

High precision initial state QED corrections in $e^+ e^-$ annihilation

A Loop Summit - new perturbative results and methods in precision physics, Cadenabbia, 2021

Kay Schönwald | July 29, 2021

TTP KARLSRUHE

in collaboration with: J. Ablinger, J. Blümlein, A. De Freitas

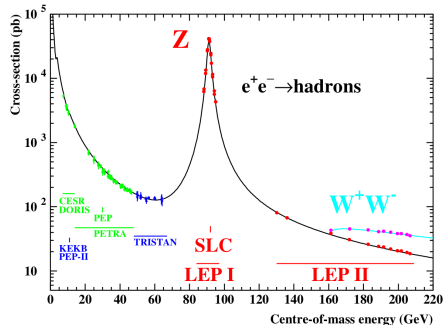
[based on: Ablinger, Blümlein, De Freitas, Schönwald (Nucl. Phys. B955 (2020))
and Blümlein, De Freitas, Schönwald (Phys. Lett. B816 (2021))]

- 1 Motivation
- 2 The Method of Massive Operator Matrix Elements
- 3 Results for the Total Cross-Section
- 4 Results for the Forward-Backward Asymmetry
- 5 Conclusions

- Corrections due to initial state radiation (ISR) can be large, especially due to large logarithmic corrections

$$L = \ln(s/m_e^2) \approx 10.$$

- These corrections are important e.g.
 - for the prediction of the Z-boson peak
 - for $t\bar{t}$ production
 - associated Higgs production through $e^+e^- \rightarrow Z^*H^0$ at future e^+e^- colliders.
- We extend the known $O(\alpha^2)$ ISR corrections up to $O(\alpha^6L^5)$, including the first three subleading logarithmic corrections at lower orders.
- We extend the ISR corrections for the forward-backward asymmetry at leading logarithmic order to $O(\alpha^6L^6)$.



- 1988: First calculation to $O(\alpha^2)$ for the LEP analysis, through expansion of the phase space integrals (BBN).
[Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))]
- 2012: New calculation up to $O(\alpha^2)$ using the method of massive operator matrix elements.
[Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))]

⇒ Calculations **do not agree** at $O(\alpha^2 L^0)$!

- Errors in one of the calculations?
- Breakdown of factorization?

- We revisited the original calculation, doing the expansion in m_e at the latest stage
[Blümlein, De Freitas, Raab, KS (Nucl. Phys. B956 (2020))]
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Our approach to the recalculation:

- **full integration over the phase space** in $d = 4$, i.e. no a-priori expansion in the mass
 - this is a four-fold integration, three integrations can be performed using standard techniques
 - integrand of the last integral contains rational, logarithmic and polylogarithmic expressions with involved argument structures
 - the last integration is performed in terms of iterated integrals after determining the minimal set of contributing letters
 - the final result is expressed as iterated integrals over involved square root valued letters of the kind

$$f_1 = \frac{y}{\sqrt{1-y}(w^2 + 4zy)w},$$
$$f_2 = \frac{y}{\sqrt{y}\sqrt{y(1-z)^2 - 16\rho^2}(w^2 + 4zw)w}$$

and related with

$$w = \sqrt{y^2(1-z^2) - 8\rho(1+z)y + 16\rho^2}, \quad z = \frac{s'}{s}, \quad \rho = \frac{m^2}{s}$$

⇒ the analytic results can be expanded in the electron mass

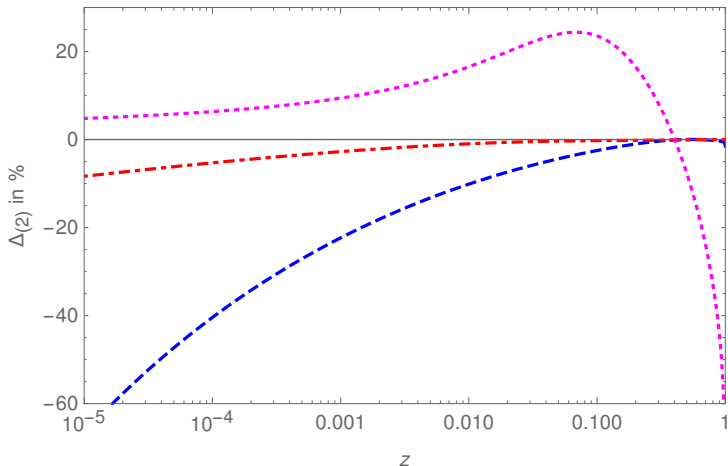
- as an example we find the **difference term to BBN** for process II:

