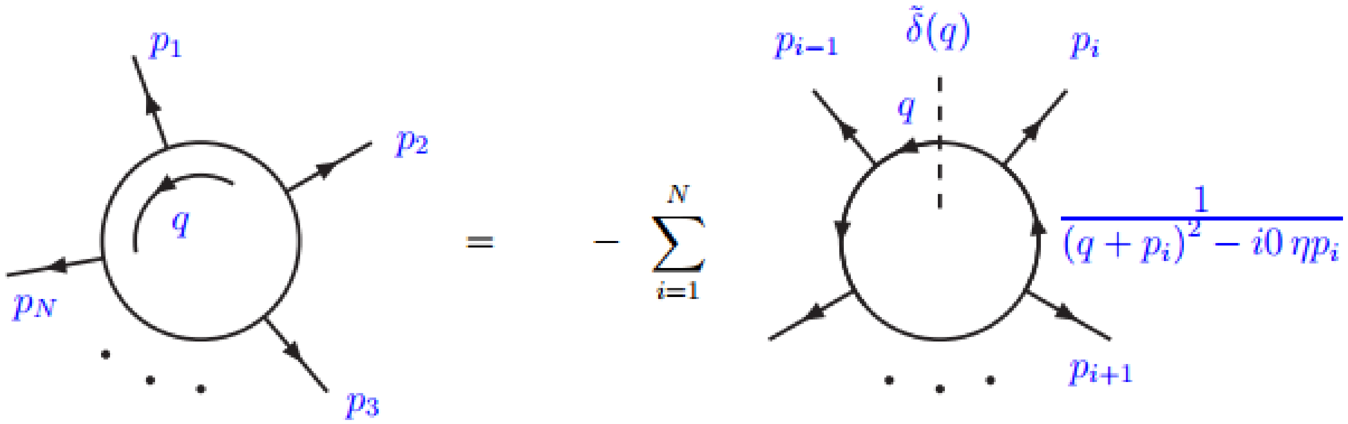
DUAL AMPLITUDES

- *On-shell cut momenta*
- *Defined in Euclidean space-time*

To be combined

REAL CONTRIBUTIONS (AND ISR/UV COUNTER-TERMS)

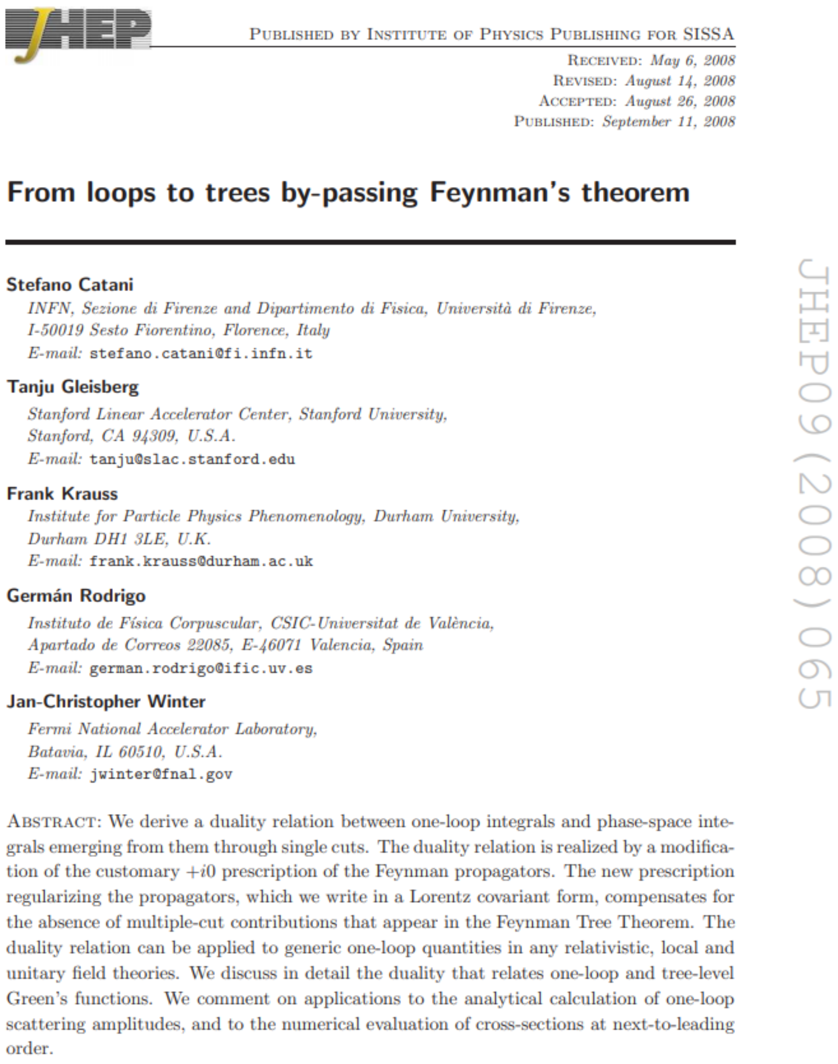
Graphical representation of one-loop opening into trees *(original idea by Catani et al '08)*



Brief history of LTD-based methods

- **Foundational paper: a new way to decompose loop amplitudes**

JHEP 09 (2008) 065



JHEP
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From loops to trees by-passing Feynman's theorem

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Abstract: We derive a duality relation between one-loop integrals and phase-space integrals emerging from them through single cuts. The duality relation is realized by a modification of the customary $+i0$ prescription of the Feynman propagators. The new prescription regularizing the propagators, which we write in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem. The duality relation can be applied to generic one-loop quantities in any relativistic, local and unitary field theories. We discuss in detail the duality that relates one-loop and tree-level Green's functions. We comment on applications to the analytical calculation of one-loop scattering amplitudes, and to the numerical evaluation of cross-sections at next-to-leading order.

JHEP09(2008)065

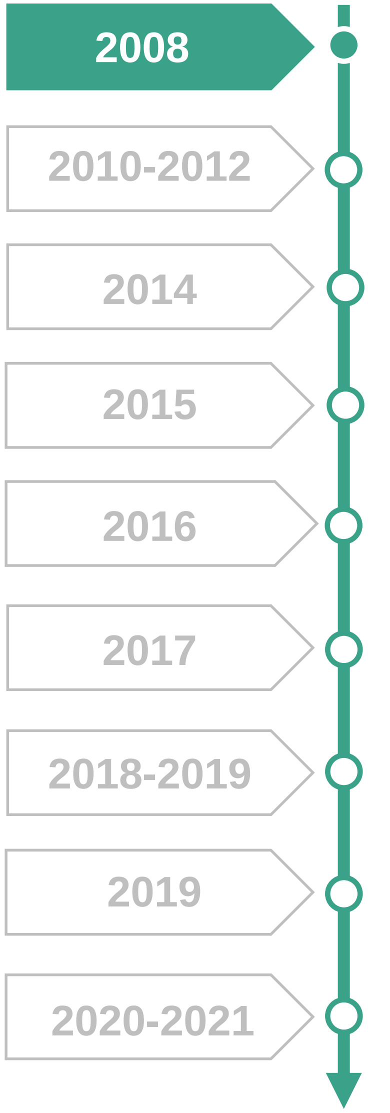
- Application of Cauchy theorem **taking care of Feynman prescription**: leads to a **new prescription!**

Feynman integral

$$L^{(1)}(p_1,\ldots,p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) = \int_\ell \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0}$$

$$L^{(1)}(p_1,\ldots,p_N) = -\sum_{i=1}^{N} \int_\ell \tilde{\delta}(q_i) \prod_{j=1,j\neq i}^{N} G_D(q_i;q_j)$$

Dual integral



Brief history of LTD-based methods

JHEP 10 (2010) 073

JHEP 03 (2013) 025

- Extension to more general amplitudes, including possible local UV counter-terms
- **Two-loop formula (2010)**

$$L^{(2)}(p_1,p_2,\ldots,p_N) = \int_{\ell_1}\int_{\ell_2} \{-G_D(\alpha_1)\,G_F(\alpha_2)\,G_D(\alpha_3)+G_D(\alpha_1)G_D(\alpha_2\cup\alpha_3)+G_D(\alpha_3)G_D(-\alpha_1\cup\alpha_2)\}$$

Uses only double-cuts!

- **Formalism for dealing with higher-order poles (2012)**

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A tree-loop duality relation at two loops and beyond

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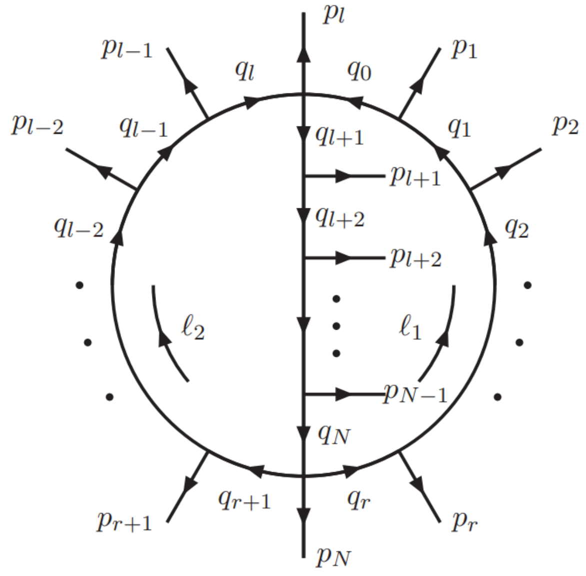
ABSTRACT: The duality relation between one-loop integrals and phase-space integrals, developed in a previous work, is extended to higher-order loops. The duality relation is realized by a modification of the customary $+i0$ prescription of the Feynman propagators, which compensates for the absence of the multiple-cut contributions that appear in the Feynman tree theorem. We rederive the duality theorem at one-loop order in a form that is more suitable for its iterative extension to higher-loop orders. We explicitly show its application to two- and three-loop scalar master integrals, and we discuss the structure of the occurring cuts and the ensuing results in detail.

KEYWORDS: NLO Computations, QCD

ARXIV EPRINT: 1007.0194

JHEP10(2010)073



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Tree-loop duality relation beyond single poles

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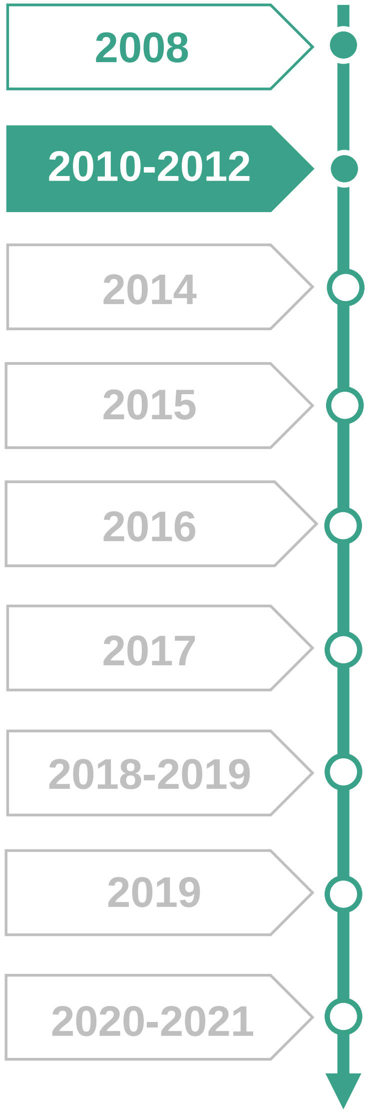
ABSTRACT: We develop the Tree-Loop Duality Relation for two- and three-loop integrals with multiple identical propagators (multiple poles). This is the extension of the Duality Relation for single poles and multi-loop integrals derived in previous publications. We prove a generalization of the formula for single poles to multiple poles and we develop a strategy for dealing with higher-order pole integrals by reducing them to single pole integrals using Integration By Parts.

KEYWORDS: QCD Phenomenology, NLO Computations

ARXIV EPRINT: 1211.5048

JHEP03(2013)025



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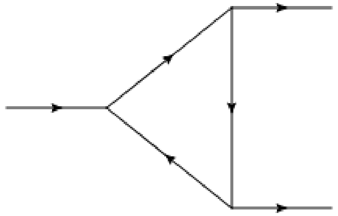
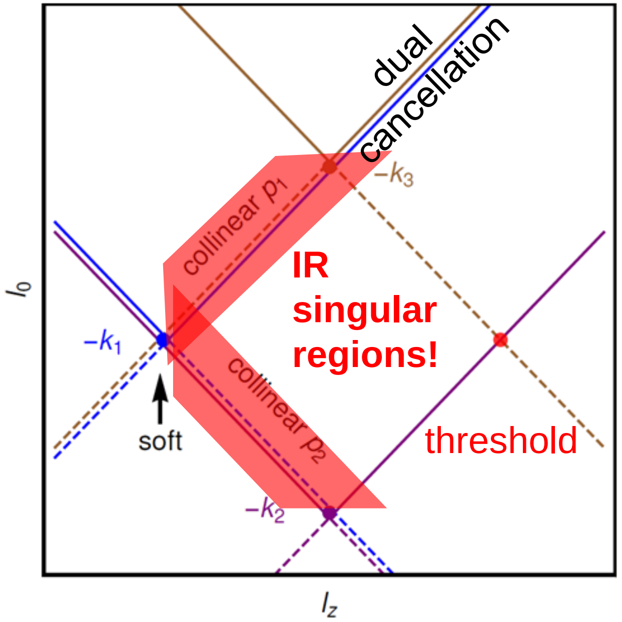
Brief history of LTD-based methods