$$\begin{aligned} \delta_{II} &= \frac{8}{3} \int_0^1 \frac{dy}{y} \sqrt{1-y} (2+y) \left[\frac{(1-z)(1-(4-z)z)y}{4z+(1-z^2)y} - \frac{1+z^2}{1-z} \ln \left(1 + \frac{(1-z)^2 y}{4z} \right) \right] \\ &= -\frac{128}{9} \left[3 + \frac{1}{(1-z)^3} - \frac{2}{(1-z)^2} - 2z \right] - 16 \left[1 + \frac{5z}{3} + \frac{8}{9} \frac{1}{(1-z)^4} - \frac{20}{9} \frac{1}{(1-z)^3} \right. \\ &\quad \left. + \frac{4}{9} \frac{1}{(1-z)^2} \right] \ln(z) + \frac{8}{3} \frac{1+z^2}{1-z} \left[\frac{10}{9} - \frac{14}{3} \ln(z) - \ln^2(z) \right], \end{aligned}$$

- in this case the difference can be attributed to the neglect of initial state electron masses
- in the pure-singlet process a calculation done for massless partons was reused
[Schellekens, van Neerven (Phys.Rev. D21 (1980))]

⇒ our results **agree** with the ones obtained using massive OMEs

Recalculation – Numerical Illustration



■ Relative deviation from BBN of process II (red), process III (blue) and process IV (magenta) contribution in %.

The initial state radiation factorizes from the born cross section:

$$\frac{d\sigma_{ij}}{ds'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{li} \left(z, \frac{\mu^2}{m_e^2} \right) \otimes \tilde{\sigma}_{lk} \left(z, \frac{s'}{\mu^2} \right) \otimes \Gamma_{kj} \left(z, \frac{\mu^2}{m_e^2} \right) + \mathcal{O} \left(\frac{m_e^2}{s} \right) = \frac{\sigma^{(0)}(s')}{s} H_{ij} \left(z, \frac{s}{m_e^2} \right)$$

with $z = s'/s$, μ the factorization scale, into: $\left[f(z) \otimes g(z) = \int_0^1 dx_1 \int_0^1 dx_2 f(x_1)g(x_2)\delta(z - x_1x_2), f(N) = \int_0^1 dz z^{N-1} f(z) \right]$

- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij} \left(z, \frac{s'}{\mu^2} \right)$
 - [Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]
 - [Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))]
 - [Duhr, Dulat, Mistelberger (Phys. Rev. Lett. 125 (2020))]
- massive operator matrix elements $\Gamma_{ij} \left(z, \frac{\mu^2}{m_e^2} \right)$, which carry all mass dependence
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Massless cross sections and massive operator matrix elements obey renormalization group equations:

- massless cross sections $\tilde{\sigma}_{ij}$

$$\left[\left(\frac{\partial}{\partial \lambda} - \beta(a) \frac{\partial}{\partial a} \right) \delta_{kl} \delta_{jm} + \frac{1}{2} \gamma_{kl}(N) \delta_{jm} + \frac{1}{2} \gamma_{jm}(N) \delta_{kl} \right] \tilde{\sigma}_{ij}(N) = 0$$

- massive operator matrix elements Γ_{ij}

$$\left[\left(\frac{\partial}{\partial \Lambda} + \beta(a) \frac{\partial}{\partial a} \right) \delta_{jl} + \frac{1}{2} \gamma_{kl}(N) \right] \Gamma_{ij}(N) = 0$$

with $\lambda = \ln(s'/\mu^2)$, $\Lambda = \ln(\mu^2/m_e^2)$, the QED β -function $\beta(a)$ and $a = \alpha/4$

- Here the usual anomalous dimensions, i.e. Mellin transforms of the splitting functions, contribute:

$$\gamma_{ij}(N) = - \int_0^1 dz z^{N-1} P_{ij}(z)$$

The Method of Massive Operator Matrix Elements

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z, L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^i a^i L^k c_{i,k}$$

The radiators:

$$c_{1,1} = -\gamma_{ee}^{(0)},$$

$$c_{1,0} = \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)},$$

$$c_{2,2} = \frac{1}{2}\gamma_{ee}^{(0)2} + \frac{\beta_0}{2}\gamma_{ee}^{(0)} + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)},$$

...

$$\begin{aligned} c_{3,1} = & -\gamma_{ee}^{(2)} - 2\Gamma_{ee}^{(0)}\gamma_{ee}^{(1)} - \Gamma_{ee}^{(0)}\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(1)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(1)} - \beta_1\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(1)}\tilde{\sigma}_{ee}^{(0)} \\ & - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{ee}^{(0)} - 2\Gamma_{ee}^{(0)}\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(1)}\tilde{\sigma}_{e\gamma}^{(0)} - \Gamma_{\gamma e}^{(0)}\gamma_{\gamma\gamma}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{\gamma e}^{(1)} + \beta_0 \left[-2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} \right. \\ & \left. - 2\tilde{\sigma}_{ee}^{(1)} - 2\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} \right] - \gamma_{ee}^{(0)} \left[\Gamma_{ee}^{(0)2} + 2\Gamma_{ee}^{(1)} + 2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} + \tilde{\sigma}_{ee}^{(1)} + \Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} \right], \end{aligned}$$

...

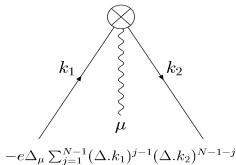
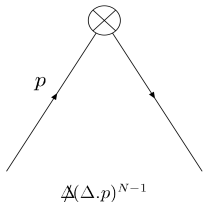
For the first three logarithmic orders we need the following ingredients:

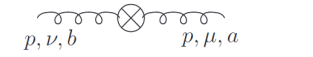
- splitting functions γ_{ij} up to three-loop order
[E.G. Floratos, D.A. Ross, C.T. Sachrajda (Nucl. Phys. B129 (1977))]
[A. Gonzalez-Arroyo, C. Lopez, F.J. Yndurain (Nucl. Phys. B153 (1979))]
...
[S. Moch, J. Vermaseren, A. Vogt (Nucl.Phys.B 688/691 (2004))]
- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij}$ up to two-loop order
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- massive operator matrix elements Γ_{ij} up to two-loop order¹
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⇒ $\Gamma_{\gamma e}$ was only considered up to one-loop order

¹In the case of massless external states massive operator matrix elements have been considered in the context of DIS.
[Buza, Matiounine, Smith, Migneron, van Neerven (Nucl. Phys. B472 (1996)),
Bierenbaum, Blümlein, Klein (Nucl. Phys. B820 (2009)), ...]

The Missing Operator Matrix Element $\Gamma_{\gamma e}$





$$\frac{1+(-1)^N}{2} (\Delta \cdot p)^{N-2} [g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu]$$

$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e | O_F^{\text{NS,S}} | e \rangle,$$

$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma | O_F^{\text{S}} | \gamma \rangle,$$

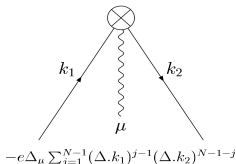
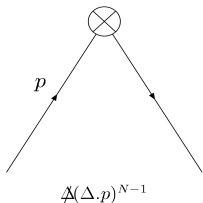
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e | O_V^{\text{S}} | e \rangle,$$

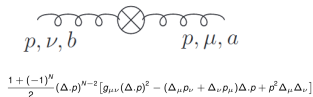
$$O_{F;\mu_1, \dots, \mu_N}^{\text{NS,S}} = i^{N-1} \text{S} [\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi] - \text{traces},$$

$$O_{V;\mu_1, \dots, \mu_N}^{\text{S}} = 2i^{N-2} \text{S} [F_{\mu_1\alpha} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^\alpha] - \text{traces}$$

- The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$.
- In the context of DIS **proven** to work at α_s^2 in the
 - non-singlet process
[Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))]
 - pure-singlet process
[Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))]

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$\frac{1+(-1)^N}{2} (\Delta.p)^{N-2} [g_{\mu\nu} (\Delta.p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta.p + p^2 \Delta_\mu \Delta_\nu]$

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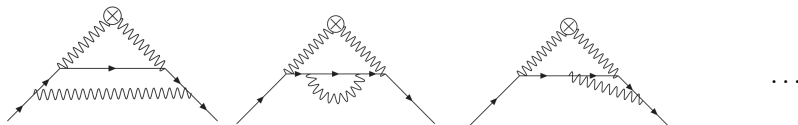
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- We have to compute on-shell 2-point functions with local operator insertions ($\Delta^2 = 0$).
- The operator can be resummed into a propagator like term:

$$\sum_{N=0}^{\infty} i^N (\Delta \cdot k)^N = \frac{1}{1 - i\Delta \cdot k}$$

- The calculation can now follow standard techniques:

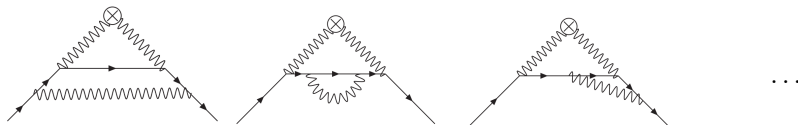
– Dimensional regularization to regulate integrals

– Feynman parameterization to combine denominators in the propagator

– Use the Mellin-Barnes representation for propagator denominators and to do the integrals

- For the calculations we make use of the packages `Sigma` [G. Schneider (Sem. Lothar. Combin.56 (2007))] and `HarmonicSums` [J. Ablinger et al. (arXiv:1011.1176)] .