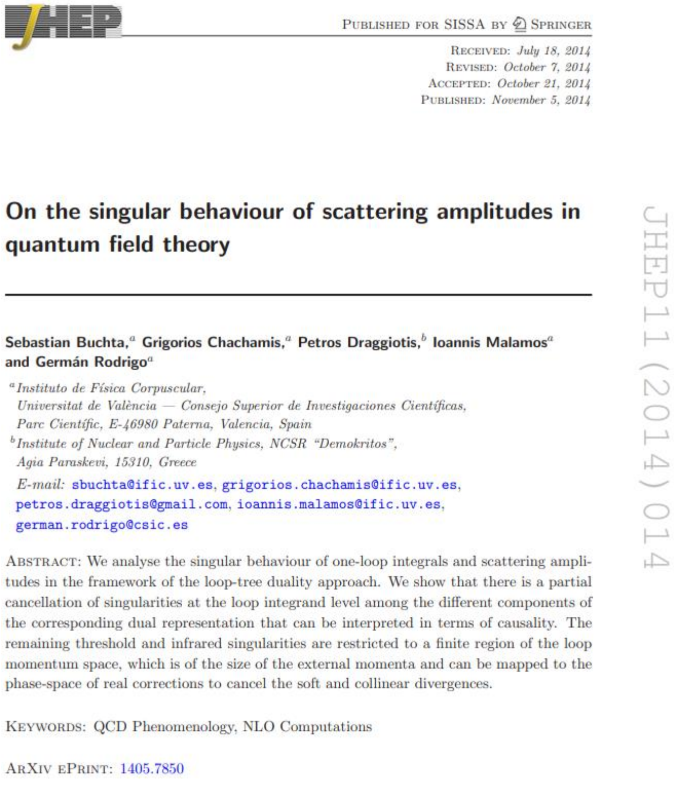
- Analysis of singular structures of loop amplitudes in LTD representation
- **First clues for real-dual integrand level combination**



Analysis of singularities in triangles



JHEP 11 (2014) 014



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On the singular behaviour of scattering amplitudes in quantum field theory

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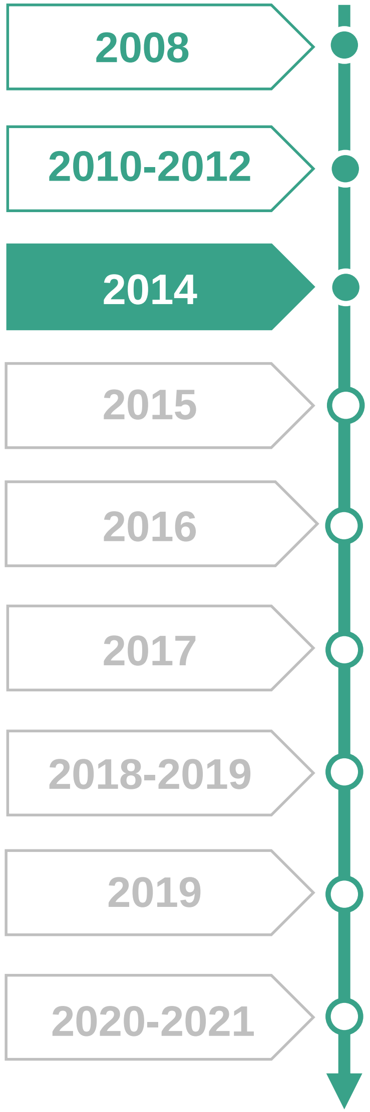
Abstract: We analyse the singular behaviour of one-loop integrals and scattering amplitudes in the framework of the loop-tree duality approach. We show that there is a partial cancellation of singularities at the loop integrand level among the different components of the corresponding dual representation that can be interpreted in terms of causality. The remaining threshold and infrared singularities are restricted to a finite region of the loop momentum space, which is of the size of the external momenta and can be mapped to the phase-space of real corrections to cancel the soft and collinear divergences.

Keywords: QCD Phenomenology, NLO Computations

ArXiv ePrint: 1405.7850

JHEP11(2014)014



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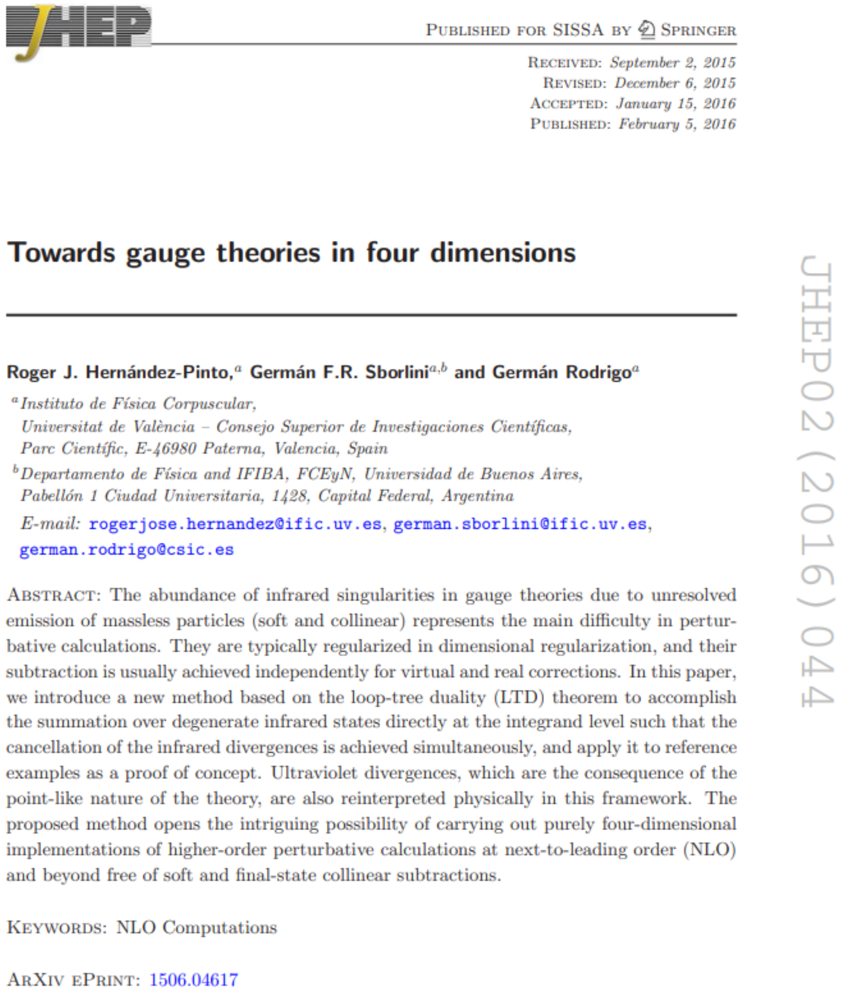
- **Forward (backward) on-shell hyperboloids associated with positive (negative) energy solutions**
- *Forward-backward intersections are physical divergences; FF cancel among them*

Brief history of LTD-based methods

JHEP 02 (2016) 044

- Towards the computation of physical observables in four space-time dimensions
- **Tested on toy scalar model; local cancellation of IR divergences**



JHEP PUBLISHED FOR SISSA BY SPRINGER

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Towards gauge theories in four dimensions

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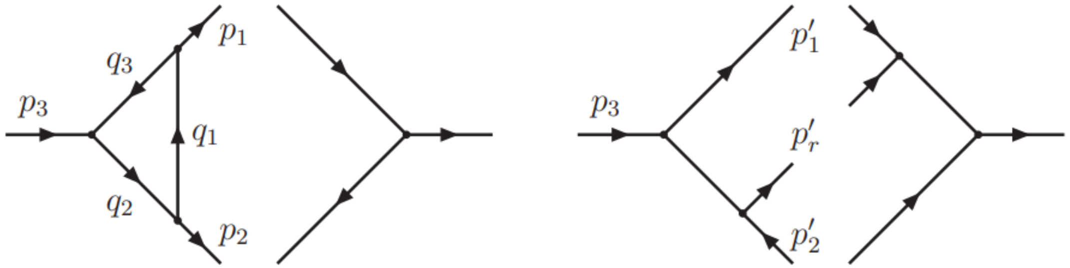
ABSTRACT: The abundance of infrared singularities in gauge theories due to unresolved emission of massless particles (soft and collinear) represents the main difficulty in perturbative calculations. They are typically regularized in dimensional regularization, and their subtraction is usually achieved independently for virtual and real corrections. In this paper, we introduce a new method based on the loop-tree duality (LTD) theorem to accomplish the summation over degenerate infrared states directly at the integrand level such that the cancellation of the infrared divergences is achieved simultaneously, and apply it to reference examples as a proof of concept. Ultraviolet divergences, which are the consequence of the point-like nature of the theory, are also reinterpreted physically in this framework. The proposed method opens the intriguing possibility of carrying out purely four-dimensional implementations of higher-order perturbative calculations at next-to-leading order (NLO) and beyond free of soft and final-state collinear subtractions.

KEYWORDS: NLO Computations

ARXIV EPRINT: 1506.04617

JHEP02(2016)044



- **Introduction of real-dual mappings, to achieve a local cancellation of IR singularities!**

$$p_r'^{\mu} = q_1^{\mu}\,, \qquad p_1'^{\mu} = -q_3^{\mu} + \alpha_1\, p_2^{\mu} = p_1^{\mu} - q_1^{\mu} + \alpha_1\, p_2^{\mu}\,,$$

$$p_2'^{\mu} = (1-\alpha_1)\, p_2^{\mu}\,, \qquad \alpha_1 = \frac{q_3^2}{2 q_3 \cdot p_2}\,,$$

- Purely four-dimensional representation of cross-sections
- **First study of dual UV *local* counter-terms:**

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{1}{\left(q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0\right)^2}$$

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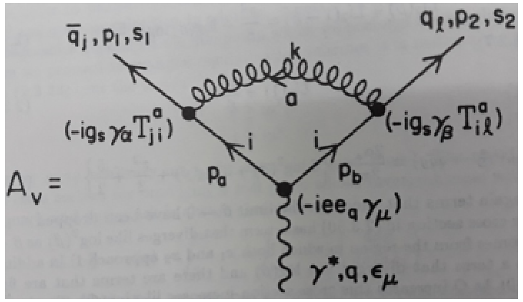
Brief history of LTD-based methods

JHEP 08 (2016) 160
JHEP 10 (2016) 162

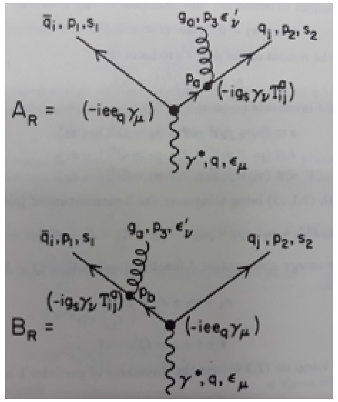
- Towards the computation of physical observables in four space-time dimensions
- **Tested on toy scalar model; local cancellation of IR divergences**

Partonic cross sections are obtained from QFT (applying perturbative methods)



Loop contributions (quantum fluctuations of vacuum)

+



Real corrections (additional particles)

+

*Appears **after** integration*

$$\frac{C_r}{\epsilon} \times d\sigma^{(0)}$$

Counter-terms (fix the problems of the other two)

=

FINITE NUMBER (compare to experiments)

CANCELLATION *AFTER* INTEGRATION

- **Integrand-level** cancellation of IR and UV singularities!
- **No need of integrated counter-terms**
- Purely four-dimensional integration (**no DREG!**)

FIRST APPROACH TO LOCAL REPRESENTATIONS!!

2008
2010-2012
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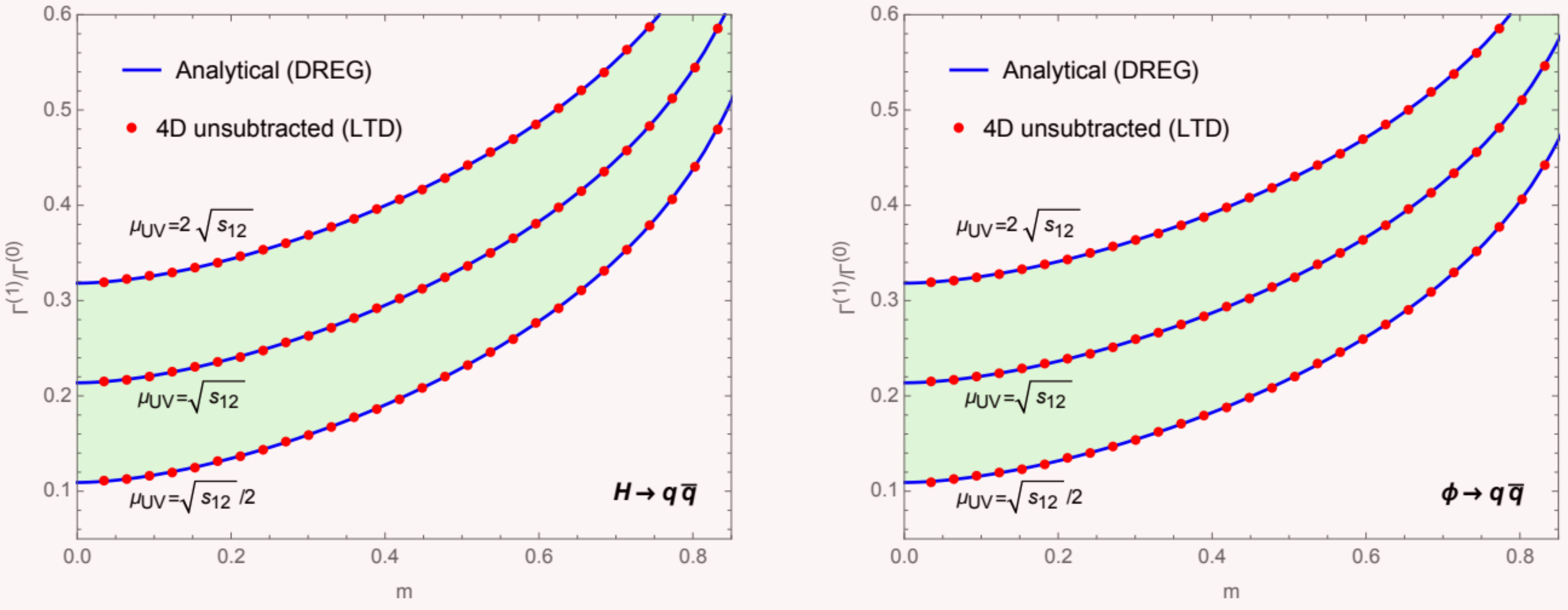
2020-2021

Brief history of LTD-based methods

JHEP 08 (2016) 160
JHEP 10 (2016) 162

- Development of the **Four Dimensional Unsubtraction (FDU)** framework @ NLO
- **Ingredients for local cancellation of IR singularities**
- Smooth numerical implementation **(massive to massless transition)**



- Integrand-level cancellation of IR and UV singularities, for physical processes!
- **No need of integrated counter-terms (up to NLO)**
- Purely four-dimensional integration (**no DREG!**)

LOCALITY!!

More studies required!