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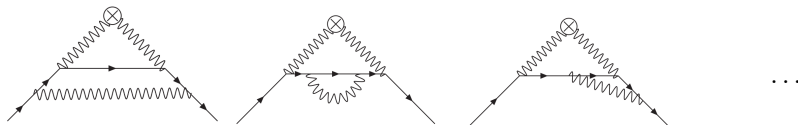


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 - Integration-By-Parts reduction to master integrals.
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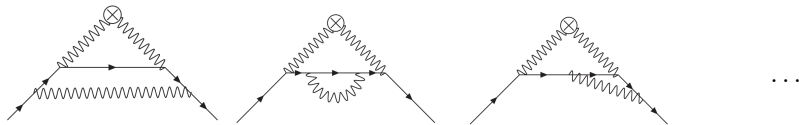


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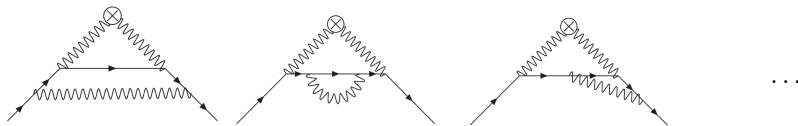


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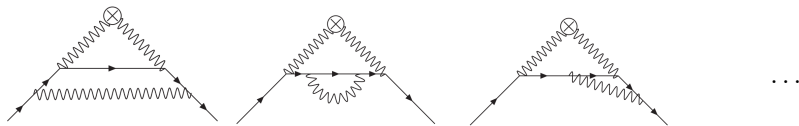


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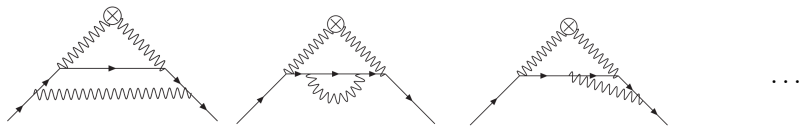


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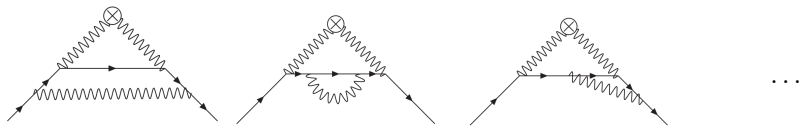


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$$\begin{aligned}
 \Gamma_{\gamma e}^{(1)}(N) = & \frac{P_8}{27(N-4)(N-3)(N-2)(N-1)N^4(N+1)^4} + \left(\frac{2P_7}{9(N-4)(N-3)(N-2)(N-1)N^3(N+1)^3} + \frac{2(N^2+N+2)}{(N-1)N(N+1)} S_2 \right) S_1 \\
 & + \frac{P_3}{3(N-2)(N-1)N(N+1)^2} S_1^2 + \frac{2(N^2+N+2)}{3(N-1)N(N+1)} S_1^3 + \frac{P_6}{3(N-2)(N-1)N^2(N+1)^2} S_2 + \frac{4(N^2+N+2)}{3(N-1)N(N+1)} S_3 \\
 & + \frac{3 \cdot 2^{6+N}}{(N-2)(N+1)^2} S_{1,1} \left(\frac{1}{2}, 1 \right) + \frac{2^{6-N} P_5}{3(N-3)(N-2)(N-1)^2 N^2} \left(S_2(2) + S_1 S_1(2) - S_{1,1}(1,2) - S_{1,1}(2,1) \right) \\
 & - \frac{32(N^2+N+2)}{(N-1)N(N+1)} \left[S_1(2) S_{1,1} \left(\frac{1}{2}, 1 \right) + S_{1,2} \left(\frac{1}{2}, 2 \right) - S_{1,1,1} \left(\frac{1}{2}, 1, 2 \right) - S_{1,1,1} \left(\frac{1}{2}, 2, 1 \right) - \frac{\zeta_2}{2} S_1(2) \right] \\
 & - \frac{48(N^2+N+2)}{(N-1)N(N+1)} S_{2,1} + \frac{4P_4}{(N-2)(N-1)N^2(N+1)^2} \zeta_2
 \end{aligned}$$

harmonic sums:

$$S_{a,\vec{b}} = S_{a,\vec{b}}(N) = \sum_{i=1}^N \frac{\text{sgn}(a)^i}{i^a} S_{\vec{b}}(i)$$

generalized harmonic sums:

$$S_{a,\vec{b}}(c, \vec{d}) = S_{a,\vec{b}}(c, \vec{d}; N) = \sum_{i=1}^N \frac{(\text{sgn}(a) \cdot c)^i}{i^a} S_{\vec{b}}(\vec{d}; i)$$