2008
2010-2012
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2018-2019
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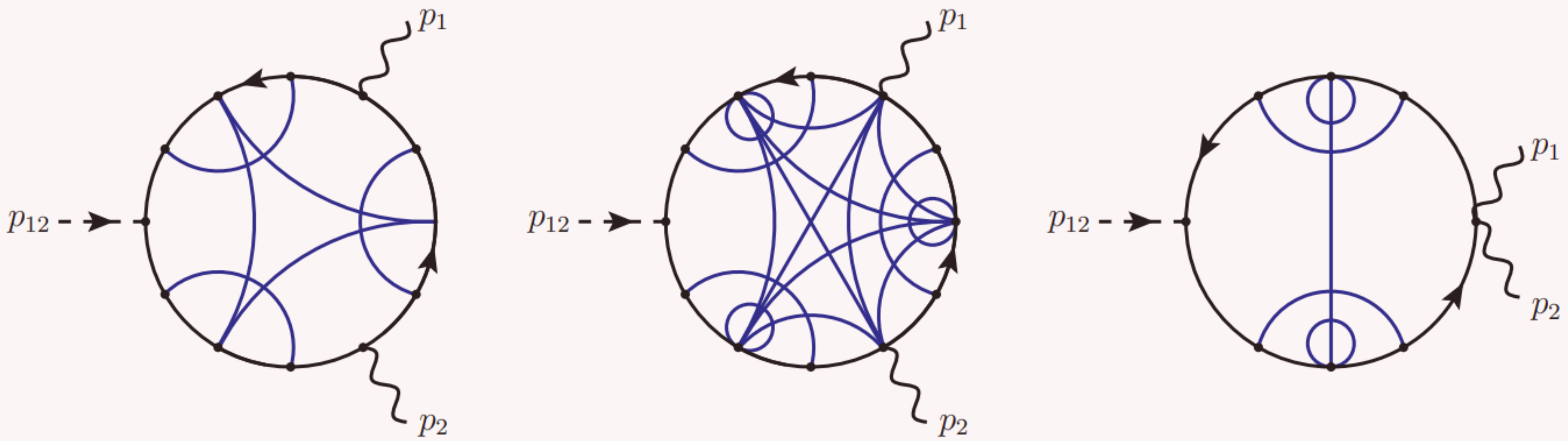
2020-2021

Brief history of LTD-based methods

JHEP 02 (2019) 143
JHEP 12 (2019) 163

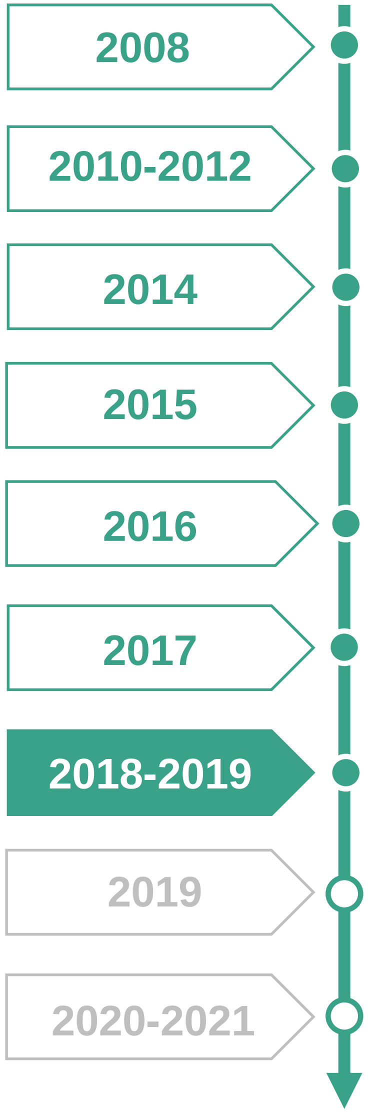
- Full analysis of Higgs decays at two-loop (inclusion of EW effects)
- **First realization of local UV counter-terms at two-loop level**

Locality explored at two-loops… what's next?



- **New singular structures arise beyond one-loop**
- More diagrams, more variables… starts to be cumbersome!
- **Explore novel representations of the integrands**
- Point towards fully local cancellations of IR/UV singularities

UNDERSTANDING SINGULARITIES IS CRUCIAL!! EXPLORE THEM!!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2020-2021

Brief history of LTD-based methods

PHYSICAL REVIEW LETTERS **124**, 211602 (2020)

Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality

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Ciudad Universitari
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(Received 16 January 2020

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arXiv:2006.112

Causal repres

Authors: J. Jesus Ag
Torres Bobadilla

Abstract: The numeric
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Submitted 19 June, 2020; **origina**
Comments: 24 pages, 8 figures
Report number: IFIC/20-27

Jun. '

heir application to multi-loop scattering

nandez-Pinto, German Rodrigo, German F. R. Sborlini, William J.

multi-loop multi-leg scattering amplitudes plays a key role to improve the precision of
s for particle physics at high-energy colliders. In this work, we focus on the mathematical
the novel integrand-level representation of Feynman integrals, which is based on the Loop-Tree Duality
. We explore the behaviour of the multi-loop iterated res... ▽ More

Submitted 24 October, 2020; **originally announced** October 2020.
Comments: 29 pages + appendices, 11 figures
Report number: IFIC/20-30; DESY 20-172; MPP-2020-184

Oct. '20

ering amplitudes to trees

er J. Hernandez-Pinto, German Rodrigo, German F. R. Sborlini, William J. Torres

bative approach to quantum field theories has made it possible to obtain incredibly accurate
predictions in high-energy physics. Although various techniques have been developed to boost the
ciency of these calculations, some ingredients remain specially challenging. This is the case of multiloop
scattering amplitudes that constitute a hard bottleneck to solve. In this Let... ▽ More

Submitted 24 June, 2020; **originally announced** June 2020.
Comments: 7 pages, 4 figures
Report number: IFIC/20-29

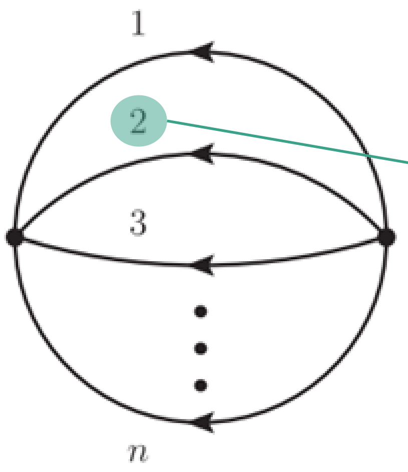
Jun. '20

NEW TECHNOLOGY FOR MULTILOOP COMPUTATIONS!!

- 2008
- 2010-2012
- 2014
- 2015
- 2016
- 2017
- 2018-2019
- 2019
- **2020-2021**

Nested residues: Details

- *Starting point:* multiloop Feynman integrals and scattering amplitudes
- ***Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta*



Multiloop diagram

Notation setup

Sets of momenta

$i_s \in s$ → $q_{i_s} = \ell_s + k_{i_s}$

Combination of external momenta

Loop momentum (integration)

Loop-energy component

$$G_F(q_{i_s}) = \frac{1}{q_{i_s,0}^2 - (q_{i_s,0}^{(+)})^2}$$

Standard propagator

$$q_{i_s,0}^{(+)} = \sqrt{\mathbf{q}_{i_s}^2 + m_{i_s}^2 - \imath 0}$$

On-shell energy (*complex number!*)

Loop space-vector (*Euclidean*)

Feynman i0 prescription

- Using this notation, we write *any* L-loop N-particle scattering amplitude:

$$\mathcal{A}_N^{(L)}(1,\dots,n) = \int_{\ell_1,\dots,\ell_L} \mathcal{N}(\{\ell_i\}_L,\{p_j\}_N)\, G_F(1,\dots,n) \quad \text{with} \quad G_F(1,\dots,n) = \prod_{i\in 1\cup\dots\cup n} (G_F(q_i))^{a_i}$$

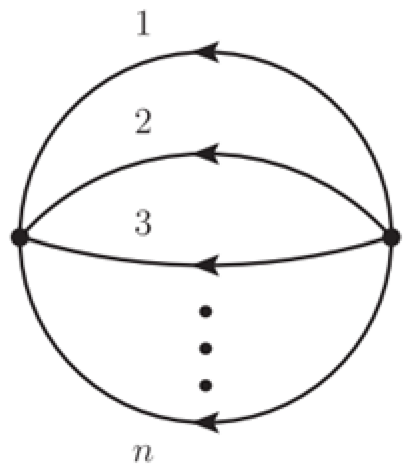
D-dimensional loop momenta (*Minkowski*)

Sets of momenta

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Nested residues: Details

- *Starting point:* multiloop Feynman integrals and scattering amplitudes
- ***Iterated*** *application of the Cauchy residue theorem to remove one DOF for each loop momenta*



Multiloop diagram

Iterated application of Cauchy's theorem

Remaining sets (no residue evaluation)

$$G_D(1,\ldots,r;n) = -2\pi\imath \sum_{i_r\in r} \mathrm{Res}(G_D(1,\ldots,r-1;r,n), \mathrm{Im}(\eta\cdot q_{i_r})<0)$$

r^{th} residue evaluation

Sum over all the elements of the r^{th} set

$(r-1)^{th}$ dual function

Depends on integration variables (q_i)

Poles could be in-or-out depending on specific momenta…

- Dual representation for L-loop amplitudes is obtained after the L^{th} residue evaluation
- *Equivalent to:* ***"Number of cuts equal number of loops"***
- ***Sum over all possible poles is implicit: some contributions vanish inside each iteration***

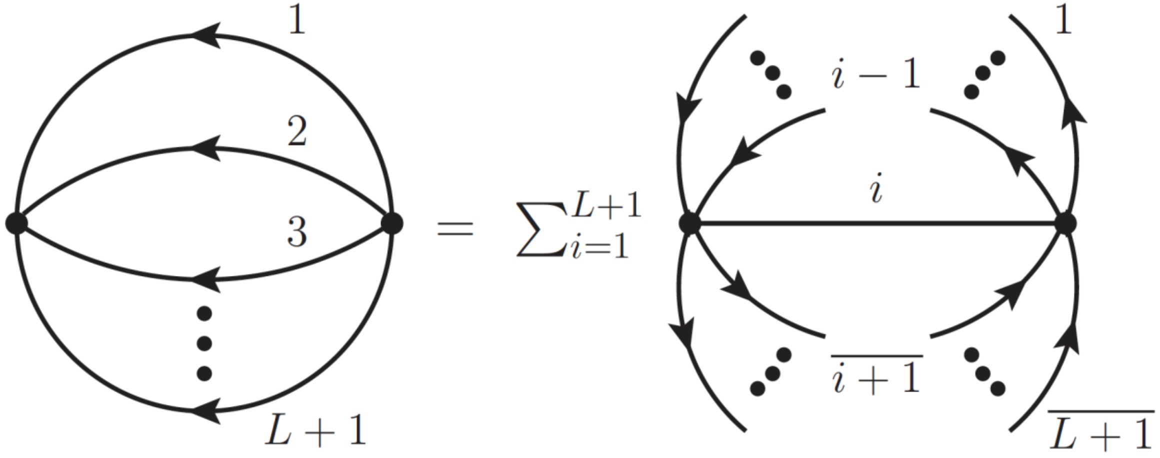
Iterated residues (all the poles) → cancellations / **"Displaced poles"** → **Nested residues *(only physical ones)***

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Nested residues: Compact representations

- **Cancellation of displaced poles leads to very compact formulae for the dual representation:**

Maximal Loop Topology (2 vertices, L+1 lines)



REMARK: External particles can be attached to each momenta set

Lines = sets of propagators

On-shell lines with reversed momenta

$$\mathcal{A}^{(L)}_{\rm MLT}(1,2,\dots,L+1) = \int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \sum_{i=1}^{L+1} \mathcal{A}_D(1,\dots,i-1,\overline{i+1},\dots,\overline{L+1};i)$$

1 off-shell line

Defined in Minkowski space

Defined in Euclidean space

On-shell lines

- We define the Maximal Loop Topology (MLT) as a building block to describe multi-loop amplitudes
- **Important:** “*Any one and two-loop amplitude can be described by MLT topologies*”

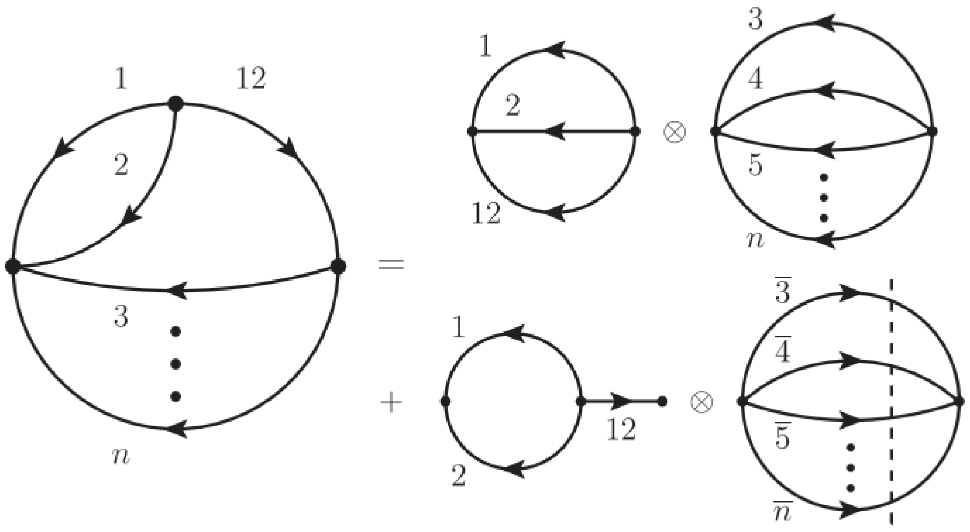
Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602 & JHEP 01 (2021) 069

Nested residues: Compact representations

- More complicated topologies can be described by convolutions with MLT-like diagrams

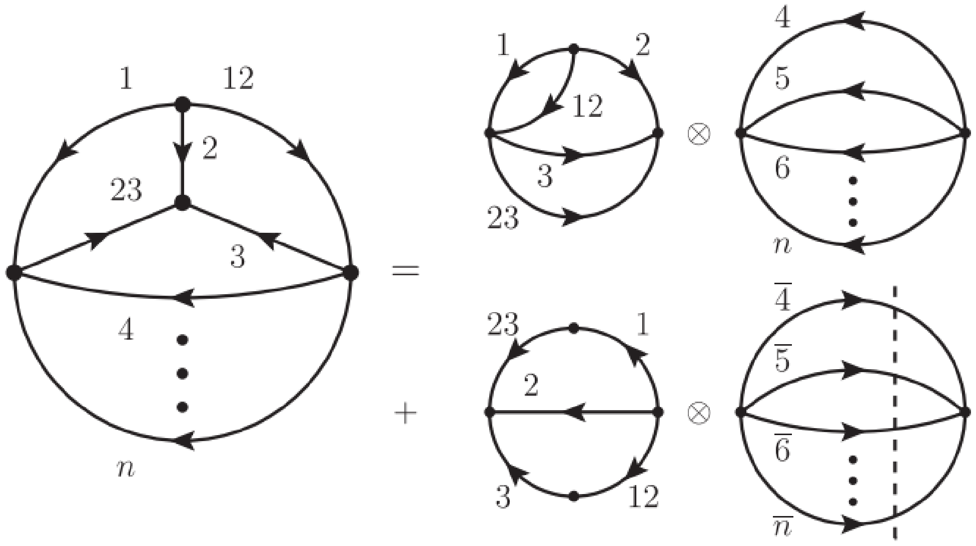
Next-to Maximal Loop Topology (3 vertices, L+2 lines)



$$\mathcal{A}^{(L)}_{\rm NMLT}(1,\dots,n,12) = \mathcal{A}^{(2)}_{\rm MLT}(1,2,12)\otimes\mathcal{A}^{(L-2)}_{\rm MLT}(3,\dots,n) + \mathcal{A}^{(1)}_{\rm MLT}(1,2)\otimes\mathcal{A}^{(0)}(12) \otimes\mathcal{A}^{(L-1)}_{\rm MLT}(\bar{3},\dots,\bar{n})$$

IMPORTANT FACTORIZATION FORMULAE
Singular and causal structure is determined by the corresponding sub-topologies

Next-to-Next-to Maximal Loop Topology (4 vertices, L+3 lines)



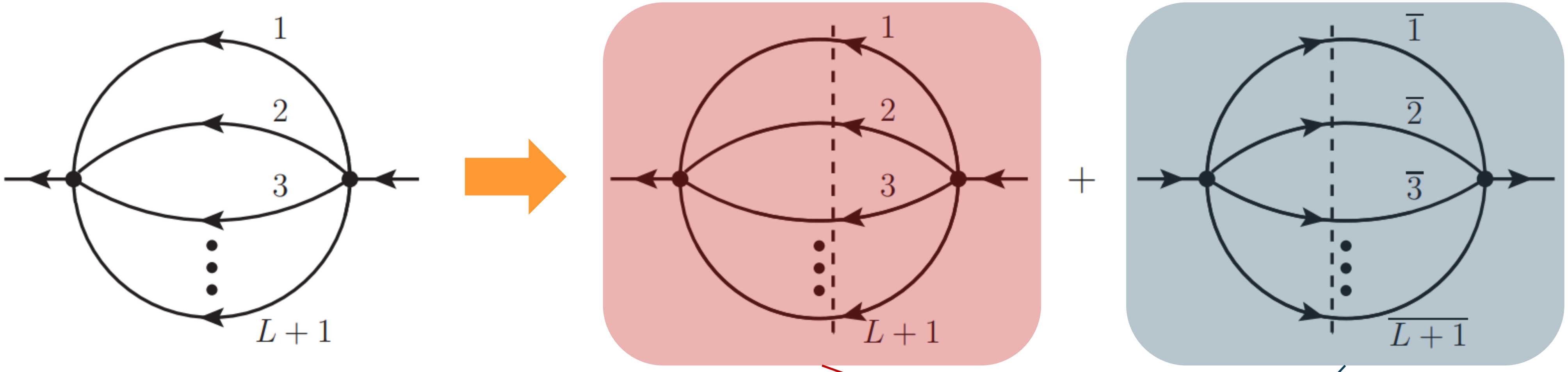
$$\mathcal{A}^{(L)}_{\rm NNMLT}(1,\dots,n,12,23) = \mathcal{A}^{(3)}_{\rm NMLT}(1,2,3,12,23)\otimes\mathcal{A}^{(L-3)}_{\rm MLT}(4,\dots,n) + \mathcal{A}^{(2)}_{\rm MLT}(1\cup 23,2,3\cup 12)\otimes\mathcal{A}^{(L-2)}_{\rm MLT}(\bar{4},\dots,\bar{n})$$

Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112

Aguilera-Verdugo et al (2020) Phys. Rev. Lett. 124, 211602

Causality at integrand level

- The cancellation of displaced poles implies un-physical terms vanish in the final representation
- Moreover, there is a strict connection between **aligned contributions** and **causal terms!!!**
- *MLT example*: If we **sum over all the possible cuts**, we get this **extremely compact** result:



$$\mathcal{A}^{(L)}_{\mathrm{MLT}}\left(1,2,\ldots,(L+1)_{-p_1}\right) = -\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{1}{x_{L+1}}\left(\frac{1}{\lambda_1^-}+\frac{1}{\lambda_1^+}\right)$$

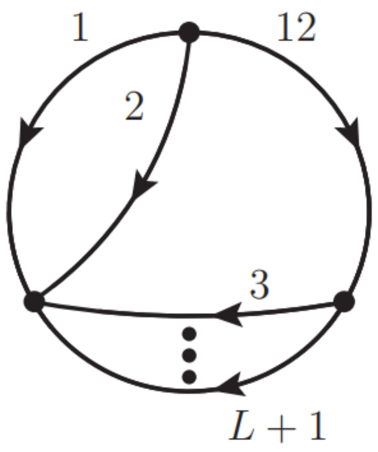
with $\lambda_1^\pm = \sum_{i=1}^{L+1} q_{i,0}^{(+)} \pm p_{1,0}$ and $x_{L+k} = 2^{L+k}\prod_{i=1}^{L+k} q_{i,0}^{(+)}$

CAUSAL PROPAGATORS

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph] & JHEP 01 (2021) 069

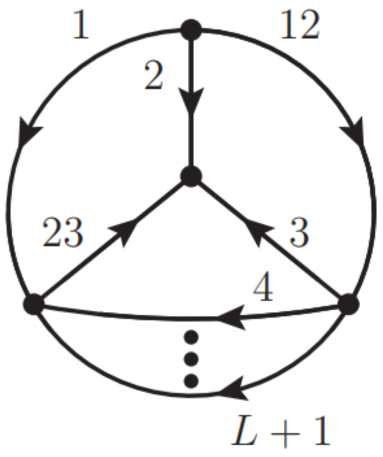
Causality at integrand level

- Similar formulae can be found for NMLT and NNMLT to all loop orders!



$$\mathcal{A}^{(L)}_{\mathrm{NMLT}}(1,2,\ldots,L+2) = \int_{\vec{\ell}_1,\ldots,\vec{\ell}_L} \frac{2}{x_{L+2}} \left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} \right)$$

with $\lambda_1 = \sum_{i=1}^{L+1} q_{i,0}^{(+)}$ $\lambda_2 = q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{L+2,0}^{(+)}$ $\lambda_3 = \sum_{i=3}^{L+2} q_{i,0}^{(+)}$



$$\mathcal{A}^{(L)}_{\mathrm{N^2MLT}}(1,2,\ldots,L+3) = -\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{2}{x_{L+3}} \left[\frac{1}{\lambda_1}\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}+\frac{1}{\lambda_5}\right) + \frac{1}{\lambda_6}\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_4}\right)\left(\frac{1}{\lambda_3}+\frac{1}{\lambda_5}\right) + \frac{1}{\lambda_7}\left(\frac{1}{\lambda_2}+\frac{1}{\lambda_5}\right)\left(\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right) \right]$$

with

$\lambda_4 = q_{2,0}^{(+)} + q_{3,0}^{(+)} + q_{L+3,0}^{(+)}$ $\lambda_6 = q_{1,0}^{(+)} + q_{3,0}^{(+)} + q_{L+2,0}^{(+)} + q_{L+3,0}^{(+)}$

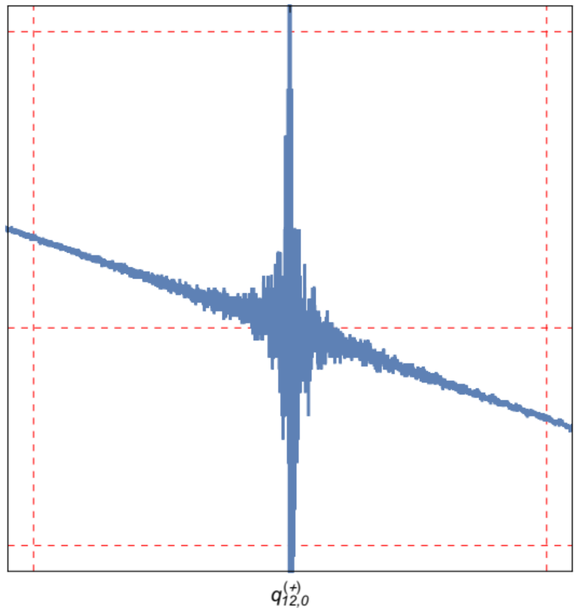
$\lambda_5 = q_{1,0}^{(+)} + q_{L+3,0}^{(+)} + \sum_{i=4}^{L+1} q_{i,0}^{(+)}$ $\lambda_7 = q_{2,0}^{(+)} + \sum_{i=4}^{L+3} q_{i,0}^{(+)}$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph] & JHEP 01 (2021) 069

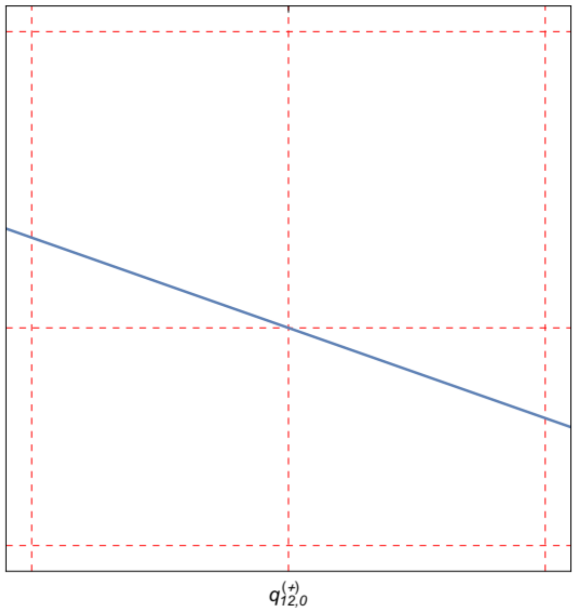
Causality at integrand level

- **This is *a* Causal Representation and exists for any QFT amplitude!**
- **<u>Advantages</u>**

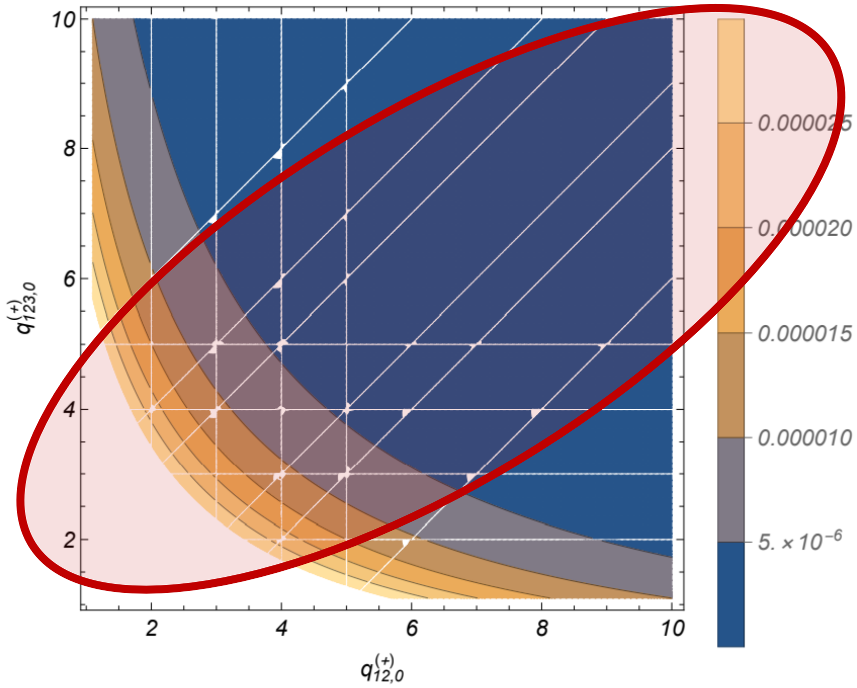
    1. Causal denominators have **same-sign combinations of on-shell energies** (positive numbers), thus are **more stable numerically!**
    2. **Only physical thresholds remain**; spurious un-physical instabilities are removed!