The Missing Operator Matrix Element $\Gamma_{\gamma e}$

Analytic Mellin-inversion with HarmonicSums:

$$\begin{aligned}
 \Gamma_{\gamma e}^{(1)}(z) = & \frac{P_9}{135z^3} - \frac{320 - 335z + 231z^2}{15z} H_0 + \frac{12 + 23z}{6} H_0^2 + \frac{2 - z}{3} H_0^3 + 32(2 - z) \left(\frac{(2 - z)^2}{3z^2} - H_0 \right) (\tilde{H}_{-1} \tilde{H}_0 - \tilde{H}_{0,-1}) \\
 & - 8(2 - z) H_{0,0,1} - \frac{96 - 190z + 118z^2 - 41z^3}{3z^2} H_1^2 - 32(2 - z) (\tilde{H}_{-1} \tilde{H}_0 - \tilde{H}_{0,-1}) \tilde{H}_1 \\
 & - \left(\frac{2(32 - 48z + 36z^2 - 13z^3)}{3z^2} + 4(2 - z) H_0 \right) H_{0,1} - \left(\frac{2P_{10}}{45z^4} - \frac{2(32 - 48z + 12z^2 + 7z^3)}{3z^2} H_0 \right) H_1 \\
 & + \frac{2(2 - 2z + z^2)}{z} \left(\frac{H_3^2}{3} + 8H_1 H_{0,1} + 16\tilde{H}_0 \tilde{H}_{0,-1} - 32\tilde{H}_{0,0,-1} - 16H_{0,1,1} + 8\tilde{H}_0 \zeta_2 \right) + \left(\frac{4(32 - 48z + 24z^2 - 3z^3)}{3z^2} \right. \\
 & \left. - 8(2 - z) (H_0 + 2\tilde{H}_1) \right) \zeta_2 + \frac{8(12 - 10z + 5z^2)}{z} \zeta_3
 \end{aligned}$$

harmonic polylogarithms of argument z and $1 - z$ ($\tilde{H}(z) = H(1 - z)$):

$$H_{a,\bar{b}} = H_{a,\bar{b}}(z) = \int_0^1 d\tau f_a(\tau) H_{\bar{b}}(\tau), \quad \text{with} \quad f_0(\tau) = \frac{1}{\tau}, \quad f_1(\tau) = \frac{1}{1 - \tau}, \quad f_{-1}(\tau) = \frac{1}{1 + \tau}$$

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z, L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^i a^i L^k c_{i,k}$$

- The radiators do not depend on the factorization scale, i.e. no collinear singularities for massive electrons.
- The analytic structures directly translate from the different ingredients.
- Radiators are distributions in z -space:

$$c_{i,j}(z) = c_{i,j}^{\delta} \delta(1-z) + c_{i,j}^+ + c_{i,j}^{\text{reg}}$$

$$c_{3,3}^{\delta} = \frac{572}{9} - \frac{704}{3} \zeta_2 + \frac{512}{3} \zeta_3,$$

$$c_{3,3}^+ = \left(\frac{5744}{27} - 256 \zeta_2 \right) \mathcal{D}_0 + \frac{1408}{3} \mathcal{D}_1 + 256 \mathcal{D}_2,$$

$$\mathcal{D}_k = \left(\frac{\ln^k(1-z)}{1-z} \right)_+,$$

$$c_{3,3}^{\text{reg}} = \left\{ \frac{16H_0 P_{104}}{9(z-1)} - \frac{4P_{131}}{27z} + \frac{8(3-19z^2)H_0^2}{3(z-1)} \right. \\ \left. + \left[\frac{16P_{105}}{9z} - \frac{128(1+z^2)H_0}{z-1} \right] H_1 - 128(1+z)H_1^2 \right. \\ \left. - \frac{352}{3}(1+z)H_{0,1} + \frac{736}{3}(1+z)\zeta_2 \right\}$$