Without causal representation



With causal representation

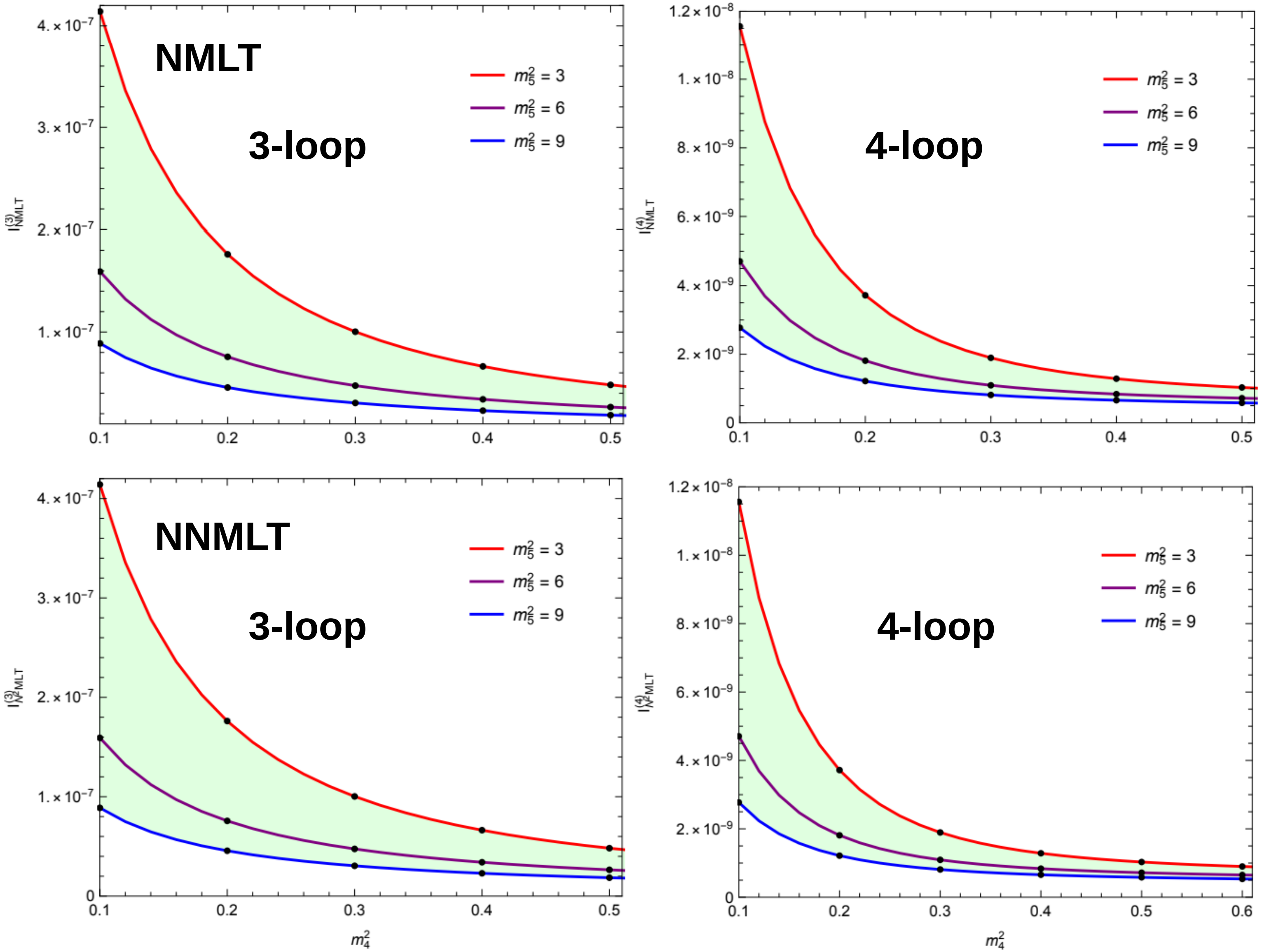


White lines = Numerical instabilities

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]
Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph]

Causality at integrand level: Implementation

- **Numerical results in D=4:**



$$\mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1^2,2^2,\ldots,L^2,L+1,\ldots,L+k)$$

$$=\prod_{i=1}^{L}\frac{\partial}{\partial(q_{i,0}^{(+)})^2}\mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1,2,\ldots,L+1,\ldots,L+k)$$

Is also causal by construction! (*derivatives preserve denominators*)

Solid lines: LTD
Dots: FIESTA

Setup:

$$\mathcal{A}^{(L)}_{\mathrm{N}^{k-1}\mathrm{MLT}}(1^2,2^2,\ldots,L^2,L+1,\ldots,L+k)$$

Mases: $\{1,2,\ldots,L\} \Longleftrightarrow m_4^2$

$\{L+1,\ldots,L+k\} \Longleftrightarrow m_5^2$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]
& JHEP 01 (2021) 069

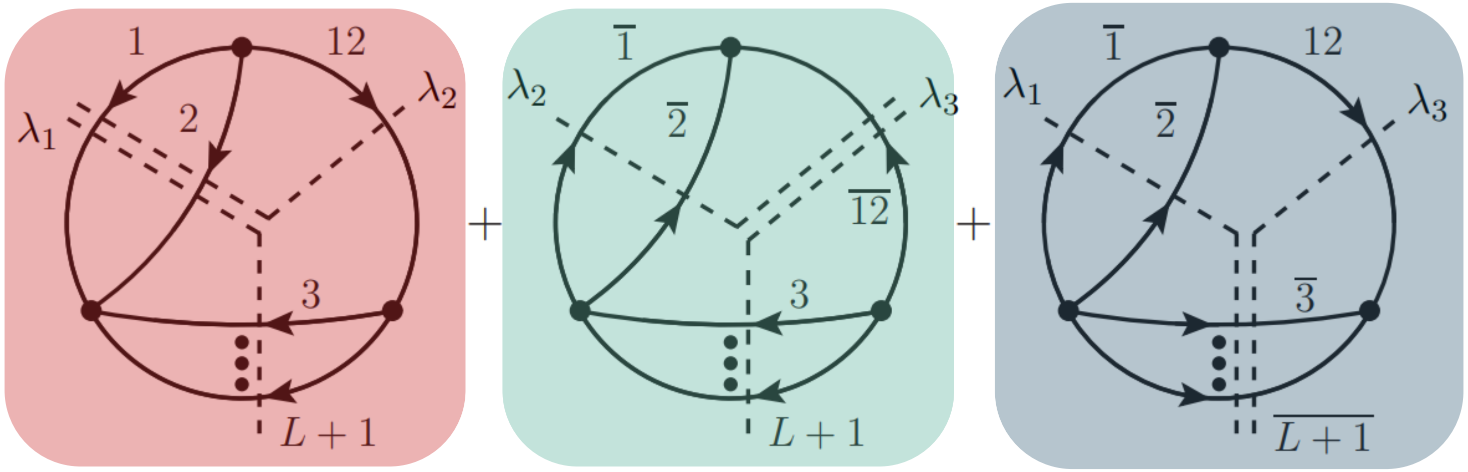
Geometric Algorithm for Causal Reconstruction

- Further studies were performed with several topological families

JHEP 01 (2021) 069, JHEP 04 (2021) 129, JHEP 04 (2021) 183, Eur.Phys.J.C 81 (2021) 6, 514

- Graphical interpretation in terms of entangled thresholds
    1. **Each causal** propagator represents a **threshold** of the diagram
    2. **Each diagram** contains **several thresholds**
    3. **The causal representation involves products of (*compatible*) thresholds**

Causal denominators (λ) are associated to *cut lines* in the diagrams: momenta flow must be adjusted to be compatible



$$\mathcal{A}^{(L)}_{\rm NMLT}(1,2,\ldots,L+2) = \int_{\vec{\ell}_1,\ldots,\vec{\ell}_L} \frac{2}{x_{L+2}} \left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} \right)$$

Aguilera-Verdugo et al (2020) arXiv:2006.11217 [hep-ph]

Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph]

Geometric Algorithm for Causal Reconstruction

- **Causal representation** obtained directly after **summing over all the nested residues**

Master formula

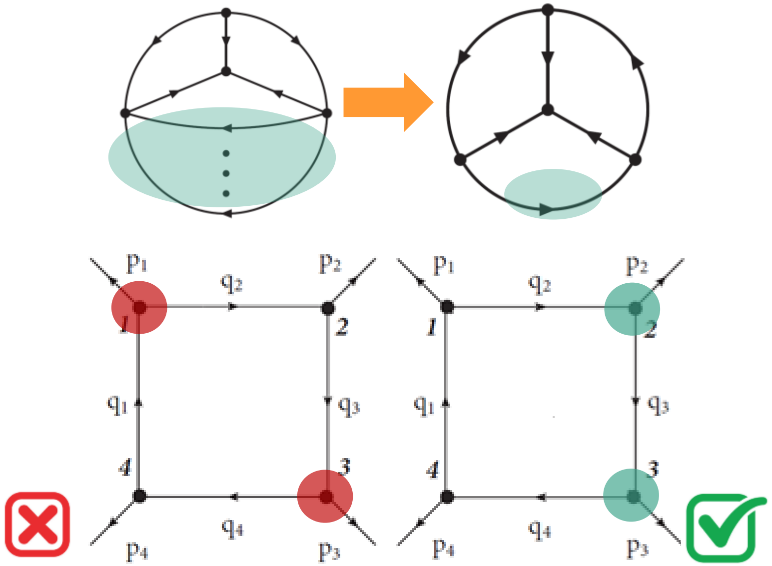
$$\mathcal{A}_N^{(L)}(1,\ldots,L+k) = \sum_{\sigma\in\Sigma}\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{\mathcal{N}_\sigma(\{q_{r,0}^{(+)}\},\{p_{j,0}\})}{x_{L+k}} \times \prod_{i=1}^{k} \frac{1}{-\lambda_{\sigma(i)}} + (\sigma \leftrightarrow \bar{\sigma})$$

Set of entangled thresholds

Products of *k* causal propagators

- *Is it possible to do it in other way?* →
  - **Geometrical reconstruction (this talk!)** **Sborlini '21**
  - **Algebraic reconstruction (Lotty)** **Torres Bobadilla '21**

- **Previous concepts**
  1. **Diagrams** are made of **vertices** and **edges** *(bunches of propagators, connecting two given vertices)*
  2. **Edges** define a **basis of momenta**, that lead to the **"vertex matrix"** → **Defines the casual structure!**
  3. **Binary partitions** are given by **subsets of vertices** that **splits in two** the original diagram → **Connected partitions!**

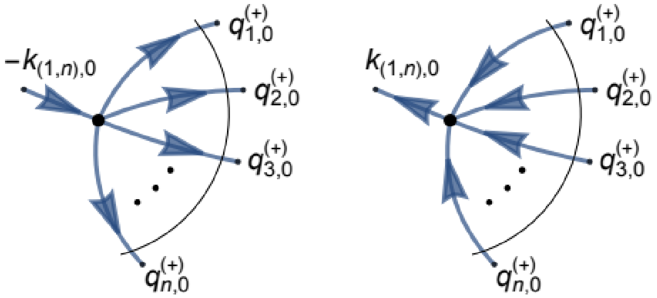


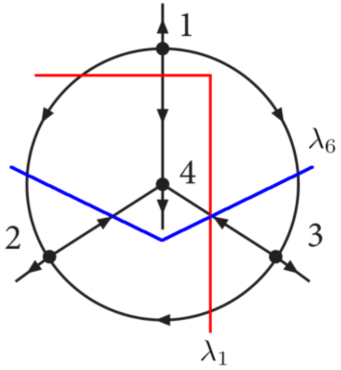
Sborlini (2021) arXiv:2102.05062 [hep-ph]

Geometric Algorithm for Causal Reconstruction

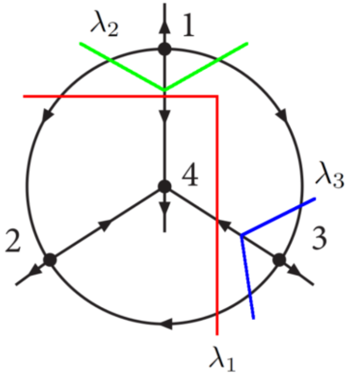
More detailed explanation arXiv:2102.05062 [hep-ph]

1. <u>**Generate causal propagators**</u>
   - Causal propagators are associated to **binary connected partitions** of the diagram, namely *"connected sub-blocks of the diagram"*
   - They encode the possible **physical thresholds**
   - Involve a **consistent (aligned) energy flow** through the cut lines
2. <u>**Order of a diagram:**</u> it quantifies the complexity of a given topology
   - *k=1 for MLT, k=2 for NMLT and so on* → ***k = vertices - 1***
   - A diagram of **order k** involves **products of k causal propagators**
3. <u>**Geometric compatibility rules:**</u> determine the entangled thresholds
   - a) **All the edges are cut at least once**
   - b) **Causal propagators do no intersect;** i.e. they are associated to disjoint or extended partitions of the diagram
   - c) **All the edges** involved in a causal threshold must carry **momenta flowing in the same direction** → Distinction λ^+ / λ^-





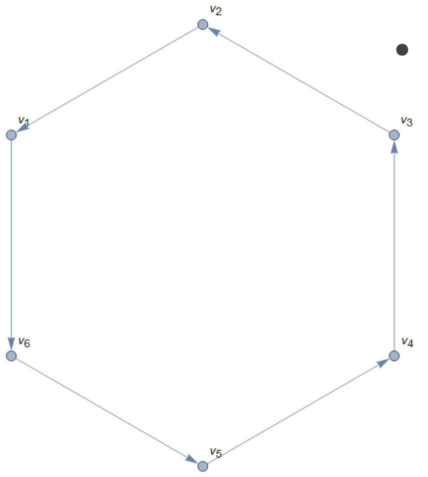
Presence of intersections



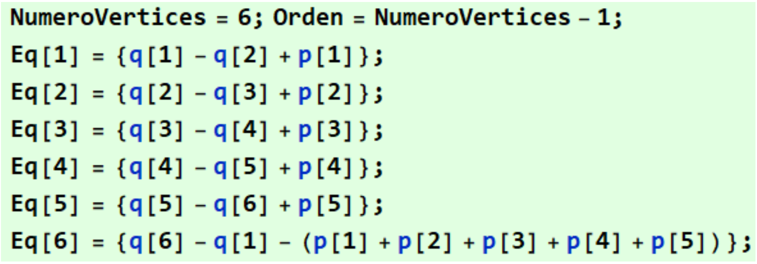
Incompatible causal flux

Sborlini (2021) arXiv:2102.05062 [hep-ph]

Geometric Algorithm for Causal Reconstruction

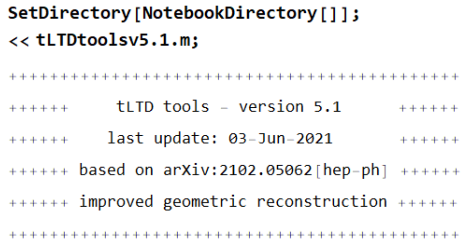


- **Example:** 1-loop hexagon (6 vertices, 1 external leg per vertex)

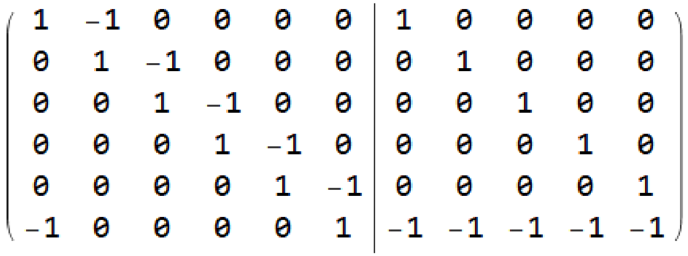


```
NumeroVertices = 6; Orden = NumeroVertices - 1;
Eq[1] = {q[1] - q[2] + p[1]};
Eq[2] = {q[2] - q[3] + p[2]};
Eq[3] = {q[3] - q[4] + p[3]};
Eq[4] = {q[4] - q[5] + p[4]};
Eq[5] = {q[5] - q[6] + p[5]};
Eq[6] = {q[6] - q[1] - (p[1] + p[2] + p[3] + p[4] + p[5])};
```

Input: vertex definition, i.e. labelling
& momentum conservation



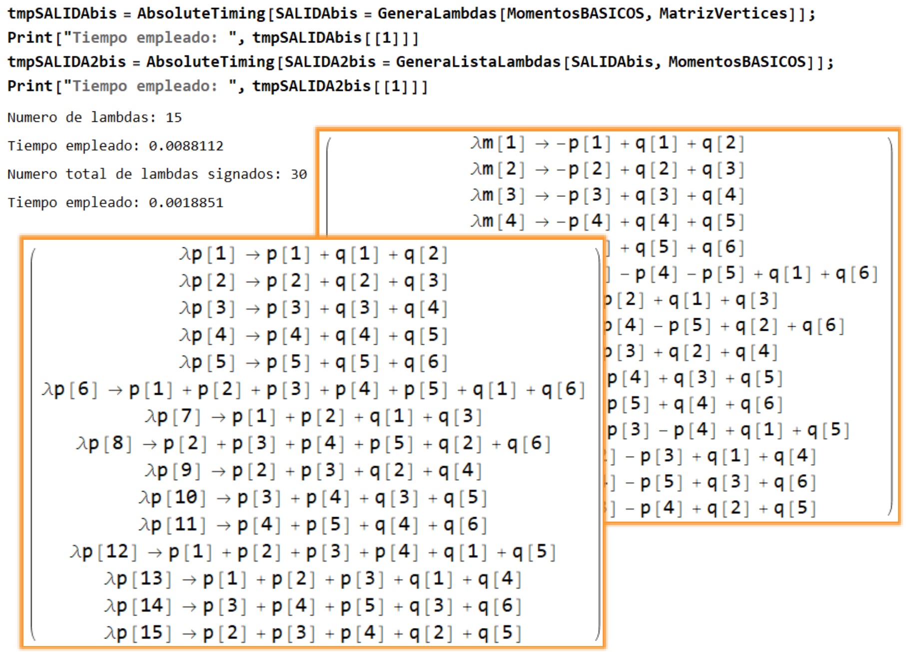
```
SetDirectory[NotebookDirectory[]];
<< tLTDtoolsv5.1.m;

++++++++++++++++++++++++++++++++++++++++++++++
++++++      tLTD tools - version 5.1     ++++++
++++++     last update: 03-Jun-2021      ++++++
++++++ based on arXiv:2102.05062[hep-ph] ++++++
++++++ improved geometric reconstruction ++++++
++++++++++++++++++++++++++++++++++++++++++++++
```



$$\left(\begin{array}{cccccc|ccccc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \end{array}\right)$$

Vertex matrix: Basic object to generate the causal representation

Generate causal propagators

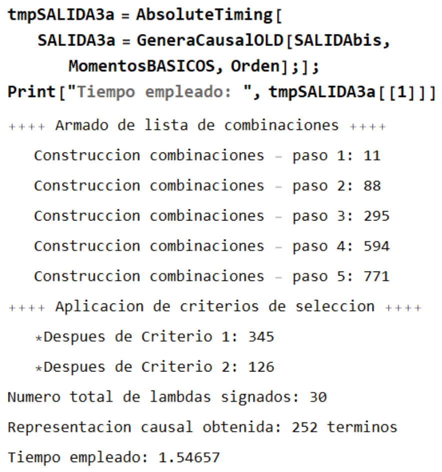


```
tmpSALIDAbis = AbsoluteTiming[SALIDAbis = GeneraLambdas[MomentosBASICOS, MatrizVertices]];
Print["Tiempo empleado: ", tmpSALIDAbis[[1]]]
tmpSALIDA2bis = AbsoluteTiming[SALIDA2bis = GeneraListaLambdas[SALIDAbis, MomentosBASICOS]];
Print["Tiempo empleado: ", tmpSALIDA2bis[[1]]]

Numero de lambdas: 15

Tiempo empleado: 0.0088112

Numero total de lambdas signados: 30

Tiempo empleado: 0.0018851
```

```
λm[1] → -p[1] + q[1] + q[2]
λm[2] → -p[2] + q[2] + q[3]
λm[3] → -p[3] + q[3] + q[4]
λm[4] → -p[4] + q[4] + q[5]
] + q[5] + q[6]
] - p[4] - p[5] + q[1] + q[6]
p[2] + q[1] + q[3]
p[4] - p[5] + q[2] + q[6]
p[3] + q[2] + q[4]
p[4] + q[3] + q[5]
p[5] + q[4] + q[6]
p[3] - p[4] + q[1] + q[5]
] - p[3] + q[1] + q[4]
] - p[5] + q[3] + q[6]
] - p[4] + q[2] + q[5]
```

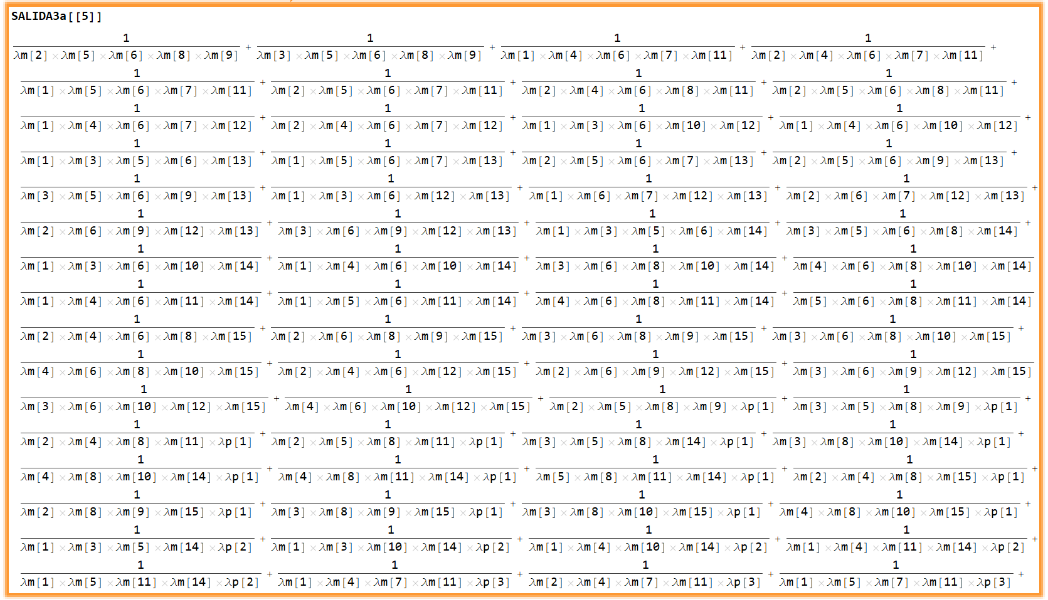
```
λp[1] → p[1] + q[1] + q[2]
λp[2] → p[2] + q[2] + q[3]
λp[3] → p[3] + q[3] + q[4]
λp[4] → p[4] + q[4] + q[5]
λp[5] → p[5] + q[5] + q[6]
λp[6] → p[1] + p[2] + p[3] + p[4] + p[5] + q[1] + q[6]
λp[7] → p[1] + p[2] + q[1] + q[3]
λp[8] → p[2] + p[3] + p[4] + p[5] + q[2] + q[6]
λp[9] → p[2] + p[3] + q[2] + q[4]
λp[10] → p[3] + p[4] + q[3] + q[5]
λp[11] → p[4] + p[5] + q[4] + q[6]
λp[12] → p[1] + p[2] + p[3] + p[4] + q[1] + q[5]
λp[13] → p[1] + p[2] + p[3] + q[1] + q[4]
λp[14] → p[3] + p[4] + p[5] + q[3] + q[6]
λp[15] → p[2] + p[3] + p[4] + q[2] + q[5]
```

Generate entangled thresholds (using selection rules)



```
tmpSALIDA3a = AbsoluteTiming[
   SALIDA3a = GeneraCausalOLD[SALIDAbis,
      MomentosBASICOS, Orden];];
Print["Tiempo empleado: ", tmpSALIDA3a[[1]]]

++++ Armado de lista de combinaciones ++++

   Construccion combinaciones - paso 1: 11

   Construccion combinaciones - paso 2: 88

   Construccion combinaciones - paso 3: 295

   Construccion combinaciones - paso 4: 594

   Construccion combinaciones - paso 5: 771

++++ Aplicacion de criterios de seleccion ++++

   *Despues de Criterio 1: 345

   *Despues de Criterio 2: 126

Numero total de lambdas signados: 30

Representacion causal obtenida: 252 terminos

Tiempo empleado: 1.54657
```

Causal representation



SALIDA3a[[5]]

$$\frac{1}{\lambda m[2]\times\lambda m[5]\times\lambda m[6]\times\lambda m[8]\times\lambda m[9]}+\frac{1}{\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[8]\times\lambda m[9]}+\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[11]}+\frac{1}{\lambda m[2]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[11]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[5]\times\lambda m[6]\times\lambda m[7]\times\lambda m[11]}+\frac{1}{\lambda m[2]\times\lambda m[5]\times\lambda m[6]\times\lambda m[7]\times\lambda m[11]}+\frac{1}{\lambda m[2]\times\lambda m[4]\times\lambda m[6]\times\lambda m[8]\times\lambda m[11]}+\frac{1}{\lambda m[2]\times\lambda m[5]\times\lambda m[6]\times\lambda m[8]\times\lambda m[11]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]}+\frac{1}{\lambda m[2]\times\lambda m[4]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]}+\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[6]\times\lambda m[10]\times\lambda m[12]}+\frac{1}{\lambda m[1]\times\lambda m[4]\times\lambda m[6]\times\lambda m[10]\times\lambda m[12]}+$$
$$\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[13]}+\frac{1}{\lambda m[1]\times\lambda m[5]\times\lambda m[6]\times\lambda m[7]\times\lambda m[13]}+\frac{1}{\lambda m[2]\times\lambda m[5]\times\lambda m[6]\times\lambda m[7]\times\lambda m[13]}+\frac{1}{\lambda m[2]\times\lambda m[5]\times\lambda m[6]\times\lambda m[9]\times\lambda m[13]}+$$
$$\frac{1}{\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[9]\times\lambda m[13]}+\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[6]\times\lambda m[12]\times\lambda m[13]}+\frac{1}{\lambda m[1]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]\times\lambda m[13]}+\frac{1}{\lambda m[2]\times\lambda m[6]\times\lambda m[7]\times\lambda m[12]\times\lambda m[13]}+$$
$$\frac{1}{\lambda m[2]\times\lambda m[6]\times\lambda m[9]\times\lambda m[12]\times\lambda m[13]}+\frac{1}{\lambda m[3]\times\lambda m[6]\times\lambda m[9]\times\lambda m[12]\times\lambda m[13]}+\frac{1}{\lambda m[1]\times\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[14]}+\frac{1}{\lambda m[3]\times\lambda m[5]\times\lambda m[6]\times\lambda m[8]\times\lambda m[14]}+$$
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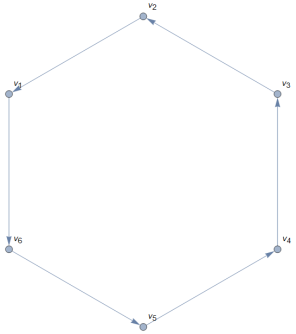
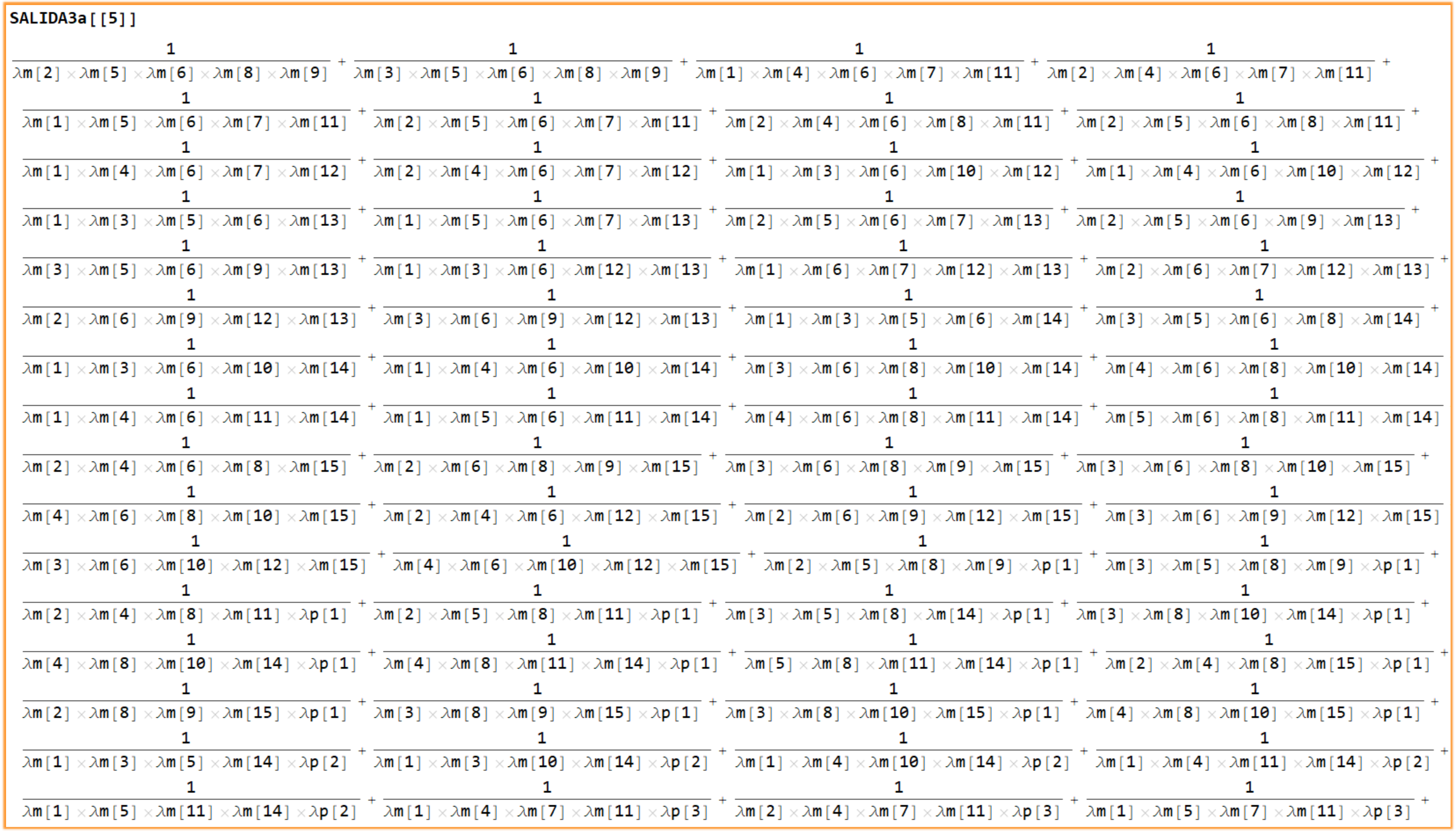
(+ similar terms …)