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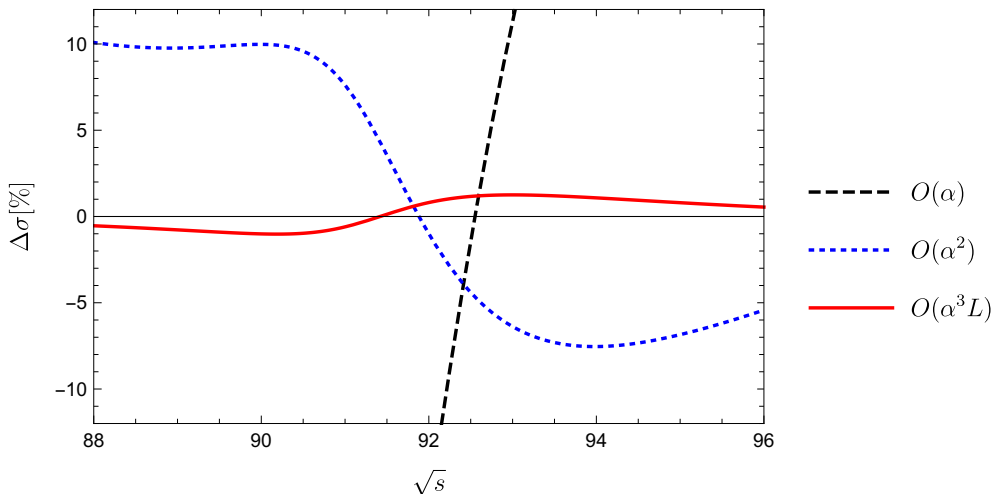
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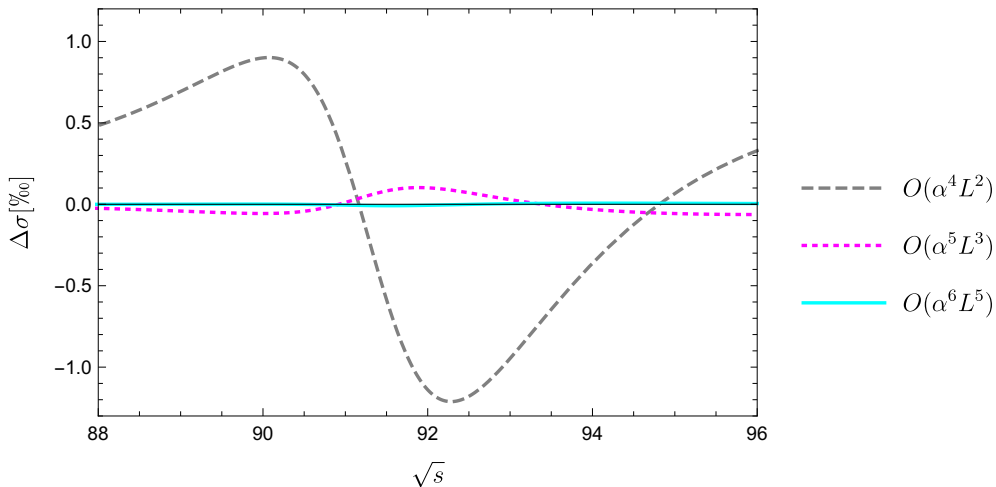
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- $\Delta\sigma$ is the change in the total cross section from one order to the other for $z_0 = 4m_\tau^2$



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- The forward-backward asymmetry is defined by:

$$A_{FB}(s) = \frac{\sigma_F(s) - \sigma_B(s)}{\sigma_F(s) + \sigma_B(s)},$$

with

$$\sigma_F(s) = 2\pi \int_0^1 d \cos(\theta) \frac{d\sigma}{d\Omega}, \quad \sigma_B(s) = 2\pi \int_{-1}^0 d \cos(\theta) \frac{d\sigma}{d\Omega},$$

and θ the angle between the incoming e^- and outgoing μ^- .

- The technique of radiators can also be used for A_{FB} : [Böhm et al. (LEP Physics Workshop 1989, p.203–234)]

$$A_{FB}(s) = \frac{1}{\sigma_F(s) + \sigma_B(s)} \int_{z_0}^1 dz \frac{4z}{(1+z)^2} H_{FB}(z) \sigma_{FB}^{(0)}(zs)$$

- Due to the angle dependence the radiators are not the same as in the total cross-section.

- At leading logarithmic (LL) accuracy the radiators are given by:

$$H_{FB}^{LL} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{(1+z)^2}{(x_1+x_2)^2} \Gamma_{ee}^{LL}(x_1) \Gamma_{ee}^{LL}(x_2) \delta(z - x_1 x_2).$$

- Due to the additional angle dependence the integral does not factorize with the Mellin-transform.
- At subleading logarithmic accuracy the integral will likely become more involved due to additional angle dependence of the cross-sections.
- The integrals can be solved analytically in Mellin and momentum fraction space.