Sborlini (2021) arXiv:2102.05062 [hep-ph]

Geometric Algorithm for Causal Reconstruction



Causal representation



SALIDA3a[[5]]

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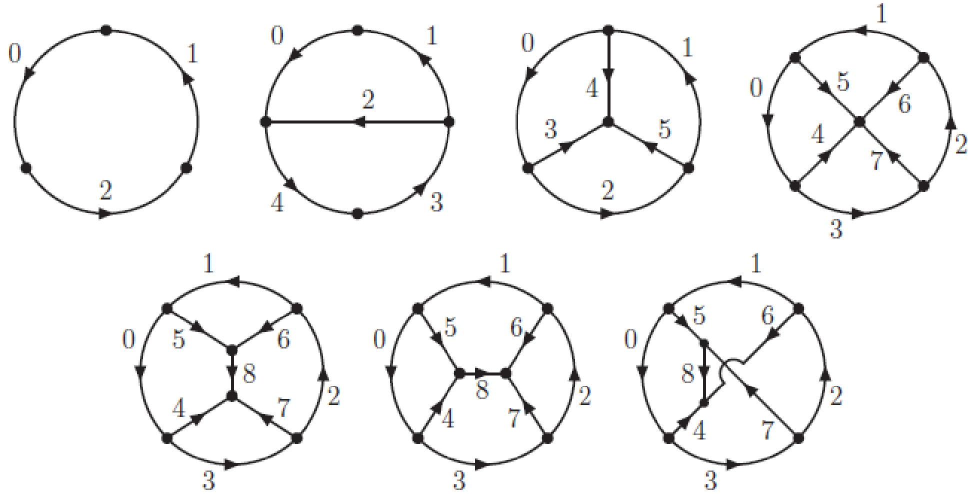
(+ similar terms …)

Sborlini (2021) arXiv:2102.05062 [hep-ph]

Quantum Algorithm for Causal Reconstruction

arXiv:2105.08703 [hep-ph]

- New technology based on **Grover's algorithm** to identify causal flux!
- We assign **1 qubit to each edge**, and impose logical conditions to select configurations without closed cycles → **Non-cyclical configurations = Causal flux**
- **Important: "loop"** refers to **integration variables; "eloop"** to loops in the **graph**



Total number of orderings (n = nº of edges) $N = 2^n$ → $|q\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$ **Quantum superposition of N flux configurations**

$$|q\rangle = \cos\theta\,|q_\perp\rangle + \sin\theta\,|w\rangle$$

$|w\rangle = \frac{1}{\sqrt{r}} \sum_{x\in w} |x\rangle$ **"Winning state" (causal flow)**

$|q_\perp\rangle = \frac{1}{\sqrt{N-r}} \sum_{x\notin w} |x\rangle$ **States with non-causal flow**

- Grover's algorithm **enhances** the probability of the **winning state** by using two operators:

$$U_w = \boldsymbol{I} - 2|w\rangle\langle w|$$

Oracle operator (changes sign of winning states)

$$U_q = 2|q\rangle\langle q| - \boldsymbol{I}$$

Diffusion operator (reflects with respect to initial state)

→ $$(U_q U_w)^t |q\rangle = \cos\theta_t\,|q_\perp\rangle + \sin\theta_t\,|w\rangle$$

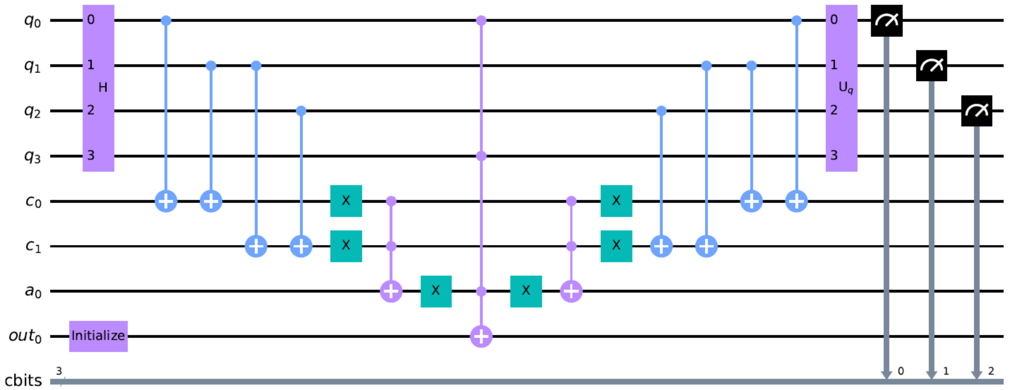
with $\boxed{\sin^2\theta_t \sim 1}$

Ramírez-Uribe et al (2021) arXiv:2105.08703 [hep-ph]

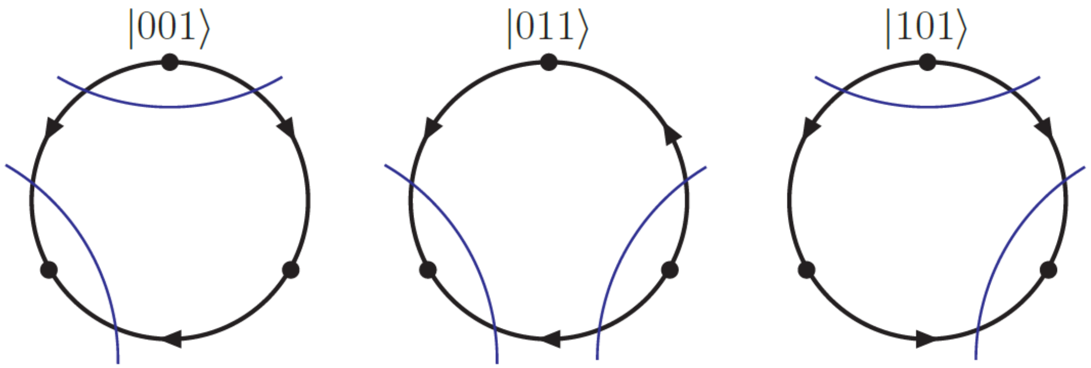
Quantum Algorithm for Causal Reconstruction

arXiv:2105.08703 [hep-ph]

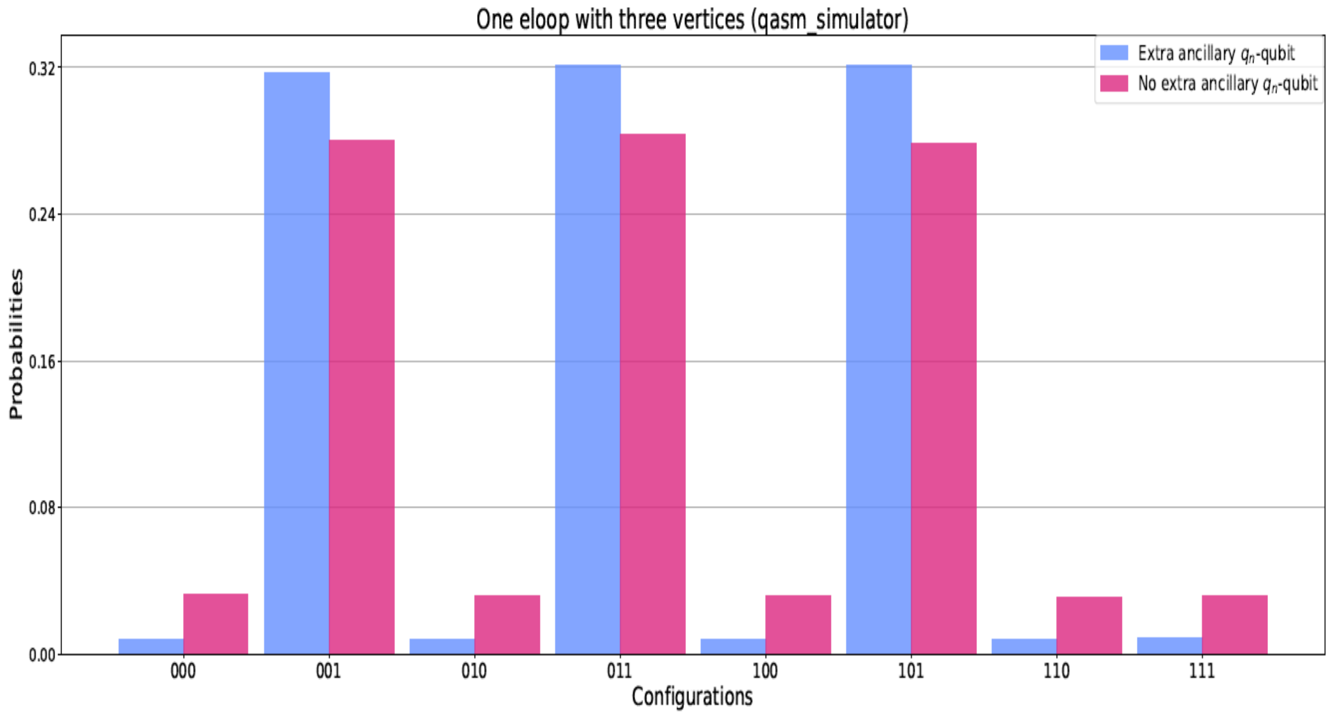
- Implemented with Qiskit and run in **IBM Q** (simulator & real QC)
- Several topologies studied!! **Enhanced performance** with extra-qubits



Quantum circuit



Causal configurations



The selected configurations are exactly |001>, |011>, |101>

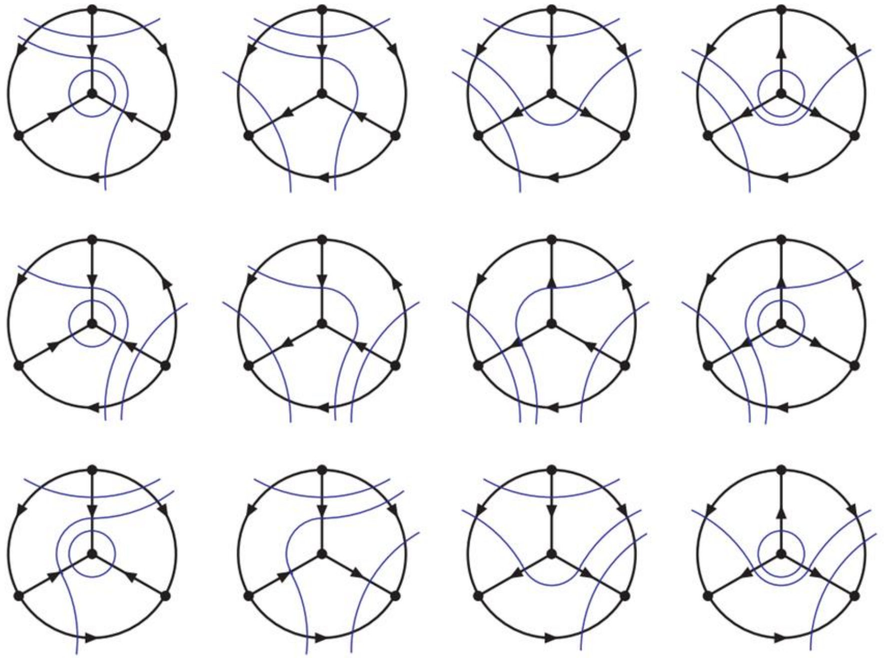
The algorithm identifies the causal fluxes, relying on geometrical concepts!

Ramírez-Uribe et al (2021) arXiv:2105.08703 [hep-ph]

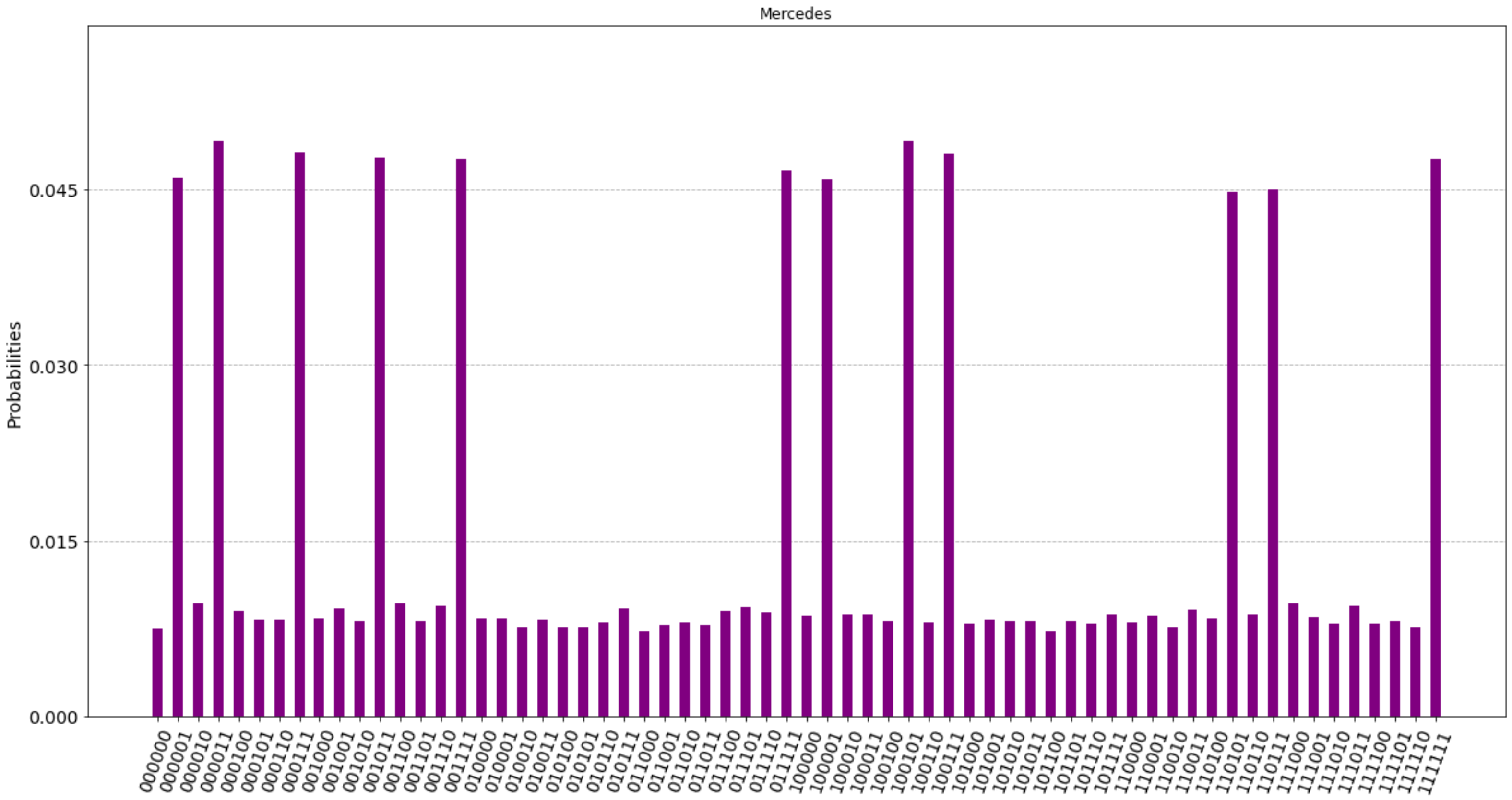
Quantum Algorithm for Causal Reconstruction

Preliminary results!!
To be published soon!!