- A direct integration in terms of iterated integrals at argument z is complicated by singularities at $z \rightarrow 0$ and $z \rightarrow 1$.
- A direct computation of the Mellin transform

$$\mathcal{M}[\tilde{H}_{FB}^{LL}(z)](N) = \int_0^1 dx_1 \int_0^1 dx_2 x_1^N x_2^N \left(\frac{(1+x_1 x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{\text{reg}}^{LL}(x_1) \Gamma_{\text{reg}}^{LL}(x_2)$$

is complicated by the symbolic powers of N .

- We chose to calculate the generating function:

$$\mathcal{G}[\tilde{H}_{FB}^{LL}(z)](t) = \sum_{N=0}^{\infty} t^N \mathcal{M}[\tilde{H}_{FB}^{LL}(z)](N) = \int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{1-t x_1 x_2} \left(\frac{(1+x_1 x_2)^2}{(x_1+x_2)^2} - 1 \right) \Gamma_{\text{reg}}^{LL}(x_1) \Gamma_{\text{reg}}^{LL}(x_2)$$

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$$f_{\{4,0\}}(\tau) = \frac{1}{1 + \tau^2},$$

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For example: $(S_{\bar{w}} \equiv S_{\bar{w}}(N))$

$$H_{FB}^{(2),LL}(N) = \frac{8(3N^2 + 3N - 1)P_1}{(N - 1)N^2(N + 1)^2(N + 2)(2N - 1)(2N + 3)} - \frac{32(4N^2 + 4N - 1)(-1)^N}{(2N - 1)(2N + 1)(2N + 3)} [S_{-1} + \ln(2)],$$

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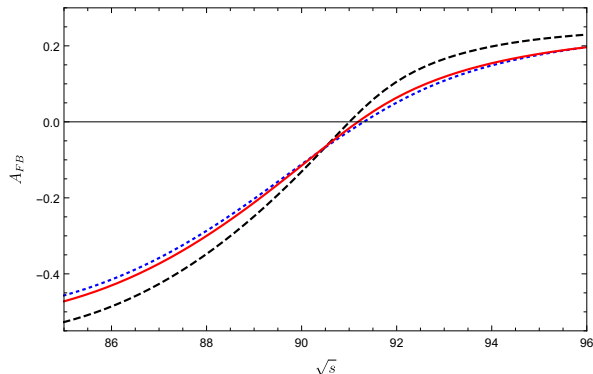
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For example: $(H_{\bar{w}} \equiv H_{\bar{w}}(\sqrt{z}))$

$$H_{FB}^{(2),LL}(z) = \frac{2(1-z)(1+z)^2}{z} + 2\pi \frac{(1-z)^2}{\sqrt{z}} - 8(1+z)H_0 - 8(1-z)^2 \frac{H_{\{4,0\}}}{\sqrt{z}},$$

$$H_{FB}^{(3),LL}(z) = \frac{64(1-z)^2}{\sqrt{z}} \left[H_{1,\{4,0\}} - H_{-1,\{4,0\}} - H_{\{4,0\},\{4,1\}} + \frac{1}{2}H_{0,\{4,0\}} \right] + \dots$$

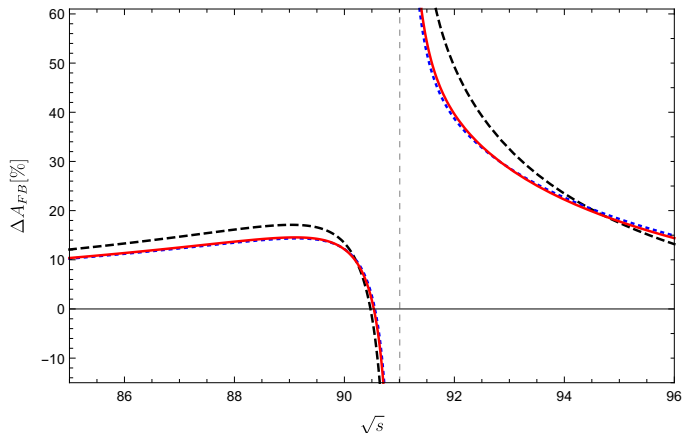
Application to A_{FB} – Numerical Results



A_{FB} evaluated at $s_- = (87.9 \text{ GeV})^2$, M_Z^2 and $s_+ = (94.3 \text{ GeV})^2$ for the cut $z > 4m_\tau^2/s$.