- **Optimized algorithm based on properties of the adjacency matrix**
- Reduced number of qubits (allows to implement more complicated topologies in current devices)
- Successful identification of causal fluxes!!



Causal fluxes
(+ possible causal entangled thresholds)



Probability distribution
(all the 12 causal fluxes identified!!)

Ramírez-Uribe et al (2021) ***in preparation***

Conclusions

- Use LTD to cleverly rewrite Feynman integrals: **Minkowski to Euclidean**
- Achieve **local integrand representations free of IR/UV** singularities for physical observables
- **Novel LTD approach** based on **nested residues leads to manifestly causal representations** of multiloop scattering amplitudes!
- Very compact formulae **with strong physical/conceptual** motivation

- **Geometrical rules** select **entangled thresholds. Complete reconstruction** of multiloop amplitudes!
- **Quantum algorithms** to speed-up **causal flux selection**. *Exploring new disruptive tools for breaking the precision frontier!!*

THANKS!

BACKUP SLIDES.

Nested residues: Displaced poles

- Practical (mathematical) example:

$$f(\vec{x}) = \frac{1}{(x_1^2 - y_1^2)\ldots(x_L^2 - y_L^2)\,(z_{L+1}^2 - y_{L+1}^2)}$$

to calculate

$$\boxed{I = \left(\prod_{i=1}^{L}\int \frac{dx_i}{2\pi\imath}\right) f(\vec{x})}$$

Complex coefficients

$$y_i \to \tilde{y}_i = \sqrt{y_i^2 - \imath 0}$$

$$z_{L+1} = -\sum_{j=1}^{L} x_j + k_{L+1}$$

Sum of integration variables (real)

- **1st step:** Apply C.R.T. in x_1, by promoting $x_1 \in \mathbb{R} \to \mathbb{C}$ (the other *x*'s remain real)

$$I = -\left(\prod_{i=2}^{L}\int \frac{dx_i}{2\pi\imath}\right) \sum_{x_{1,j}\in \mathrm{Poles}[f,x_1]} \mathrm{Res}\left(f(\vec{x}), \{x_1, x_{1,j}\}\right) \theta(-\mathrm{Im}(x_{1,j})) \Rightarrow I = -\left(\prod_{i=2}^{L}\int \frac{dx_i}{2\pi\imath}\right) \sum_{x_{1,j}\in \mathrm{Poles}^{(+)}[f,x_1]} \mathrm{Res}\left(f(\vec{x}), \{x_1, x_{1,j}\}\right)$$

Theta functions removed

$$\mathrm{Poles}^{(+)}[f, x_1] = \{y_1, y_{L+1} - k_{L+1} - x_2 - \ldots - x_L\}$$

Subset of poles with negative imaginary part

IMPORTANT! x's are real, y's are complex

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

Nested residues: Displaced poles

- Practical (mathematical) example:

$$I = -\left(\prod_{i=2}^{L}\int\frac{dx_i}{2\pi\imath}\right)\sum_{x_{1,j}\in\mathrm{Poles}^{(+)}[f,x_1]}\mathrm{Res}\left(f(\vec{x}),\{x_1,x_{1,j}\}\right)$$

$$\mathrm{Poles}^{(+)}[f,x_1] = \{y_1, y_{L+1}-k_{L+1}-x_2-\ldots-x_L\}$$

$$\begin{aligned}\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\}) &= \frac{1}{2y_1\left(x_2^2-y_2^2\right)\ldots\left(x_L^2-y_L^2\right)\left((y_1+x_2+\ldots+x_L-k_{L+1})^2-y_{L+1}^2\right)} \\ &+ \frac{1}{2y_{L+1}\left((y_{L+1}+k_{L+1}-x_2-\ldots-x_L)^2-y_1^2\right)(x_2^2-y_2^2)\ldots(x_L^2-y_L^2)}\end{aligned}$$

Sum of the residues in x_1 (negative imaginary part)

- **2nd step:** Apply C.R.T. in x_2, by promoting $x_2 \in \mathbb{R} \to \mathbb{C}$ (the other *x*'s remain real)

$$\mathrm{Res}(\,\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\})\,,\{x_2,\mathrm{Im}(x_2)<0\})$$

$$= \sum_{x_{2,l}\in\mathrm{Poles}[f,x_1,x_2]}\mathrm{Res}(\,(\mathrm{Res}(f,\{x_1,\mathrm{Im}(x_1)<0\}),\{x_2,x_{2,l}\}\,)\,\theta(-\mathrm{Im}(x_{2,l}))$$

Theta functions remain!

$$\mathrm{Poles}[f,x_1;x_2] = \{\pm y_2, \pm y_1+y_{L+1}-x_3-\ldots-x_L+k_{L+1}, \pm y_{L+1}-y_1-x_3-\ldots-x_L+k_{L+1}\}$$

All the possible poles:
SIGN OF IMAGINARY PART + or - !!!

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

Nested residues: Displaced poles

- Practical (mathematical) example:

$$\text{Res}(\,\text{Res}(f,\{x_1,\text{Im}(x_1)<0\})\,,\{x_2,\text{Im}(x_2)<0\}) \;=\; \sum_{x_{2,l}\in\text{Poles}[f,x_1,x_2]} \text{Res}(\,(\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,x_{2,l}\}\,)\,\theta(-\text{Im}(x_{2,l}))$$

- **3$^{\text{rd}}$ step:** Collect the different contributions according to $\theta(-\text{Im}(x_{2,l}))$:

$$\left\{\begin{aligned}
&\text{Res}(\,(\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,y_2\}\,)\\
&\qquad=\frac{1}{4y_1y_2\,(x_3^2-y_3^2)\ldots(x_L^2-y_L^2)((y_1+y_2+x_3+\ldots+x_L-k_{L+1})^2-y_{L+1}^2)}\\
&\qquad+\frac{1}{4y_{L+1}y_2\,((y_{L+1}-y_2-x_3-\ldots-x_L+k_{L+1})^2-y_1^2)\ldots(x_L^2-y_L^2)}\\
&\text{Res}(\,\text{Res}(f,\{x_1,\text{Im}(x_1)<0\}),\{x_2,y_1+y_{L+1}-x_3-\ldots-x_L+k_{L+1}\}\,)\\
&\qquad=\frac{1}{4y_1y_3\,((y_1+y_{L+1}-x_3-\ldots-x_L+k_{L+1})^2-y_2^2)(x_3^2-y_3^2)\ldots(x_L^2-y_L^2)}
\end{aligned}\right.$$

Theta functions are trivially 1: y's have negative imaginary part, x's are real → **Only sums of y's!!! ALIGNED CONTRIBUTIONS**

$$\left\{\begin{aligned}
&[\text{Res}(\,\text{Res}(f,\{x_1,y_1\}),\{x_2,y_{L+1}-y_1-x_3-\ldots-x_L+k_{L+1}\}\,)\\
&+\text{Res}(\,\text{Res}(f,\{x_1,y_{L+1}-x_2-\ldots-x_L+k_{L+1}\}),\\
&\qquad\{x_2,y_{L+1}-y_1-x_3-\ldots-x_L+k_{L+1}\}\,)]\;\theta(\text{Im}(y_1-y_{L+1}))
\end{aligned}\right.$$

Different-sign combinations of y's: **NON-TRIVIAL THETA!** → **DISPLACED POLES: VANISH!!**

Aguilera-Verdugo et al (2020) arXiv:2010.12971 [hep-ph]

Nested residues: Displaced poles

- *Theorem*: Given a generic* rational function $F(x_i, x_j) = \dfrac{P(x_i, x_j)}{((x_i - a_i)^2 - y_i^2)^{\gamma_i}((x_i + x_j - a_{ij})^2 - y_k^2)^{\gamma_k}}$

  then: $$\mathrm{Res}(\mathrm{Res}(F(x_i,x_j), \{x_i, y_i + a_i\}), \{x_j, y_k - y_i + a_{ij} - a_i\})$$
  $$= -\mathrm{Res}\left(\mathrm{Res}\left(F(x_i, x_j), \{x_i, y_k - x_j + a_{ij}\}\right), \{x_j, y_k - y_i + a_{ij} - a_i\}\right)$$

- **Physical consequences:**

  1. **Displaced poles** are associated to **un-physical** contributions:

     "they can not be mapped into cuts"

  2. After applying C.R.T. to all the loop momenta and **summing over the physical poles**:

     "only same-sign combinations of $q_{k,0}^{(+)}$ *remain"*

Cancellation of displaced poles $\xrightarrow{\text{“Aligned contributions”}}$ **Causal propagators**

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- **Mathematical consequences:**
  1. In each iteration of C.R.T., contributions with **different sign combinations of *y*'s vanish**
  2. Thus, after iterating over all integration variables, **only same-sign combinations of *y*'s remain**

Example:
$L = 2$

$$\text{Res}(\,\text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, \text{Im}(x_2) < 0\}\,)$$
$$= \frac{1}{4y_1y_2\left((y_1 + y_2 - k_3)^2 - y_3^2\right)} + \frac{1}{4y_2y_3\left((y_3 + y_1 + k_3)^2 - y_2^2\right)}$$
$$+ \frac{1}{4y_1y_3\left((y_3 - y_2 + k_3)^2 - y_1^2\right)}$$
$$= -\frac{1}{8y_1y_2y_3}\left(\frac{1}{\boxed{y_1 + y_2 + y_3} - k_3} + \frac{1}{\boxed{y_1 + y_2 + y_3} + k_3}\right)$$

Connection to QFT

y_i ↔ $q_{i,0}^{(+)} = \sqrt{\boldsymbol{q}_i^2 + m_i^2 - \imath 0}$

x_i ↔ $q_{i,0}$

a_i ↔ $\{k_{m,0}\}$

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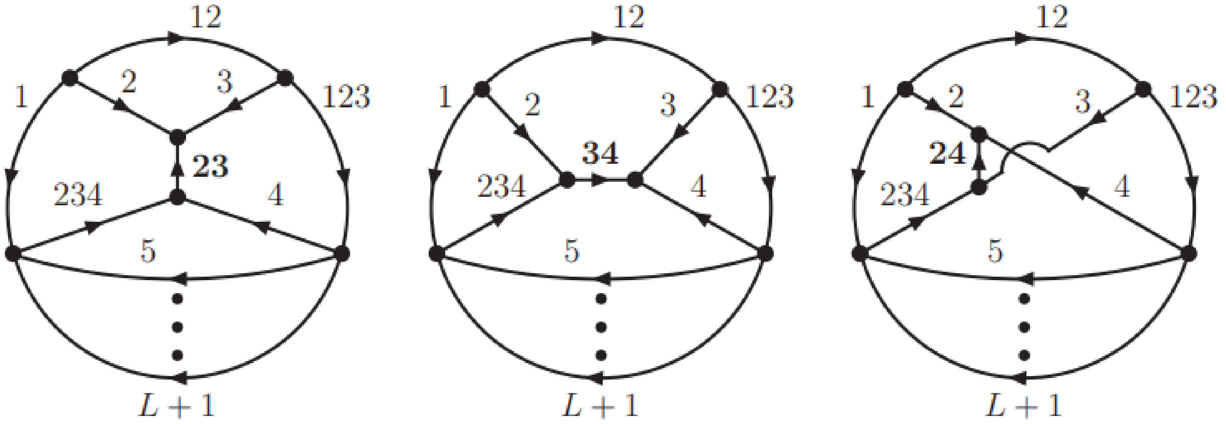
Nested residues: Compact representations

- It works also for (much) more complicated topologies!!!

NNNN Maximal Loop Topologies (6 vertices, L+5 lines)



Thanks to factorization properties, the singular and **causal** structure is given in terms of simpler objects

Ramírez Uribe et al (2020) arXiv:2006.13818 [hep-ph] & JHEP 04 (2021) 129

Lines = sets of propagators

$$
\begin{aligned}
\mathcal{A}^{(L)}_{\mathrm{N^4MLT}}(1,\ldots,L+1,12,123,234,J)
&= \mathcal{A}^{(4)}_{\mathrm{N^4MLT}}(1,2,3,4,12,123,234,J) \\
&\quad \otimes \mathcal{A}^{(L-4)}_{\mathrm{MLT}}(5,\ldots,L+1) \\
&+ \mathcal{A}^{(3)}_{\mathrm{N^2MLT}}(1\cup 234,2,3,4\cup 123,12,J) \\
&\quad \otimes \mathcal{A}^{(L-3)}_{\mathrm{MLT}}(\bar{5},\ldots,\overline{L+1})
\end{aligned}
$$

N^{4}MLT universal topology