	$A_{FB}(s_-)$	$A_{FB}(M_Z^2)$	$A_{FB}(s_+)$
$\mathcal{O}(\alpha^0)$	-0.3564803	0.0225199	0.2052045
$+\mathcal{O}(\alpha L^1)$	-0.2945381	-0.0094232	0.1579347
$+\mathcal{O}(\alpha L^0)$	-0.2994478	-0.0079610	0.1611962
$+\mathcal{O}(\alpha^2 L^2)$	-0.3088363	0.0014514	0.1616887
$+\mathcal{O}(\alpha^3 L^3)$	-0.3080578	0.0000198	0.1627252
$+\mathcal{O}(\alpha^4 L^4)$	-0.3080976	0.0001587	0.1625835
$+\mathcal{O}(\alpha^5 L^5)$	-0.3080960	0.0001495	0.1625911
$+\mathcal{O}(\alpha^6 L^6)$	-0.3080960	0.0001499	0.1625911

A_{FB} and its initial state QED corrections as a function of \sqrt{s} . Black (dashed) the Born approximation, blue (dotted) the $\mathcal{O}(\alpha)$ improved approximation, red (full) also including the leading-log improvement up to $\mathcal{O}(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.



$$\Delta A_{FB} = 1 - \frac{A_{FB}^{(l)}}{A_{FB}^{(0)}},$$

where (l) denotes the order of ISR-corrections considered

ΔA_{FB} in % as a function of \sqrt{s} . Black (dashed) the $O(\alpha)$ improved approximation, blue (dotted) the $O(\alpha^2 L^2)$ improved approximation, red (full) also including the leading-log improvement up to $O(\alpha^6)$ for $s'/s \geq 4m_\tau^2/s$.

Conclusions:

- We calculated the ISR corrections to the process $e^+e^- \rightarrow \gamma^*/Z^*$ up to $O(\alpha^6 L^5)$ including the first three logarithmic terms at lower orders.
- We calculated the leading logarithmic ISR corrections to the forward-backward asymmetry up to $O(\alpha^6 L^6)$.
- The corrections can become important at future e^+e^- machines running at high luminosities at or close to the Z -pole.
- The radiators can be used for other processes like $e^+e^- \rightarrow t\bar{t}$ and $e^+e^- \rightarrow ZH$.

Outlook:

- The massless Drell-Yan cross sections are known up to $O(\alpha^3)$
⇒ An extension to the first four logarithmic orders is possible, but needs the calculation the operator matrix elements up to $O(\alpha^3)$ and the 4-loop splitting functions.
- The technique can be extended to subleading logarithmic corrections of A_{FB} .
- The method can be extended to QCD to study e.g. the heavy-quark initiated Drell-Yan process.

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Backup

- Usually it is more convenient to work in Mellin space:

$$M[f(z)](N) = \int_0^1 dz z^{N-1} f(z)$$

- Here the convolution \otimes

$$f(z) \otimes g(z) = \int_0^1 dz_1 \int_0^1 dz_2 f(z_1) g(z_2) \delta(z - z_1 z_2)$$

factorizes:

$$M[f(z) \otimes g(z)](N) = M[f(z)](N) \cdot M[g(z)](N)$$

	L^6	L^5	L^4	L^3	L^2	L	L^0
$O(\alpha)$						✓	✓
$O(\alpha^2)$					✓	✓	✓
$O(\alpha^3)$				✓	✓	✓	-
$O(\alpha^4)$			✓	✓	✓	-	-
$O(\alpha^5)$		✓	✓	✓	-	-	-
$O(\alpha^6)$	✓	✓	-	-	-	-	-

	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2 L^2)$:	-96.894	-177.147	-95.342	-176.235
$O(\alpha^2 L)$:	6.982	22.695	6.841	21.896
$O(\alpha^2)$:	0.176	-2.218	0.174	-2.001
$O(\alpha^3 L^3)$:	23.265	38.560	22.968	38.081
$O(\alpha^3 L^2)$:	-1.507	-1.888	-1.491	-1.881
$O(\alpha^3 L)$:	-0.152	0.105	-0.151	-0.084
$O(\alpha^4 L^4)$:	-1.857	0.206	-1.858	0.146
$O(\alpha^4 L^3)$:	0.131	-0.071	0.132	-0.065
$O(\alpha^4 L^2)$:	0.048	-0.001	0.048	0.001
$O(\alpha^5 L^5)$:	0.142	-0.218	0.144	-0.212
$O(\alpha^5 L^4)$:	-0.000	0.020	-0.001	0.020
$O(\alpha^5 L^3)$:	-0.008	0.009	-0.008	0.008
$O(\alpha^6 L^6)$:	-0.007	0.027	-0.007	0.027
$O(\alpha^6 L^5)$:	-0.001	0.000	-0.001	0.000

Table 1: Shifts in the Z -mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of $\Gamma_Z = 2.4952$ GeV and s -dependent width using $M_Z = 91.1876$ GeV and $s_0 = 4m_e^2$.

Application to A_{FB} – Numerical Results

- ΔA_{FB} is the change of the forward-backward asymmetry from one order to the other for $z_0 = 4m_\tau^2$

